Structure of finite groups with two real conjugacy class sizes

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Abstract. In this short note, we deteremine the structure of a finite group G satisfying $cs_r(G) = \{1, 2\}$, where $cs_r(G)$ denotes the set of real conjugacy class sizes of G.
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1. Introduction

All groups considered in this paper are finite. We called $x \in G$ a real element, if $x^g = x^{-1}$ for some $g \in G$, moreover, x^G is called a real conjugacy class size of G. We denote by $Re(G)$ the set of all real elements of G, and by $cs_r(G)$ the set of the real conjugacy class sizes of G. All unexplained notation and terminology are standard(see [4]).

There are many results illustrating the relationship between the structure of a group and the arithmetic property of real conjugacy class sizes. In [6], S. Dolfi, E. Pacifici and L. Sanus gave the structure of group G when $cs_r(G) = \{1, 2\}$:

Theorem A. *Let G be a group. Then $cs_r(G) = \{1, 2\}$ if and only if $G = A \times O$, where $cs_r(O) = \{1\}$, and either*

(a) A is a 2-group with $cs_r(A) = \{1, 2\}$; or

(b) $A = MP$, where M is a normal abelian 2-complement of A and P is a Sylow 2-subgroup of A, $C = C_P(M)$ has index 2 in P, and $Re(P) \subseteq Z(C)$.

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In this note, we point out that in case (b) of Theorem A above, M can be replaced by a subgroup generating by all real elements of odd order of G. Also, we will give a new proof of Theorem A.

2. Preliminaries

In this section we list some lemmas which will be used in the sequel.

Lemma 2.1 ([5, Theorem C]). *Let G be a group. All real classes of G have 2-power size if and only if G has a normal 2-complement K and $Re(G) \subseteq C_G(K)$.*

Lemma 2.2 ([1, Proposition 6.4]). *Let G be a group. Then every nontrivial real element in G has even order if and only if G has a normal Sylow 2-subgroup.*

Lemma 2.3 ([3, Theorem B]). *Let K be a group of odd order that acts on a 2-group P, and assume that K fixes all elements of order 2 in P and all real elements of order 4. Then K acts trivially on P.*

Lemma 2.4 ([2, Lemma 2.2]). *Let $N \trianglelefteq G$ and suppose that Nx is a real element in G/N. Assume that $|N|$ or the order of Nx in G/N is odd. Then $Nx = Ny$ for some real element y of G (of odd order if the order of Nx is odd).*

3. Proof of the Theorem A

Proof. Since $cs_r(G) = \{1, 2\}$, Lemma 2.1 implies that G has a normal 2-complement, say H. Hence H is solvable, so is G.

Let P be a Sylow 2-subgroup of G. Assume first P is normal in G. Now we consider the action of H on P. Clearly, $|G : C_G(x)| = 1$ or 2 for every real element x of P, forcing $H \leq C_G(x)$. By Lemma 2.3, it follows that H acts on P trivially. Hence $G = P \times H$. Further, $Re(H) = \{1\}$ by Lemma 2.2, Statement (a) of Theorem A holds.

Now suppose that P is not normal in G. By Lemma 2.2, there exists at least one non-trivial real element of odd order. Let Ω be the set of all non-trivial odd order real elements of G. Then $|\Omega| \geq 1$. Let $M := \langle \Omega \rangle$. Obviously, $M \trianglelefteq G$. Further, for every $w_1, w_2 \in \Omega$, $|G : C_G(w_i)| = 1$ or 2 for $i = 1, 2$. Hence $w_i \in C_G(w_j)$ for $i, j = 1, 2$, showing $M \leq O_{2'}(G)$ and $M \leq C_G(w)$. Consequently, M is abelian.

Let $\widetilde{G} := G/M$. By Lemma 2.4, we have that $\widetilde{G}$ has no non-trivial odd order real element. Therefore, $\widetilde{P} \trianglelefteq \widetilde{G}$ by Lemma 2.2. As the same in the first paragraph of our proof, we see that $\widetilde{G} = \widetilde{P} \times \widetilde{H}$, where $\widetilde{H}$ is the Hall $2'$-subgroup of $\widetilde{G}$ and $MP \trianglelefteq G$. Then $H \trianglelefteq G$ and $G = H \rtimes P$.

Now consider the action of P on H. By [4, 8.2.7], we have $H = [H, P]C_H(P)$. Note that $[H, P] \leq M$ and $M \leq Z(H)$, leading to $[H, P, H] \leq [M, H] = 1$. Thus $[P, H, H] = 1$. By the Three Subgroups Lemma, we ontain that $[H, H, P] = 1$, leading to $H' \leq C_H(P)$ and thus $C_H(P) \trianglelefteq H$. Note that $\widetilde{H}$ has only one real element $\widetilde{1}$, we have that $M \leq [H, P]$. Therefore, $M = [H, P]$. Since $M \leq Z(H)$, we have that $C_M(P) \leq Z(G)$. By [4, 8.4.2], we have $M = [M,P] \times C_M(P)$. Since $C_M(P) \leq Z(G)$, we may assume that $C_M(P) = 1$. Hence $M = [M,P]$. Now we have $M \cap C_H(P) = 1$ and $H = M \times C_H(P)$. Then $G = MP \times C_H(P)$.

Let $C := C_P(M)$. We claim that $C_G(a) = C_G(b)$ for every $a, b \in \Omega \setminus Z(G)$. Assume there exist $a, b \in \Omega \setminus Z(G)$ such that $C_G(a) \neq C_G(b)$. Then $G = C_G(a)C_G(b)$. Hence $(ab)^G = a^G b^G$ and $ab \in \Omega$ by [6, Lemma 2.5(i)]. It is easy to see that $ab \notin Z(G)$. Since $|(ab)^G| = |a^G| = |b^G|$, we have that $(ab)^G = \{ab, (ab)^{-1}\}$, $a^G = \{a, a^{-1}\}$ and $b^G = \{b, b^{-1}\}$. So we have that $a^G b^G = \{ab, ab^{-1}, a^{-1}b, a^{-1}b^{-1}\}$. It follows that $ab = ab^{-1}$ or $a^{-1}b$, that is, $b^2 = 1$ or $a^2 = 1$, a contradiction. This proves that $C_G(a) = C_G(b)$ for every $a, b \in \Omega \setminus Z(G)$, as required.

Furthermore, $C_P(a) = C_P(b) = C$, which implies that $|P : C| = 2$. It is easy to see that $Re(P) \subseteq C$. We assert that $C_G(z) = C_G(e)$ for every $z \in Re(P) \setminus Z(G), e \in \Omega \setminus Z(G)$. Otherwise, by the same reason as above, we $o(z) = 2$. Since $|G : C_G(z)| = 2$, we get that $\langle z \rangle \trianglelefteq G$. In particular, $z \in Z(G)$, a contradiction. Hence $C_G(z) = C_G(e)$. This shows that $C_P(z) = C$ and thus $z \in Z(C)$. So we get that $Re(P) \subseteq Z(C)$. □

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