Improved reverses Young type inequalities

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Abstract. The main purpose of this paper is to present new reverses Young type inequalities for numbers. Then, we use these inequalities to establish corresponding reverses Young type inequalities for operators.
Keywords: reverse Young type inequality, Kantorovich constant, positive operator.

1. Introduction

Throughout this paper, $\mathcal{B}(\mathcal{H})$ stands for the C^*-algebra of all bounded linear operators on a Hilbert space $(\mathcal{H}, <\cdot,\cdot>)$ and m, m', M, M' are scalars. An operator A is said to be positive (denoted $A \geq 0$) if $< Ax, x > \geq 0$ for all $x \in H$, and A is said to be strictly positive (denoted $(A > 0)$) if A is positive and invertible.

For $A, B > 0$ and $0 \leq v \leq 1$, the v-weighted arithmetic mean $A\nabla_v B$ and the v-weighted geometric mean $A\sharp_v B$ are defined by

$$A\nabla_v B = (1-v)\,A + vB, A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}},$$

when $v = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity for $A\nabla_{\frac{1}{2}}B$ and $A\sharp_{\frac{1}{2}}B$, respectively.

It is well-known that

$$A\sharp B \leq H_v(A,B) \leq A\nabla B,$$

where $H_v(A,B) = \frac{A\sharp_v B + A\sharp_{1-v}B}{2}$ is called Heinz mean.

The classical Young inequality says that if $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$a^v b^{1-v} \leq va + (1-v)b \tag{1.1}$$

with equality if and only if $a = b$. When $v = \frac{1}{2}$, we have $\sqrt{ab} \leq \frac{a+b}{2}$.

In 2011, Zou et al. [1] proved the refined version of (1.1)

$$va + (1-v)b \geq K(h,2)^r a^v b^{1-v}, \tag{1.2}$$

where $K(h,2) = \frac{(h+1)^2}{4h}$ is called Kantorovich constant, $h = \frac{a}{b}$ and $r = \min\{v, 1-v\}$.

Hu [2] obtained the refinement of the inequality (1.1) as follows:

$$v^2a^2 + (1-v)^2b^2 \geq v^2(a-b)^2 + \left[(va)^v b^{1-v}\right]^2, \quad 0 \leq v \leq \frac{1}{2} \tag{1.3}$$

and

$$v^2a^2 + (1-v)^2b^2 \geq (1-v)^2(a-b)^2 + \left[a^v\left((1-v)b\right)^{1-v}\right]^2, \quad \frac{1}{2} \leq v \leq 1. \tag{1.4}$$

Moreover, Nasiri and Liao [3] derived the following reverse Young inequalities:

$$(1-v)^{2v}\left[(1-2v)a + 2vb\right] + (1-v)^{2-2v}a^{2v}b^{1-2v}K(h,2)^{-r} \geq 2(1-v)\sqrt{ab}, \tag{1.5}$$

$$(1-v)^{2-2v}\left[2va + (1-2v)b\right] + (1-v)^{2v}a^{1-2v}b^{2v}K(h,2)^{-r} \geq 2(1-v)\sqrt{ab}, \tag{1.6}$$

where $0 \leq v \leq \frac{1}{2}$ and $r = \min\{2v, 1-2v\}$.

By (1.5) and (1.6), Nasiri and Liao [3] gave the following inequalities:

$$\begin{aligned}&(1-v)^{2v}\left(\frac{a+b}{2}\right) + (1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v} + a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-r}\\ &\geq 2(1-v)\sqrt{ab},\end{aligned} \tag{1.7}$$

$$\begin{aligned}&(1-v)^{2-2v}\left(\frac{a+b}{2}\right) + (1-v)^{2v}\left(\frac{a^{2v}b^{1-2v} + a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-r}\\ &\geq 2(1-v)\sqrt{ab},\end{aligned} \tag{1.8}$$

where $0 \leq v \leq \frac{1}{2}$ and $r = \min\{2v, 1-2v\}$.

For some related of Young inequalities, the reader is also referred to recent papers [4-8], and references therein.

Motivated by the aforementioned discussion, we will obtain sharper results than (1.5)-(1.8) in this paper .

2. Main results

In this section, we will give the reverses Young type inequalities for numbers and operators.

Theorem 1. *Let* $a, b \geq 0$ *and* $0 \leq v \leq \frac{1}{2}$, *then*

$$(1-v)^{2v}\left[(1-2v)a + 2vb\right] + (1-v)^{2-2v}a^{2v}b^{1-2v} \geq 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab} \tag{2.1}$$

and

$$(1-v)^{2-2v}\left[2va + (1-2v)b\right] + (1-v)^{2v}a^{1-2v}b^{2v} \geq 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab}, \tag{2.2}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $h = \frac{a}{b}$ *and* $r = \min\{2v, 1-2v\}$.

Proof. For $0 \le v \le \frac{1}{2}$, by inequality (1.2), we have

$$\begin{aligned}
&(1-v)^{2v}\left[(1-2v)a + 2vb\right] K(h,2)^{-\frac{r}{2}} \\
&+ (1-v)^{2-2v} a^{2v} b^{1-2v} K(h,2)^{-\frac{r}{2}} - 2(1-v)\sqrt{ab} \\
&\ge (1-v)^{2v} a^{1-2v} b^{2v} K(h,2)^{\frac{r}{2}} + (1-v)^{2-2v} a^{2v} b^{1-2v} K(h,2)^{-\frac{r}{2}} - 2(1-v)\sqrt{ab} \\
&= \left[(1-v)^{v} a^{\frac{1-2v}{2}} b^{v} K(h,2)^{\frac{r}{4}} - (1-v)^{1-v} a^{v} b^{\frac{1-2v}{2}} K(h,2)^{-\frac{r}{4}}\right]^2 \\
&\ge 0.
\end{aligned}$$

Thus, (2.1) holds.

Similarly, we have

$$\begin{aligned}
&(1-v)^{2-2v}\left[2va + (1-2v)b\right] K(h,2)^{-\frac{r}{2}} \\
&+ (1-v)^{2v} a^{1-2v} b^{2v} K(h,2)^{-\frac{r}{2}} - 2(1-v)\sqrt{ab} \\
&\ge (1-v)^{2-2v} a^{2v} b^{1-2v} K(h,2)^{\frac{r}{2}} + (1-v)^{2v} a^{1-2v} b^{2v} K(h,2)^{-\frac{r}{2}} - 2(1-v)\sqrt{ab} \\
&= \left[(1-v)^{1-v} a^{v} b^{\frac{1-2v}{2}} K(h,2)^{\frac{r}{4}} - (1-v)^{v} a^{\frac{1-2v}{2}} b^{v} K(h,2)^{-\frac{r}{4}}\right]^2 \\
&\ge 0.
\end{aligned}$$

Thus, (2.2) holds.

This completes the proof. □

Corollary 1. *Let* $a, b \ge 0$ *and* $0 \le v \le \frac{1}{2}$, *then*

$$(1-v)^{2v}\left[(1-v)a + vb\right] + (1-v)^{2-2v} a^{v} b^{1-v} \ge 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab}$$

and

$$(1-v)^{2-2v}\left[va + (1-v)b\right] + (1-v)^{2v} a^{1-v} b^{v} \ge 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab},$$

where $K(h) = \frac{(h+1)^2}{4h}$, $h = \frac{a}{b}$ *and* $r = \min\{v, 1-v\}$.

Corollary 2. *Let* $a, b \ge 0$ *and* $0 \le v \le \frac{1}{2}$, *then*

$$\begin{aligned}
&(1-v)^{2v}\left(\frac{a+b}{2}\right) + (1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v} + a^{1-2v}b^{2v}}{2}\right) \\
&\ge 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab}
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&(1-v)^{2-2v}\left(\frac{a+b}{2}\right) + (1-v)^{2v}\left(\frac{a^{2v}b^{1-2v} + a^{1-2v}b^{2v}}{2}\right) \\
&\ge 2(1-v)K(h,2)^{\frac{r}{2}}\sqrt{ab},
\end{aligned} \tag{2.4}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $h = \frac{a}{b}$ *and* $r = \min\{2v, 1-2v\}$.

Remark 1. Our results are better than (1.5)-(1.8). In fact, we have

$$
\begin{aligned}
&(1-v)^{2v}\left(\frac{a+b}{2}\right) + (1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-r} \\
&-(1-v)^{2v}\left(\frac{a+b}{2}\right)K(h,2)^{-\frac{r}{2}} - (1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-\frac{r}{2}} \\
&= (1-v)^{2v}\left(\frac{a+b}{2}\right)\left[1-K(h,2)^{-\frac{r}{2}}\right] \\
&-(1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-\frac{r}{2}}\left[1-K(h,2)^{-\frac{r}{2}}\right] \\
&= \left[1-K(h,2)^{-\frac{r}{2}}\right]\left[(1-v)^{2v}\left(\frac{a+b}{2}\right)\right. \\
&\left.-(1-v)^{2-2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-\frac{r}{2}}\right] \\
&\geq \left[1-K(h,2)^{-\frac{r}{2}}\right]\left[(1-v)^{2v}\left(\frac{a+b}{2}\right) - (1-v)^{2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)\right] \\
&\geq 0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&(1-v)^{2-2v}\left(\frac{a+b}{2}\right) + (1-v)^{2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-r} \\
&\geq (1-v)^{2-2v}\left(\frac{a+b}{2}\right)K(h,2)^{-\frac{r}{2}} + (1-v)^{2v}\left(\frac{a^{2v}b^{1-2v}+a^{1-2v}b^{2v}}{2}\right)K(h,2)^{-\frac{r}{2}}.
\end{aligned}
$$

Theorem 2. *Let* A, B *be two operators satisfy* $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$ *where* m, m', M', M *are positive real numbers such that* $m < m' < M' < M$ *. Then for* $0 \leq v \leq \frac{1}{2}$,

$$(1-v)^{2v}(A\nabla B) + (1-v)^{2-2v}H_{2v}(A,B) \geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B \tag{2.5}$$

and

$$(1-v)^{2-2v}(A\nabla B) + (1-v)^{2v}H_{2v}(A,B) \geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B, \tag{2.6}$$

where $K(h) = \frac{(h+1)^2}{4h}$, $h = \frac{M}{m}$, $h' = \frac{M'}{m'}$ *and* $r = \min\{2v, 1-2v\}$.

Proof. If $0 \leq v \leq \frac{1}{2}$ and $x \geq 0$, by inequality (2.1), we have

$$(1-v)^{2v}\left[(1-2v)+2vx\right] + (1-v)^{2-2v}x^{1-2v} \geq 2(1-v)K(x,2)^{\frac{r}{2}}\sqrt{x}.$$

Let $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. It is easy to know that

$$0 < h'I \leq X \leq hI.$$

By the monotonicity of operator functions and (2.1), we have

$$(1-v)^{2v}\left[(1-2v)+2vA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right]+(1-v)^{2-2v}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-2v}$$
$$\geq 2(1-v)K(h,2)^{\frac{r}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}$$

and so

$$(1-v)^{2v}\left[(1-2v)A+2vB\right]+(1-v)^{2-2v}A\sharp_{1-2v}B\geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B. \tag{2.7}$$

Replacing A and B by B and A respectively in (2.7), we have

$$(1-v)^{2v}\left[(1-2v)B+2vA\right]+(1-v)^{2-2v}A\sharp_{2v}B\geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B. \tag{2.8}$$

Summing (2.7) and (2.8), we have

$$(1-v)^{2v}(A\nabla B)+(1-v)^{2-2v}H_{2v}(A,B)\geq 2(1-v)K(h,2)^{\frac{r}{2}}A\sharp B.$$

Similarly, (2.6) holds.

This completes the proof. □

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