Some types of filters and states on hyper NM-algebras

Xiao Li Gao
School of Mathematics
Northwest University
Xi'an, 710127
China
xiligao@stumail.nuw.edu.cn

Xiao Long Xin*
School of Mathematics
Northwest University
Xi'an, 710127
China
xlxin@nwu.edu.cn

Jian Ming Zhan
Department of Mathematics
Hubei University for Nationalities
Enshi, 445000
China
zhanjianming@hotmail.com

Abstract. In this paper, we apply the hyper structures theory to NM-algebras and define the notion of hyper NM-algebras, which is a generalization of NM-algebras, and obtain some related results. We also state that any hyper NM-algebra is a hyper residuated lattice and the converse may not hold. Furthermore, we put forth some types of filters and deductive systems of hyper NM-algebras, such as (weak)h-filters, weak h-deductive systems and (positive) implicative weak h-deductive systems. Especially, we focus on discussing relationships of them. Finally, we give the definitions of sup-state, inf-state and hyper state on hyper NM-algebras and obtain there exists a hyper NM-algebra with sup-state and inf-state.

Keywords: hyper NM-algebra, weak h-filter, weak h-deductive system, sup-state, inf-state.

1. Introduction

In past several years, the study of fuzzy logic and fuzzy reasoning has greatly increased. Some scholars obtained many interesting results on t-norm based fuzzy logics. Hájek's basic logic system BL is an ideal formalization or logical framework of continuous t-norms and their residua [13]. Based on Hájek's work, Esteva and Godo proposed a new formal deductive system MTL, called monoidal t-norm based logic, intended to cope with left-continuous t-norms and

∗. Corresponding author

their residua [9]. As important schematic extensions of MTL, they gave weak nilpotent minimum logic WNM, the involutive monoidal t-norm based logic IMTL and nilpotent minimum logic NM. The system NM is a common schematic extension of WNM and IMTL which is an ideal formalization of Fodor's nilpotent minimums. We know that nilpotent minimums are important t-norms and possess many excellent properties which are proposed by Fodor in 1995 [10].

In order to provide a logic foundation for fuzzy reasoning and to reduce the gap between fuzzy reasoning and artificial intelligence, a new formal deductive system $\mathcal{L}^*$ for fuzzy propositional calculus is proposed by G. J. Wang [22]. D. W. Pei introduced R_0-algebras as the algebraic completeness of $\mathcal{L}^*$. He also proved the standard completeness of $\mathcal{L}^*$ with respect to R_0-algebras [21], and nilpotent minimum algebras are equivalent to R_0-algebras [20]. Consequently, NM-algebras play an important role in studying fuzzy logics and the corresponding algebraic structures.

The hyper structure theory (called also multialgebra) was introduced by F. Marty in 1934 at 8th congress of Scandinavian Mathematicians [18]. In his definition, a function $f : A \times A \to P^*(A)$, of the set $A \times A$ into the set of all nonempty subsets of A, is called a binary hyperoperation, and the pair (A, f) is called a hypergroupoid. If f is associative, A is called a semihypergroup, and it is said to be commutative if f is commutative. Also, an element $1 \in A$ is called the unit or the neutral element if $a \in f(1, a)$, for all $a \in A$. Since then many researchers have worked on this area [1, 6, 7]. Zahiri et al. applied the hyperstructures to lattices and residuated lattice [2, 27]. Borzooei et al. introduced and studied hyper K-algebras [5]. Ghorbani et al. applied the hyperstructures to MV-algebras, named hyper MV-algebras which are a generalization of MV-algebras, and defined some new types of deductive systems on hyper MV-algebras, such as (weak)hyper MV-deductive systems and (weak) implicative hyper MV-deductive systems, and studied their relations [11, 12, 15, 16]. Xin et al. considered the concepts of states, state operators and state-morphism operators on hyper BCI/BCK-algebras [3, 4, 14, 23, 24, 25].

Dvurečenskij et al.[8] introduced hyper effect algebras as a generalization of effect algebras and presented basic notions like states on hyper effect algebras. They also provided a representation of any finite linearly ordered hyper effect algebra. Xin et al.[26] defined the concept of hyper BL-algebras which is a generalization of BL-algebras. In particular, they defined the concept of regular compatible congruence on hyper BL-algebras and construct the quotient structure in hyper BL-algebras. Finally, they discussed the conditions in which a quotient hyper BL-algebra is an MV-algebra.

At present, the research papers about hyper NM-algebras have not been proposed. Hence, in order to enrich the theories of hyper structures, it is meaningful to construct hyper NM-algebras as a generalization of the concept of NM-algebras. We show that hyper NM-algebras are more complicated structures than NM-algebras, because in general, they are not necessarily transitive.

In NM-algebras, the operation of two elements is an element, while in hyper NM-algebras the operation of two elements is a subset.

This paper is structured in five sections. In order to make the paper as self-contained as possible, in Section 2, we recapitulate some basic notions and some results of some hyperstructures. In Section 3, we exhibit an axiom system of hyper NM-algebras and study some properties of them. Also, we state that any hyper NM-algebra is hyper residuate lattice. In Section 4, we investigate and prove some propositions about (weak) h-filters and weak h-deductive systems on hyper NM-algebras. In Section 5, we give three types of states on hyper NM-algebras and study some basic properties of them.

2. Preliminaries

In this section, we recall some definitions and results which will be used in the following sections.

Definition 2.1 ([9]). *An algebra* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *is called an NM-algebra if it satisfies the following conditions: for all* $x, y, z \in L$,

(1) $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
(2) $(L, \odot, 1)$ *is a commutative monoid,*
(3) $x \odot y \leq z$ *if and only if* $x \leq y \rightarrow z$,
(4) $(x \rightarrow y) \vee (y \rightarrow x) = 1$,
(5) $((x \odot y) \rightarrow 0) \vee ((x \wedge y) \rightarrow (x \odot y)) = 1$,
(6) $(x \rightarrow 0) \rightarrow 0 = x$.

R_0-algebras were introduced by Prof.Wang as an algebraic structure counterpart of the formal logic $\mathcal{L}^*$ in 1996.

Definition 2.2 ([22]). *Let* M *be an algebra of type* $(\neg, \vee, \rightarrow)$. *If* $(M, \leq, 0, 1)$ *is a bounded distributive lattice with a partial order* $\leq$ *(0 and 1 are the least element and the greatest element of* M *with respect to* $\leq$*, respectively),* $\vee$ *is the supermum operator, and* $\neg$ *is an order-reserving involution, then* M *is called an* R_0*-algebra, if the following conditions are satisfied: for all* $x, y, z \in M$,

$(M1)$ $\neg x \rightarrow \neg y = y \rightarrow x$,
$(M2)$ $1 \rightarrow x = x, x \rightarrow x = 1$,
$(M3)$ $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,
$(M4)$ $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
$(M5)$ $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$, $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$,
$(M6)$ $(x \rightarrow y) \wedge ((x \rightarrow y) \rightarrow (\neg x \vee y)) = 1$.

Remark 2.3. Let L be an NM-algebras. For all $x, y \in L$, we define $x \odot y = \neg(x \rightarrow \neg y)$. Then R_0-algebras are equivalent to NM-algebras in [20]. Therefore, they have similar properties.

Proposition 2.4 ([22]). *Let* L *be an* R_0*-algebra. The following properties hold: for all* $x, y, z \in L$,

(1) $x \leq y$ *if and only if* $x \to y = 1$,
(2) $x \leq y \to x$,
(3) $\neg x = x \to 0$,
(4) $x \leq y$ *implies* $y \to z \leq x \to z$,
(5) $x \leq y$ *implies* $z \to x \leq z \to y$,
(6) $((x \to y) \to y) \to y = x \to y$,
(7) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(8) $x \odot \neg x = 0$,
(9) $x \odot y \leq x \wedge y$ *and* $x \odot (x \to y) \leq x \wedge y$,
(10) $(x \odot y) \to z = x \to (y \to z)$,
(11) $x \leq y \to (x \odot y)$,
(12) $x \leq y$ *implies* $x \odot z \leq y \odot z$,
(13) $x \to y \leq (y \to z) \to (x \to z)$,
(14) $(x \to y) \odot (y \to z) \leq x \to z$,
(15) $x \to y = x \to (x \wedge y)$,
(16) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

Definition 2.5 ([18]). *Let H be a nonempty set and "$\circ$" be a function from H^2 to $P(H) \setminus \{\emptyset\}$. Then "$\circ$" is called a hyper operation on H.*

Note that for any two subsets A and B of H, denote the set $\bigcup_{a \in A, b \in B}\{a \circ b\}$ by $A \circ B$. We use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

In a classical algebraic structure, the operation of two elements is an element, while in an algebraic hyper structure the operation of two elements is a subset.

Definition 2.6 ([19]). *A super lattice is a partially ordered set $(S, \leq)$ endowed with two binary hyperoperations $\vee$ and $\wedge$ satisfying the following properties: for all $x, y, z \in S$,*
$(SL1)$ $x \in (x \vee x) \cap (x \wedge x)$,
$(SL2)$ $x \vee y = y \vee x$, $x \wedge y = y \wedge x$,
$(SL3)$ $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
$(SL4)$ $x \in ((x \vee y) \wedge x) \cap ((x \wedge y) \vee x)$,
$(SL5)$ $x \leq y$ *implies* $y \in x \vee y$ *and* $x \in x \wedge y$,
$(SL6)$ *if* $x \in x \wedge y$ *or* $y \in x \vee y$, *then* $x \leq y$.

Proposition 2.7 ([19]). *Let $(L, \leq)$ be a partially ordered set with the least element 0 and the largest element 1. Define two binary hyperoperations $\vee$ and $\wedge$ on L as follows: $a \vee b = \{c | a \leq c, b \leq c\}$ and $a \wedge b = \{c | c \leq a, c \leq b\}$, for all $a, b \in L$. Then $(L, \vee, \wedge)$ is a bounded super lattice.*

Definition 2.8 ([1]). *Let A be a set, $\odot$ be a binary hyperoperation on A and $1 \in A$. $(A, \odot, 1)$ is called a commutative semihypergroup with 1 as an identity if it satisfies the following properties: for all $x, y, z \in A$,*
$(CSHG1)$ $x \odot (y \odot z) = (x \odot y) \odot z$,
$(CSHG2)$ $x \odot y = y \odot x$,

$(CSHG3)$ $x \in 1 \odot x$.

Moreover if $a \in A$ *such that* $|a \odot x| = 1$ *for all* $x \in A$, a *is called a scalar element of* A, *simply a scalar of* A.

Definition 2.9 ([27]). *By a hyper residuated lattice, we mean a nonempty set* L *endowed with four binary hyperoperations* $\vee, \wedge, \odot, \rightarrow$ *and two constants* 0 *and* 1 *satisfying the following conditions: for all* $x, y, z \in L$,

$(HRL1)$ $(L, \vee, \wedge, 0, 1)$ *is a bounded super lattice,*

$(HRL2)$ $(L, \odot, 1)$ *is a commutative semihypergroup with* 1 *as an identity,*

$(HRL3)$ $x \odot z \ll y$ *if and only if* $z \ll x \rightarrow y$,

where $A \ll B$ *means that there exist* $a \in A$ *and* $b \in B$ *such that* $a \leq b$, *for all nonempty subsets* A *and* B *of* L.

Definition 2.10 ([17]). *Let* L *be an* R_0*-algebra. A state on* L *is a function* $s : L \rightarrow [0, 1]$ *such that the following conditions hold: for all* $x, y \in L$,

(1) $s(0) = 0, s(1) = 1$,

(2) $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)$.

Proposition 2.11 ([17]). *Let* s *be a state on* R_0*-algebra* L. *Then the following properties hold:*

(1) *if* $x \leq y$, *then* $s(x) \leq s(y)$,

(2) $s(\neg x) = 1 - s(x)$,

(3) $s(x \oplus y) + s(x \odot y) = s(x) + s(y)$,

(4) *if* $x \leq \neg y$, *then* $s(x \oplus y) = s(x) + s(y)$,

(5) *if* $x = \neg x$, *then* $s(x) = \frac{1}{2}$,

(6) $ker(s) = \{x : s(x) = 1\}$ *is a proper filter of* L.

Proposition 2.12 ([17]). *Let* L *be an* R_0*-algebra and* $s : L \rightarrow [0, 1]$ *be a function. Then* s *is a state if and only if it satisfies Definition 2.10(1) and Proposition 2.11(3).*

Thanks to R_0-algebras that have been shown to be equivalent to NM-algebras, we can get that the properties of state on NM-algebras are similar to R_0-algebras. In section 5, we will not repeat to introduce the properties of state on NM-algebras.

3. Hyper NM-algebras

In this section, we will introduce the notion and some properties of hyper NM-algebras.

Definition 3.1. *By a hyper NM-algebra we mean a nonempty set* L *endowed with four binary hyperoperations* $\vee, \wedge, \odot, \rightarrow$ *and two constants 0 and 1 satisfying the following conditions: for all* $x, y, z \in L$,

$(HNM1)$ $(L, \vee, \wedge, 0, 1)$ *is a bounded super lattice,*

$(HNM2)$ $(L, \odot, 1)$ *is a commutative semihypergroup with 1 as an identity,*

($HNM3$) *$x \odot y \ll z$ if and only if $x \ll y \to z$,*
($HNM4$) *$1 \in (x \to y) \vee (y \to x)$,*
($HNM5$) *$1 \in ((x \odot y) \to 0) \vee ((x \wedge y) \to (x \odot y))$,*
($HNM6$) *$x \in (x \to 0) \to 0$,*
where $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$, for all nonempty subsets A and B of L.

In the following, we give some examples of hyper NM-algebras.

Example 3.2. Let $L = \{0, a, b, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < a < b < 1$. Consider the following tables:

$\vee$	0	a	b	1
0	{0, a, b, 1}	{a, b, 1}	{b, 1}	{1}
a	{a, b, 1}	{a, b, 1}	{b, 1}	{1}
b	{b, 1}	{b, 1}	{b, 1}	{1}
1	{1}	{1}	{1}	{1}

$\wedge$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{0}	{0, a}	{0, a}	{0, a}
b	{0}	{0, a}	{0, a, b}	{0, a, b}
1	{0}	{0, a}	{0, a, b}	{0, a, b, 1}

$\to$	0	a	b	1
0	{1}	{1}	{1}	{1}
a	{a, b, 1}	{a, 1}	{a, b, 1}	{1}
b	{a, b, 1}	{a, b, 1}	{b, 1}	{1}
1	{0, a, b, 1}	{a, b, 1}	{a, b, 1}	{0, 1}

Let $\odot = \wedge$. We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Example 3.3. Let $L = [0, 1]$. Define a unary operation"$\neg$" on L by $\neg x = 1 - x$, and four binary hyperoperations $\vee, \wedge, \odot$ and $\to$ as follows: $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $x \to y = \begin{cases} 1, & \text{x} \leq \text{y} \\ \neg x \vee y, & \text{x} > \text{y} \end{cases}$, and $x \odot y = [\neg(x \to \neg y), 1]$, for all $x, y \in L$. Then we can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Example 3.4. Let $L = [0, 1]$. Define a unary operation"$\neg$" by $\neg x = 1 - x$, and four binary hyperoperations $\vee, \wedge, \odot$ and $\to$ as follows: $x \vee y = \{z \mid x \leq z, y \leq z\}$, $x \odot y = x \wedge y = \{z \mid z \leq x, z \leq y\}$, and $x \to y = \neg x \vee y$, for all $x, y \in L$. We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Proposition 3.5. (1) *Let $(L, \vee, \wedge, \odot, \to, 0, 1)$ be an NM-algebra. We define $x \bar{\circ} y = \{x \circ y\}$, for any $\circ \in \{\vee, \wedge, \odot, \to\}$. Then $(L, \bar{\vee}, \bar{\wedge}, \bar{\odot}, \bar{\to}, 0, 1)$ is a hyper NM-algebra.*

(2) *Let* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *be a hyper NM-algebra satisfying that four hyperoperations* $\vee, \wedge, \odot, \rightarrow$ *are all binary operations. Then* L *is an NM-algebra.*

(3) *Any hyper NM-algebra is a hyper residuated lattice.*

Proof. (1) Straightforward.

(2) Let L be a hyper NM-algebra satisfying that all binary hyperoperations are binary operations. Then by (HNM1)-(HNM3), we have L is a residuated lattice. By (HNM4)-(HNM6), we can get $1 = (x \rightarrow y) \vee (y \rightarrow x)$, $1 = ((x \odot y) \rightarrow 0) \vee ((x \wedge y) \rightarrow (x \odot y))$, $x = (x \rightarrow 0) \rightarrow 0$, respectively. Therefore, L is an NM-algebra.

(3) Straightforward. □

Remark 3.6. (1) From Proposition 3.5(1), we have that any NM-algebra can be seen as a hyper NM-algebra. And from Proposition 3.5(2), we know that the notion of hyper NM-algebras is a generalization of the notion of NM-algebras.

(2) From Proposition 3.5(3), we know that any hyper NM-algebra is a hyper residuated lattice. L is a residuated lattice while all hyperoperations of a hyper NM-algebra L are binary operations.

In the following example we show that any hyper NM-algebra is hyper residuated lattice, but the converse may be not.

Example 3.7. Let $L = \{0, a, b, c, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < c < a < b < 1$. Define the binary hyperoperations $\vee, \wedge, \odot$ on L as follows: $x \vee y = \{z \mid x \leq z, y \leq z\}$ and $x \odot y = x \wedge y = \{z \mid z \leq x, z \leq y\}$, for all $x, y \in L$. Now, let $\rightarrow$ be a hyperoperation on L defined by the following table.

→	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{0, 1}	{1}	{1}	{c, 1}	{1}
b	{0, 1}	{a, b, 1}	{1}	{c, 1}	{1}
c	{0, 1}	{1}	{1}	{1}	{1}
1	{0, 1}	{a, b, 1}	{b, 1}	{c, 1 }	{1}

We can check that $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a hyper residuated lattice ([27]). But it isn't a hyper NM-algebra, since $a \notin (a \rightarrow 0) \rightarrow 0$.

Now, we characterize some basic properties of hyper NM-algebras.

Proposition 3.8. *Let* L *be a hyper NM-algebra. Then the following properties hold: for any* $x, y, z \in L$ *and for any nonempty subsets* A, B, C *of* L,

(1) $1 \ll A$ *implies* $1 \in A$, $A \ll 0$ *implies* $0 \in A$,

(2) $A \odot B \ll C$ *if and only if* $A \ll B \rightarrow C$,

(3) $A \ll x \ll B$ *implies* $A \ll B$,

if $A \cap B \neq \emptyset$, *then* $A \ll B$ *and* $B \ll A$,

(4) $x \odot (x \rightarrow y) \ll y$, $x \ll y \rightarrow (x \odot y)$,

(5) $x \ll (x \to y) \to y$, $A \ll (A \to B) \to B$,

(6) $x \to y \ll ((x \to y) \to y) \to y$,

(7) $x \leq y$ *implies* $1 \in x \to y$, *and if* 1 *is a scalar element of* L, *the converse holds,*

$A \ll B$ *implies* $1 \in A \to B$,

(8) $1 \in x \to x$, $1 \in x \to 1$, *and* $1 \in 0 \to x$,

(9) *if* 1 *is a scalar element, then* $x \in 1 \to x$ *and* x *is the greatest element of* $1 \to x$,

(10) $(x \odot y) \to z \ll x \to (y \to z)$,

$(x \odot y) \to z \ll y \to (x \to z)$,

(11) $x \to (y \to z) \ll (x \odot y) \to z$,

$y \to (x \to z) \ll (x \odot y) \to z$,

(12) $x \to (y \to z) \ll y \to (x \to z)$,

(13) $x \leq y$ *implies* $x \odot z \ll y \odot z$, $z \to x \ll z \to y$, *and* $y \to z \ll x \to z$,

(14) $x \leq y$ *implies* $y^- \ll x^-$, *where* $x^- = x \to 0$,

(15) $x \odot y \ll x$, $x \odot y \ll y$. *Particularly,* $0 \in x \odot 0$,

(16) $x \ll y \to x$, $1 \in x \to (y \to x)$,

(17) $y \ll (x \to y) \to y$, $B \ll (A \to B) \to B$,

(18) $0 \in x \odot x^-$,

(19) $x \leq y$ *and* $x \leq z$ *imply* $x \ll y \wedge z$,

$y \leq x$ *and* $z \leq x$ *imply* $y \vee z \ll x$,

(20) $x \wedge y \ll x$, $x \ll x \vee y$, $A \wedge B \ll A$, $A \ll A \vee B$,

(21) $x \to (x \wedge y) \ll x \to y$,

(22) $x \ll y^- \Leftrightarrow 0 \in x \odot y$.

Proof. (1) If $1 \ll A$, then $1 \leq a$, for some $a \in A$. Hence $1 = a \in A$. If $A \ll 0$, then $a \leq 0$, for some $a \in A$. Hence $0 = a \in A$.

(2) Let $A \ll B \to C$. Then there exist $a \in A$, $b \in B$, and $c \in C$ such that $a \ll b \to c$. Therefore $a \odot b \ll c$. This means $A \odot B \ll C$. Conversely, let $A \odot B \ll C$. Then $a \odot b \ll c$, for some $a \in A$, $b \in B$, and $c \in C$. Therefore $a \ll b \to c$. We get that $A \ll B \to C$.

(3) Straightforward.

(4) It follows from $x \to y \ll x \to y$ and $x \odot y \ll x \odot y$ that $x \odot (x \to y) \ll y$ and $x \ll y \to (x \odot y)$.

(5) It is clear by (2) and (4).

(6) It is clear by (5).

(7) Let $x \leq y$. Note that $x \in 1 \odot x$. Hence $1 \odot x \ll x$. Thus $1 \odot x \ll y$, $1 \ll x \to y$. Therefore $1 \in x \to y$. Let $1 \in x \to y$. Then $1 \ll x \to y$ and $1 \odot x \ll y$ by (HNM3). By (CSHG3) $x \in 1 \odot x$ and 1 is a scalar, we have $x = 1 \odot x$. Therefore $x \leq y$. Moreover, let $A \ll B$. Then $a \leq b$ for some $a \in A, b \in B$. Hence $1 \in a \to b \subseteq A \to B$.

(8) It is clear by (7).

(9) For any $u \in 1 \to x$, we have $u \ll 1 \to x$. So $\{u\} = u \odot 1 \ll x$. Thus $u \leq x$, and x is the greatest element of $1 \to x$. Since $x \in x \odot 1$, we get $x \odot 1 \ll x$.

Hence $x \ll 1 \to x$. There exists $u \in 1 \to x$ such that $x \leq u$. So $x \leq u \leq x$. Therefore, $x \in 1 \to x$.

(10) Let $u \in (x \odot y) \to z$. Then $u \ll (x \odot y) \to z \Leftrightarrow u \odot (x \odot y) \ll z \Leftrightarrow (u \odot x) \odot y \ll z \Leftrightarrow u \odot x \ll y \to z \Leftrightarrow u \ll x \to (y \to z)$. So $(x \odot y) \to z \ll x \to (y \to z)$. Moreover, we have $(x \odot y) \to z = (y \odot x) \to z \ll y \to (x \to z)$.

(11) Let $t \in x \to (y \to z)$. Then there exixts $u \in y \to z$ such that $t \in x \to u$. Hence we have $t \ll x \to u$, $x \odot t \ll u$. Thus $x \odot t \ll y \to z$. Hence $s \ll y \to z$ for some $s \in x \odot t$. By (HNM3), we have $s \odot y \ll z$. It is clear that $(x \odot t) \odot y \ll z$ or $(x \odot y) \odot t \ll z$. Then $v \odot t \ll z$ for some $v \in x \odot y$. By (HNM3) again we get $t \ll v \to z$. Thus $t \ll (x \odot y) \to z$. This shows that $x \to (y \to z) \ll (x \odot y) \to z$.

Similarly, we have $y \to (x \to z) \ll (x \odot y) \to z$.

(12) Let $u \in x \to (y \to z)$. Then $u \ll x \to (y \to z)$. We get $u \ll x \to a$, for some $a \in y \to z$. Hence $u \odot x \ll a$. So $b \leq a$, for some $b \in u \odot x$. Since $a \in y \to z$, we get $b \ll y \to z$ and $b \odot y \ll z$. Hence $(u \odot x) \odot y = (u \odot y) \odot x \ll z$. Hence $u \odot y \ll x \to z$. Therefore, $u \ll y \to (x \to z)$. Then $x \to (y \to z) \ll y \to (x \to z)$.

(13) Since $y \ll z \to (y \odot z)$ and $x \leq y$, we have $x \ll z \to (y \odot z)$. Hence we get $x \odot z \ll y \odot z$.

Let $u \in z \to x$. Then $u \ll z \to x$ and $u \odot z \ll x$. Since $x \leq y$, we get $u \odot z \ll y$ and $u \ll z \to y$. Therefore, $z \to x \ll z \to y$.

Let $u \in y \to z$. Then $u \ll y \to z$ and $y \ll u \to z$. Since $x \leq y$, we get that $x \ll u \to z$. Hence we get $u \ll x \to z$. Therefore $y \to z \ll x \to z$.

(14) It is clear by (13).

(15) Since $y \leq 1 \in x \to x$, we get $y \ll x \to x$. So $x \odot y \ll x$. Similarly, we have $x \odot y \ll y$. Particularly, $x \odot 0 \ll 0$. Therefore $0 \in x \odot 0$.

(16) Since $x \odot y \ll x$, we get $x \ll y \to x$. Then $x \leq u$ for some $u \in y \to x$. It follows from (7) that $1 \in x \to u$. This shows that $1 \in x \to (y \to x)$.

(17) Since $y \odot (x \to y) \ll y$, we get $y \ll (x \to y) \to y$. Therefore $B \ll (A \to B) \to B$.

(18) Since $x^- \ll x^-$, we have $x \odot x^- \ll 0$. Then $0 \in x \odot x^-$.

(19) Let $x \leq y$ and $x \leq z$. Then $x \in x \wedge y$ and $x \in x \wedge z$ by (SL5). Thus $x \in (x \wedge y) \wedge z = x \wedge (y \wedge z)$. This shows that there exists $a \in y \wedge z$ such that $x \in x \wedge a$. It follows from (SL6) that $x \leq a$. Hence $x \ll y \wedge z$. Similarly we can prove that $y \leq x$ and $z \leq x$ imply $y \vee z \ll x$.

(20) From the properties of super lattices, it is known that $x \wedge y \ll x$ and $x \ll x \vee y$. Since $a \wedge b \ll a$ for any $a \in A, b \in B$, we have $A \wedge B \ll A$. Similarly, $a \ll a \vee b$, for any $a \in A, b \in B$, implies $A \ll A \vee B$.

(21) It is clear by (13) and (20).

(22) Note that $x \ll y^- = y \to 0 \Leftrightarrow x \odot y \ll 0 \Leftrightarrow 0 \in x \odot y$. □

4. Hyper filters on hyper NM-algebras

In this section we set up the theory of hyper filters on hyper NM-algebras.

Definition 4.1. *Let L be a hyper NM-algebra. A nonempty subset F of L satisfying condition*

$$(F) \quad x \leqslant y \text{ and } x \in F \text{ imply } y \in F, for\ all\ x, y \in L,$$

is called

(1) *an h-filter of L if it satisfies: (HF) $x \odot y \subseteq F$, for all $x, y \in F$,*

(2) *a weak h-filter of L if it satisfies: (WHF) $F \ll x \odot y$, for all $x, y \in F$.*

Definition 4.2. *Let D be a nonempty subset of a hyper NM-algebra L satisfying conditions:*

$$\begin{cases} (DS) & 1 \in D, \\ (WDS) & x \in D \text{ and } D \ll x \rightarrow y \text{ imply } y \in D. \end{cases}$$

Then D is called a weak h-deductive system.

Remark 4.3. It is easy to see that any h-filter of hyper NM-algebra L is a weak h-filter. Moreover, for any (weak) h-filter F of L, we know that $1 \in F$.

Proposition 4.4. *Let L be a hyper NM-algebra. Then the following properties hold:*

(1) *every weak h-deductive system satisfies (F),*

(2) *if D is a non-empty subset of L containing* 1*, then D is a weak h-deductive system of L if and only if D satisfies the following condition,*

$$(D) \quad (x \rightarrow y) \cap D \neq \emptyset \text{ and } x \in D \text{ imply } y \in D,\ for\ all\ x, y \in L,$$

(3) *every weak h-deductive system is a weak h-filter,*

(4) $\{1\}$ *is a weak h-filter of L,*

(5) *if* 1 *is scalar, then* $\{1\}$ *is a weak h-deductive system of L.*

Proof. (1) Let F be a weak h-deductive system of L, $x \leq y$ and $x \in F$, for all $x, y \in L$. Then by Proposition 3.8(7), we get $1 \in x \rightarrow y$. So $F \ll x \rightarrow y$. Now, it follows from (WDS) that $y \in F$. Therefore, F satisfies condition (F).

(2) ($\Rightarrow$) Let D be a weak h-deductive system of L. Then $1 \in D$ by (DS). Let $(x \rightarrow y) \cap D \neq \emptyset$ and $x \in D$. Then there exists $a \in (x \rightarrow y) \cap D$. So $D \ll x \rightarrow y$. By (WDS) we have $y \in D$.

($\Leftarrow$) Let D be a non-empty subset of L containing 1 and satisfying (D). First we prove that D satisfies (F). Let $x \leq y$ and $x \in D$. Then by Proposition 3.8(7), we get $1 \in x \rightarrow y$. So $(x \rightarrow y) \cap D \neq \emptyset$. Therefore $y \in D$ by (D). This means that (F) holds. Now, let $x \in D$ and $D \ll x \rightarrow y$. Then there exist $d \in D$ and $u \in x \rightarrow y$ such that $d \leq u$. So by (F), we get $u \in D$. Hence $(x \rightarrow y) \cap D \neq \emptyset$ and $y \in D$. Therefore, D is a weak h-deductive system of L.

(3) Let F be a weak h-deductive system of L. Then (F) holds by (1). Now, let $x, y \in F$. By Proposition 3.8(4) $y \ll x \rightarrow (x \odot y)$, we get $y \leq u$ for some

$u \in x \to (x \odot y)$. Hence $u \in F$. So $F \ll x \to v$ for some $v \in x \odot y$. Since $x \in F$, we have $v \in F$. Therefore $F \ll x \odot y$.

(4) Straightforward.

(5) Let 1 be scalar and $D = \{1\}$. Clearly (DS) is true. Assume $x \in D$ and $D \ll x \to y$. Then $x = 1$ and $1 \ll x \to y$. Therefore $1 \in x \to y$. By Proposition 3.8(7), we have $x \le y$. Hence $y = 1$. This shows that $y \in D$, that is, (WDS) holds. □

The following example shows that the converse of Proposition 4.4 (3) may not hold.

Example 4.5. Consider the hyper NM-algebra L given in Example 3.2. Let $F = \{b, 1\}$. It is easy to see that F is a weak h-filter of L but F is not a weak h-deductive system since $b \in F, F \ll b \to a$ but $a \notin F$.

In general, if 1 is not a scalar element of a hyper NM-algebra L, then $\{1\}$ may be not a weak h-deductive system of L. We give the following counter example.

Example 4.6. Consider the hyper NM-algebra L given in Example 3.2, in which 1 is not a scalar element of L. Then $D = \{1\}$ is not a weak h-deductive system of L since $1 \in D$ and $D \ll 1 \to b$ but $b \notin D$.

Proposition 4.7. *Let $\{F_i \mid i \in I\}$ be a family of nonempty subsets of a hyper NM-algebra L. Then the following properties hold,*

(1) if F_i is a weak h-deductive system, for all $i \in I$, then $\bigcap_{i \in I} F_i$ is a weak h-deductive system of L,

(2) let $\{F_i \mid i \in I\}$ be a chain. if F_i is a weak h-filter (weak h-deductive system), for all $i \in I$, then $\bigcup_{i \in I} F_i$ is a weak h-filter (weak h-deductive system) of L.

Proof. Straightforward.

□

Recall that a weak h-filter F (weak h-deductive system D) is called proper if $F \neq L$ $(D \neq L)$.

Definition 4.8. *Let F be a proper weak h-filter (weak h-deductive system) of a hyper NM-algebra L. Then F is said to be maximal, if $F \subseteq J \subseteq L$ implies $F = J$ or $J = L$, for all weak h-filters (weak h-deductive systems) J of L.*

Proposition 4.9. *Let L be a hyper NM-algebra. Then every proper weak h-filter (weak h-deductive system) of L is contained in a maximal weak h-filter (weak h-deductive system) of L.*

Proof. Let F be a proper weak h-filter of L and S be the collection of all proper weak h-filter of L containing F. Then $F \in S$ and $(S, \subseteq)$ is a poset. Let $\{F_i \mid i \in I\}$ be a chain in S. Then $\bigcup_{i \in I} F_i$ is a weak h-filter of L containing F. If

$0 \in \bigcup_{i \in I} F_i$, then there exists $i \in I$ such that $0 \in F_i$, which is impossible. Hence, $\bigcup_{i \in I} F_i$ is a proper weak h-filter of L containing F and $\bigcup_{i \in I} F_i \in S$. Hence, every chain of elements of S has an upper bounded in S. By Zorn's lemma, S has a maximal element such as M. We shall show that M is a maximal weak h-filter of L. Let $M \subseteq J \subseteq L$, for some weak h-filter J of L. If $J \neq L$, then $J \in S$. Since M is a maximal element of S we get $M = J$. Therefore, M is a maximal weak h-filter of L.

It is similar that the proof of every proper weak h-deductive system of L is contained in a maximal weak h-deductive system of L. □

Recall that a hyper NM-algebra L is called nontrivial, if $L \neq \{1\}$. Using Propositions 4.4 and 4.9, we have the following corollaries.

Corollary 4.10. *Every nontrivial hyper NM-algebra has a maximal weak h-filter.*

Corollary 4.11. *If 1 is a scalar element, every nontrivial hyper NM-algebra has a maximal weak h-deductive system.*

Definition 4.12. *Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a hyper NM-algebra and D be a nonempty subset of L containing 1. Then D is called*

(1) *an implicative weak h-deductive system (IWDS for short) if $(x \rightarrow (y \rightarrow z)) \cap D \neq \emptyset$ and $(x \rightarrow y) \cap D \neq \emptyset$ imply $(x \rightarrow z) \cap D \neq \emptyset$, for all $x, y, z \in L$,*

(2) *a positive implicative weak h-deductive system (PIWDS for short) if $(x \rightarrow ((y \rightarrow z) \rightarrow y)) \cap D \neq \emptyset$ and $x \in D$ imply $y \in D$ for all $x, y, z \in L$.*

Example 4.13. Consider the hyper NM-algebra L given in Example 3.2. Let $D = \{a, b, 1\}$. Then D is an IWDS but not a PIWDS. Since $a \in D$ and $(a \rightarrow ((0 \rightarrow b) \rightarrow 0)) \cap D = \{a, b, 1\} \cap D \neq \emptyset$ but $0 \notin D$.

Open Problem: Is there a PIWDS which is not IWDS?

Proposition 4.14. *Let D be a nonempty subset of a hyper NM-algebra L. Then*

(1) if D is an IWDS of L and an upset, and 1 is scalar, then D is a weak h-deductive system,

(2) if D is a PIWDS of L and 1 is scalar, then D is a weak h-deductive system.

Proof. (1) It is clear that $1 \in D$. Let $(x \rightarrow y) \cap D \neq \emptyset$ and $x \in D$. Then $(1 \rightarrow (x \rightarrow y)) \cap D \neq \emptyset$ and $(1 \rightarrow x) \cap D \neq \emptyset$ by Proposition 3.8(9). Since D is a *IWDS* of L, we get $(1 \rightarrow y) \cap D \neq \emptyset$. Hence there exists $a \in D$ and $a \in 1 \rightarrow y$ such that $a \ll 1 \rightarrow y$. So $1 \odot a \ll y$. Since 1 is scalar, $a = 1 \odot a \ll y$. Note that D is an upset, then we get $y \in D$. Therefore, D is a weak h-deductive system.

(2) It is clear that $1 \in D$. Let $(x \rightarrow y) \cap D \neq \emptyset$ and $x \in D$. Then by Proposition 3.8(8) and Proposition 3.8(9), $x \rightarrow y \subseteq x \rightarrow (1 \rightarrow y) \subseteq x \rightarrow ((y \rightarrow 1) \rightarrow y)$. So $(x \rightarrow ((y \rightarrow 1) \rightarrow y)) \cap D \neq \emptyset$. Since $x \in D$ and D is a *PIWDS* of L, we conclude that $y \in D$. Therefore D is a weak h-deductive system by Proposition 4.4(2). □

Proposition 4.15. *Let D be a non-empty subset of a hyper NM-algebra L. Then the following properties hold,*

(1) *if 1 is a scalar, then D is a PIWDS of L if and only if D is a weak h-deductive system such that $((x \to y) \to x) \cap D \neq \emptyset$ implies $x \in D$, for all $x, y \in L$,*

(2) *D is an IWDS of L if and only if $1 \in D$ and $D_x = \{u \in L \mid (x \to u) \cap D \neq \emptyset\}$ is a weak h-deductive system of L, for all $x \in L$.*

Proof. (1) Let D be a PIWDS. Then by Proposition 4.14(2), D is a weak h-deductive system. Now, let $((x \to y) \to x) \cap D \neq \emptyset$. Then there exists $u \in ((x \to y) \to x) \cap D$. It follows from 1 is scalar and Proposition 3.8(9) that $u \in 1 \to u \subseteq (1 \to ((x \to y) \to x))$. Hence $(1 \to ((x \to y) \to x)) \cap D \neq \emptyset$. Since $1 \in D$ and D is a $PIWDS$, we get $x \in D$.

Conversely, let $(x \to ((y \to z) \to y)) \cap D \neq \emptyset$ and $x \in D$. Since D is a weak h-deductive system and $x \in D$, we get $(y \to z) \to y)) \cap D \neq \emptyset$. So $y \in D$. Therefore D is a PIWDS.

(2) Let D be an IWDS of L and $x \in L$. By Proposition 3.8(8), $1 \in D_x$. Now, let $(a \to b) \cap D_x \neq \emptyset$ and $a \in D_x$, for some $a, b \in L$. Then $(x \to a) \cap D \neq \emptyset$ and $(x \to (a \to b)) \cap D \neq \emptyset$. Since D is a $IWDS$, we get $(x \to b) \cap D \neq \emptyset$. So $b \in D_x$. Hence D_x satisfies (D) and D_x is a weak h-deductive system.

Conversely, let $1 \in D$ and $D_x = \{u \in L \mid (x \to u) \cap D \neq \emptyset\}$ be a weak h-deductive system of L, for all $x \in L$. If $(x \to (y \to z)) \cap D \neq \emptyset$ and $(x \to y) \cap D \neq \emptyset$, for $x, y, z \in L$, then $y \in D_x$ and $(y \to z) \cap D_x \neq \emptyset$. Since D_x is a weak h-deductive system of L, we get that $z \in D_x$. So $(x \to z) \cap D \neq \emptyset$. Therefore, D is an IWDS of L. □

Theorem 4.16. *Let D be a non-empty subset of a hyper NM-algebra L and 1 is a scalar. Then the following statements are equivalent:*

(1) *D is an IWDS and a maximal weak h-deductive system of L, for all $x, y \in L \setminus D$,*

(2) *D is a weak h-deductive system and $(x \to y) \cap D \neq \emptyset$, and $(y \to x) \cap D \neq \emptyset$, for all $x, y \in L \setminus D$.*

Proof. (1)⇒(2) Let D be an IWDS and a maximal weak h-deductive system of L. By Proposition 4.15(2), we have $1 \in D$, and $(x \to x) \cap D \neq \emptyset$, $(y \to y) \cap D \neq \emptyset$. Thus $x \in D_x$, $y \in D_y$. Since $x, y \in L \setminus D$, we get $D \subset D_x \subseteq L$ and $D \subset D_y \subseteq L$. Moreover, by Proposition 4.15(2) we get D_x and D_y are weak h-deductive systems of L. Hence by assumption $D_x = L = D_y$ and so $y \in D_x, x \in D_y$. Therefore, $(x \to y) \cap D \neq \emptyset$, and $(y \to x) \cap D \neq \emptyset$. Clearly D is a weak h-deductive system.

(2)⇒(1) Let $x, y \in L \setminus D$. Suppose D_a is not a weak h-deductive system of L, for some $a \in D$. Then there exist $x, y \in L$ and $(x \to y) \cap D_a \neq \emptyset$, $x \in D_a$ such that $y \notin D_a$. Hence for some $u \in x \to y$, $(a \to u) \cap D \neq \emptyset$ and $(a \to x) \cap D \neq \emptyset$, but $(a \to y) \cap D = \varnothing$. Since D is a weak h-deductive system, we get $x \in D$ and $u \in D$, for some $u \in x \to y$. Hence $(x \to y) \cap D \neq \emptyset$. It is clear that $y \in D$, which is a contradiction. Therefore, D_a is a weak h-deductive system of L, for any $a \in L$. By Proposition 4.15(2), we have D is an IWDS.

Now, we show that D_a is the least weak h-deductive system of L containing $D \cup \{a\}$, for all $a \in L \setminus D$. Let D' be a weak h-deductive system of L containing $D \cup \{a\}$, for all $a \in L \setminus D$. Then $(a \to u) \cap D \neq \emptyset$, for all $u \in D_a$. Since $D \subseteq D'$, we get $(a \to u) \cap D' \neq \emptyset$. Hence $u \in D'$. Therefore $D_a \subseteq D'$. That is D_a is the least weak h-deductive system of L containing $D \cup \{a\}$ and $D \subset D_a \subseteq L$. We show that D is a maximal weak h-deductive system of L. Since $a \in L \setminus D$, by assumption we get $D_a = L$. Therefore, D is a maximal weak h-deductive system of L. □

5. Some types of states on hyper NM-algebras

Inspired by the notion of states on hyper effect algebras [8], we will introduce three types of states on hyper NM-algebras such as sup-states, inf-states, and hyper states, furthermore, study the relationships of them in the following.

Definition 5.1. *Let L be a hyper NM-algebra. A sup-state on L is a mapping $s : L \to [0, 1]$ such that the following conditions hold: for all $x, y \in L$,*

(1) $s(0) = 0, s(1) = 1$,

(2) $s(x) + s^*(x \to y) = s(y) + s^*(y \to x)$.

where $s^*(A)$ is defined by $s^*(A) = \sup\{s(t) \mid t \in A\}$, for any $A \subseteq L$.

Example 5.2. Let $L = \{0, a, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < a < 1$. Define the binary hyperoperations $\vee, \wedge, \odot$ and $\to$ by the following tables:

$\vee$	0	a	1
0	$\{0, a, 1\}$	$\{a, 1\}$	$\{1\}$
a	$\{a, 1\}$	$\{a, 1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

$\wedge$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0, a\}$
1	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$

$\odot$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

$\to$	0	a	1
0	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{a\}$	$\{a, 1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Let $s : L \to [0, 1]$ be a mapping and $s(0) = 0$, $s(a) = \frac{1}{2}$, $s(1) = 1$. It is easy to verify that s is a sup-state on the hyper NM-algebra L.

Proposition 5.3. *Let s be a sup-state on hyper NM-algebras L. Then the following properties hold: for any $x, y \in L$,*

(1) *if $x \leq y$ implies $s(x) = s(y) + s^*(y \to x) - 1$,*

(2) *if $x \leq y$ implies $s(x) \leq s(y)$,*

(3) $s(x) = 1 - s^*(x \to 0)$,

(4) $s(x) = s^*(1 \to x)$,

(5) $s^*(x \to 0) + s^*(1 \to x) = 1$.

Proof. (1) If $x \le y$, $1 \in x \to y$ by Proposition 3.8(7). We get $s(x) + s(1) = s(y) + s^*(y \to x)$. Therefore $s(x) = s(y) + s^*(y \to x) - 1$.

(2) It follows from (1) and $x \le y$ that $s^*(y \to x) = s(x) + s(1) - s(y) \le 1$. Therefore $s(x) \le s(y)$.

(3) It follows from Proposition 3.8(8) $1 \in 0 \to x$ that $s(x) + s^*(x \to 0) = s(0) + s(1)$. Therefore, $s(x) = 1 - s^*(x \to 0)$.

(4) It follows from Proposition 3.8(8) $1 \in x \to 1$ that $s(x) + s(1) = s(1) + s^*(1 \to x)$. Therefore, $s(x) = s^*(1 \to x)$.

(5) It is clear by (3) and (4). □

Proposition 5.4. *Let s be a sup-state on hyper NM-algebras L. Define $K = Ker(s) = \{a \in L \mid s(a) = 1\}$ which is called the kernel of the sup-state s. Then K is a weak h-deductive system of L.*

Proof. Clearly, $1 \in K$. Let $x \in K$ and $K \ll x \to y$. Then $s(x) = 1$. There exist $v \in K$ and $u \in x \to y$ such that $v \le u$. Since s is order-preserving, we have $1 = s(v) \le s(u)$. Hence $s(u) = 1$, i.e. $s^*(x \to y) = 1$. Also note that $x \ll y \to x$, so $1 = s(x) \le s^*(y \to x)$. This shows that $s^*(y \to x) = 1$. Then we obtain that $s(y) = 1$. Therefore, $y \in K$. □

Now, we define another type of states on hyper NM-algebras.

Definition 5.5. *Let L be a hyper NM-algebra. An inf-state on L is a mapping $s : L \to [0,1]$ such that the following conditions hold: for all $x, y \in L$,*

(1) $s(0) = 0, s(1) = 1$,

(2) $s(x) + s_*(x \to y) = s(y) + s_*(y \to x)$.

where $s_*(A)$ is defined by $s_*(A) = \inf\{s(t) \mid t \in A\}$ for any $A \subseteq L$.

Example 5.6. Let $L = \{0, a, 1\}$ and $(L, \le)$ be a partially ordered set such that $0 < a < 1$. Define the binary hyperoperations $\vee, \wedge, \odot$ and $\to$ by the following tables:

$\vee$	0	a	1
0	$\{0,a,1\}$	$\{a,1\}$	$\{1\}$
a	$\{a,1\}$	$\{a,1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

$\wedge = \odot$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0,a\}$	$\{0,a\}$
1	$\{0\}$	$\{0,a\}$	$\{0,a,1\}$

$\to$	0	a	1
0	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{0,a,1\}$	$\{a,1\}$	$\{1\}$
1	$\{0,a,1\}$	$\{a,1\}$	$\{0,1\}$

We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Let $s : L \to [0,1]$ be a mapping and $s(0) = 0$, $s(a) = s(1) = 1$. It is easy to verify that s is an inf-state on the hyper NM-algebra L. But it isn't sup-state, indeed, $s(0) = 0$, $s^*(0 \to 1) = 1$, $s(1) = 1$, $s^*(1 \to 0) = 1$. Therefore $s(0) + s^*(0 \to 1) \ne s(1) + s^*(1 \to 0)$.

Example 5.7. Consider the hyper NM-algebra L given in Example 5.2. We can check that s is both a sup-state and an inf-state on the hyper NM-algebra L.

Proposition 5.8. *Let s be a sup-state and an inf-state on hyper NM-algebra L. For some $x, y \in L$, if $0, 1 \in x \to y$, then $s^*(x \to y) = s^*(y \to x) = 1, s_*(x \to y) = s_*(y \to x) = 0$.*

Proof. Let s be a sup-state and inf-state. Then for some $x, y \in L$, we have $s(x) + s^*(x \to y) = s(y) + s^*(y \to x)$, $s(x) + s_*(x \to y) = s(y) + s_*(y \to x)$. Since $0, 1 \in x \to y$, it is obvious that $s^*(x \to y) = 1$, $s_*(x \to y) = 0$. We also get that $s(x) + 1 = s(y) + s^*(y \to x)$, $s(x) + 0 = s(y) + s_*(y \to x)$. Hence, $s^*(y \to x) - s_*(y \to x) = 1$. This shows that $s^*(x \to y) = s^*(y \to x) = 1, s_*(x \to y) = s_*(y \to x) = 0$. □

Proposition 5.9. *Let L be a hyper NM-algebra. If s is a sup-state and an inf-state, then $s^*(x \to y) - s^*(y \to x) = s_*(x \to y) - s_*(y \to x)$, for any $x, y \in L$.*

Proof. If s is a sup-state and inf-state, then for any $x, y \in L$, we have $s(x) + s^*(x \to y) = s(y) + s^*(y \to x)$, $s(x) + s_*(x \to y) = s(y) + s_*(y \to x)$. Hence, $s^*(x \to y) - s^*(y \to x) = s_*(x \to y) - s_*(y \to x)$.

Open problem: How can S be a sup-state and inf-state at the same time? □

We introduce another definition of states on hyper NM-algebras.

Definition 5.10. *Let L be a hyper NM-algebra. A hyper state on L is a mapping $s : L \to [0, 1]$ such that the following conditions hold: for all $x, y \in L$,*

(1) $s(0) = 0, s(1) = 1$,

(2) *there exist $a \in x \to y$, $b \in y \to x$, for which $s(x) + s(a) = s(y) + s(b)$.*

Example 5.11. Let $L = \{0, a, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < a < 1$. Define the binary hyperoperations $\vee, \wedge, \odot$ and $\to$ by the following tables:

$\vee$	0	a	1
0	$\{0, a, 1\}$	$\{a, 1\}$	$\{1\}$
a	$\{a, 1\}$	$\{a, 1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$

$\wedge$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0, a\}$
1	$\{0\}$	$\{0, a\}$	$\{0, a, 1\}$

$\odot$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

$\to$	0	a	1
0	$\{1\}$	$\{1\}$	$\{1\}$
a	$\{0, a\}$	$\{a, 1\}$	$\{1\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

We can check that $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a hyper NM-algebra.

Let $s : L \to [0,1]$ be a mapping and $s(0) = 0$, $s(a) = \frac{1}{2}$, $s(1) = 1$. It is easy to verify that s is a sup-state and a hyper state on the hyper NM-algebra L. But it isn't inf-state, indeed, $s(0) = 0$, $s_*(0 \to a) = 1$, $s(a) = \frac{1}{2}$, $s_*(a \to 0) = 0$. Therefore $s(0) + s_*(0 \to a) \neq s(a) + s_*(a \to 0)$.

Let $s : L \to [0,1]$ be a mapping and $s(0) = 0$, $s(a) = s(1) = 1$. It is easy to verify that s is an inf-state and a hyper state on the hyper NM-algebra L. But it isn't sup-state, indeed, $s(0) = 0$, $s^*(0 \to a) = 1$, $s(a) = 1$, $s^*(a \to 0) = s(a) = 1$, therefore $s(0) + s^*(0 \to a) \neq s(a) + s^*(a \to 0)$.

Proposition 5.12. *Let s be a state on NM-algebra $(L, \vee, \wedge, \odot, \to, 0, 1)$ and let $(L_h, \vee_h, \wedge_h, \odot_h, \to_h, 0, 1)$ be a hyper NM-algebra, where $L_h = L$, $x \vee_h y := \{x \vee y\}$, $x \wedge_h y := \{x \wedge y\}$, $x \odot_h y := \{x \odot y\}$, $x \to_h y := \{x \to y\}$. Define $s_h(x) := s(x), x \in L$ on hyper NM-algebra L_h, then s_h is a hyper state on the hyper NM-algebra L_h.*

Proof. Straightforward. □

Let X be a nonempty set and $s : L \to [0,1]$ be a mapping. We say that the mapping s has supremum property, if for all subset A of X, $\sup\{s(a) | a \in A\} = s(t)$, for some $t \in A$. Similarly, we say that the mapping s has infimum property, if for all subset A of X, $\inf\{s(a) | a \in A\} = s(t)$, for some $t \in A$.

Proposition 5.13. *Let s be a hyper state on hyper NM-algebra L. Then the following properties hold,*

(1) *if s with supremum property, then every sup-state is a hyper state,*
(2) *if s with infimum property, then every inf-state is a hyper state.*

Proof. Straightforward. □

The following example shows that a hyper state may be neither sup-state nor inf-state on hyper NM-algebras.

Example 5.14. Consider the hyper NM-algebra L given in Example 3.2. Let $s : L \to [0,1]$ be a mapping and $s(0) = s(a) = 0$, $s(b) = s(1) = 1$. We can check that s is a hyper state rather than sup-state and inf-state on the hyper NM-algebra L, since $s(a) + s^*(a \to b) \neq s(b) + s^*(b \to a)$ and $s(a) + s_*(a \to b) \neq s(b) + s_*(b \to a)$.

6. Conclusions

In this paper, we introduce the concept of hyper NM-algebras, which is a generalization of the concept of NM-algebras. Also, we state that hyper NM-algebras are special hyper residuated lattices. Furthermore, we give some properties, related results and relations between (weak) h-filters and weak h-deductive systems on hyper NM-algebras. Finally, we preliminarily define the notions of sup-states, inf-states and hyper states on hyper NM-algebras. In the next task, we will focus on some related properties of those states for further studying the algebraic structure of hyper NM-algebras.

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