Characterizations of obstinate filters in semihoops

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Abstract. In this paper, we consider fundamental properties of obstinate filters in semihoops and give some characterizations of them. Also, we discuss the relationship between obstinate filters and other types of filters (maximal, implicative, positive implicative, normal and fantastic filters) of semihoops and prove that a filter is an obstinate filter if and only if it is a maximal filter and positive implicative filter. Finally, we give some characterizations of simple semihoop by obstinate filter and prove that the homomorphic image of obstinate filters are also obstinate filters. These results will provide a more general algebraic foundation for inference rule in fuzzy logic based on left continuous t-norms.
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1. Introduction

Much of human reasoning and decision making is based on an environment of imprecision, uncertainty, incompleteness of information, partiality of truth and partiality of possibility-in short, on an environment of imperfect information. Hence how to represent and simulate human reasoning become a crucial problem in information science field. For this reason, various logical algebras have been proposed as the semantical systems of non classical logic systems, for example, MV-algebras, BL-algebras, MTL-algebras, residuated latticese, hoops and semihoops. Among these logical algebras, semihoops [1] are very basic algebraic structures and contain all logical algebras based on residuated lattices. Semihoops are generalizations of hoops which were introduced by Bosbach. In the last few years, the theory of hoops has been enriched with deep structure theorems[2, 3, 4, 5, 6, 7]. Many of these results have a strong impact with fuzzy logics. In particular, from the structure theorem of finite basic hoops, one obtains an elegant short proof of the completeness theorem for propositional basic

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logic, which introduced by Häjek [8]. As a more general structure, a semihoop is a hoop without the condition $x \odot (x \to y) = y \odot (y \to x)$. It follows that a semihoop does not satisfy the divisibility condition $x \wedge y = x \odot (x \to y)$. Compared to hoops contains all algebraic structures that induce by continuous t-norms [10], semihoops contains all algebraic structures that induce by left continuous t-norms. Therefore, semihoops play an important role in studying fuzzy logics and the related algebraic structures.

The filter theory of the semihoops plays an important role in studying these algebras and the completeness of the correspondence logic system. From a logic point of view, various filters have natural interpretation as various sets of provable formulas. The filters on semihoops have been widely studied and some important results have been obtained [7, 11, 12]. In particular, Block introduced the idea of filters in semihoops and investigate some important properties of it [7]. After then, the concepts of prime, primary and prefect filters were defined and characterized in semihoops in [11]. In [12], Kondo was the first to systematically study filter theory in hoops, in which the relations between kinds of filters were obtained and and some of their characterizations were presented. In the last few years, in order to study the consequence operators and MP rules in Basic Logic (BL for short), Saeid [14] introduced a new type filter in BL-algebras, called an obstinate filter, and obtained some important result about this filters. Since obstinate filters were successful in several distinct tasks respect to inference rule and provide a sold algebraic foundation for inference rule in fuzzy logic, it has been extended to other logical algebras such as MV-algebras[15], residuated lattices [13] and so on. As we have mentioned in the above, obstinate filters have been widely studied on BL-algebras, residuated lattices and MV-algebras, etc. All the above mentioned algebraic structures are the special case of semihoops. In fact, semihoops are the widest possible residuated structure. Therefore, it is interesting to study the obstinate filters on semihoops for providing a more general algebraic foundation for inference rule in fuzzy logic based on left continuous t-norms. This is the motivation for us to investigate obstinate filters on semihoops.

In this paper, we introduce and study obstinate filters in semihoops and give some characterizations of them. Also, we discuss the relationship between obstinate filters and some types of filters (maximal, implicative, positive implicative, normal and fantastic filters) of semihoops and prove that for any semihoop L and an obstinate filter F, L/F is a Boolean algebra. Besides, we prove that a filter F is an obstinate filter if and only if $x \in F$ or $x^* \in F$, for all $x \in L$. This results are used in the rest of the paper to analyzing various filters in semihoops. Finally, we give a characterization of simple semihoop by obstinate filter and prove that the homomorphic image of obstinate filters are also obstinate filters. These results will provide a more general algebraic foundation for inference rule in fuzzy logic based on left continuous t-norms.

2. Preliminaries

In this section, we summarize some definitions and results about semihoops which will be used in the following sections.

Definition 2.1 ([2, 10]). *An algebra* $(L, \odot, \to, \wedge, 1)$ *of type (2,2,2,0) is called a semihoop if it satisfies the following conditions:*

(1) $(L, \wedge, 1)$ *is a* $\wedge$*-semilattice with upper bounded 1,*

(2) $(L, \odot, 1)$ *is a commutative monoid,*

(3) $(x \odot y) \to z = x \to (y \to z)$*, for all* $x, y, z \in L$.

In what follows, by L we denote the universe of a hoop $(L, \odot, \to, \wedge, 1)$. For any $x \in L$, we define $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for any natural number n.

On a semihoop L, we define $x \leq y$ if and only if $x \to y = 1$ for all $x, y \in L$. It is easy to check that $\leq$ is a partial order relation on L and for all $x \in L$, $x \leq 1$. Moreover, an algebra L is a bounded semihoop if L is a semihoop and there exists an element $0 \in L$ such that $0 \leq x$ for all $x \in L$. In a bounded semihoop L, we define the negation $* : x^* = x \to 0$ for all $x \in L$. If $x \odot x = x$, that is, $x^2 = x$ for all $x \in L$, then the semihoop L is said to be idempotent. It is easy to check that an idempotent semihoop is equivalent to a Brouwerian semilattice [11].

Proposition 2.2 ([2, 3, 5, 6, 7, 9]). *In any semihoop* L*, the following properties hold: for any* $x, y, z \in L$,

(1) $x \leq y \to x$,

(2) $x \to 1 = 1$,

(3) $1 \to x = x$,

(4) $x \leq y \Rightarrow x \to z \geq y \to z$,

(5) $x \leq y \Rightarrow z \to x \leq z \to y$,

(6) $x \to (y \to z) = y \to (x \to z)$,

(7) $x \odot y \leq z$ *iff* $x \leq y \to z$,

(8) $x \odot y \leq x, y$,

(9) $x \odot y \leq x \wedge y$,

(10) $x \to y \leq (z \to x) \to (z \to y)$,

(11) $x \to y \leq (y \to z) \to (x \to z)$,

(12) $(x \to y) \odot (y \to z) \leq x \to z$.

Proposition 2.3 ([2, 3])**.** *In any bounded semihoop L, the following properties hold: for any $x, y, z \in L$,*

(1) $0^* = 1$, $1^* = 0$,

(2) $x \leq y \Rightarrow x^* \geq y^*$,

(3) $x \odot x^* = 0$.

Definition 2.4 ([2])**.** *Let L_1 and L_2 be two semihoops. A function $f : L_1 \longrightarrow L_2$ is called a homomorphism of semihoops if and only if:*

(1) $f(1) = 1$,

(2) $f(x \odot y) = f(x) \odot f(y)$,

(3) $f(x \to y) = f(x) \to f(y)$.

Let L be a semihoop. A non-empty set F of L is called a filter of L if it satisfies: (1) $x, y \in F$ implies $x \odot y \in F$; (2) $x \in F$, $y \in H$ and $x \leq y$ implies $y \in F$. A filter of L is called a proper filter if $F \neq L$. A non-empty set F of L is a filter if and only if it satisfies $x, x \to y \in F$ implies $y \in F$, for any $x, y, z \in L$. A proper filter of L is called a prime filter of L, if for any filters F_1,F_2 of L such that $F_1 \cap F_2 \subseteq F$, then $F_1 \subseteq F$ or $F_2 \subseteq F$. A proper filter F of L is called a maximal filter of L, if it is not properly contained in any other proper filters of L. A subset $F \subseteq L$ is called an implicative filter if it satisfies: (1) $1 \in F$; (2) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$ for all $x, y, z \in L$. A subset $F \subseteq L$ is called a positive implicative filter if it satisfies: (1) $1 \in F$; (2) $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$ for all $x, y, z \in L$. A subset $F \subseteq L$ is called a fantastic filter if it satisfies: (1) $1 \in F$; (2) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$ for all $x, y, z \in L$. A subset $F \subseteq L$ is called a normal filter if it satisfies: (1) $1 \in f$; (2) $z \to ((y \to x) \to x) \in F$ and $z \in F$ imply that $(x \to y) \to y \in F$, for all $x, y, z \in L$[11, 12].

Proposition 2.5 ([11, 12])**.** *Let L be a semihoop and F a filter of L. Then F is a normal filter if and only if $D(F) \subseteq F$, where $D(F) = \{x \in L | x^{**} \in F\}$.*

Proposition 2.6 ([12])**.** *Let L be a semihoop and F a filter of L. Then F is a positive implicative filter if and only if it is an implicative and fantastic filter.*

Proposition 2.7 ([12])**.** *Let L be a bounded semihoop and F a filter of L. Then F is a positive implicative filter if and only if L/F is a Boolean algebra.*

3. Obstinate filters in semihoops

Definition 3.1. *Let L be a semihoop and F be a proper filter of L. F is called an obstinate filter of L if it satisfies: $x, y \notin F$ imply $x \to y \in F$ and $y \to x \in F$, for any $x, y \in L$.*

The following example shows that any filter may not be an obstinate filter in semihoops.

Example 3.2. Let $L = \{0, a, b, c, 1\}$ with $0 \leq a, b \leq c \leq 1$. Define $\odot$ and $\to$ as follows,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(L, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop. One can easily to check that $F = \{a, c, 1\}$ is an obstinate filter of L. Moreover, the set $G = \{1\}$ is a filter of L while it is not an obstinate filter of L, since $a, b \notin G$ and $a \to b = b \notin G$.

In what follows, some conditions for a proper filter become an obstinate filter of bounded semihoops are given.

Theorem 3.3. *Let L be a bounded semihoop and F be a proper filter of L. Then the following conditions are equivalent:*

(1) *F is an obstinate filter of L,*

(2) *for any $x \in L$, if $x \notin F$, then there exists $n \geq 1$ such that $(x^*)^n \in F$.*

Proof. (1) $\Rightarrow$ (2) Suppose that F is an obstinate filter and $x \notin F$. Since $0 \notin F$, then $1 = 0 \to x \in F$ and $x^* = x \to 0 \in F$. Therefore $(x^*)^n \in F$ for $n = 1$.

(2) $\Rightarrow$ (1) Suppose that $x, y \notin F$. By hypothesis $(x^*)^n \in F$ and $(y^*)^m \in F$, for some $n, m \geq 1$. We know that $(x^*)^n \leq x^*$ and $(y^*)^m \leq y^*$. By filter property, $x^* \in F$ and $y^* \in F$. By Proposition 2.2, we have $x^* \leq x \to y$ and $y^* \leq y \to x$. Hence, $x \to y \in F$ and $y \to x \in F$. Therefore F is an obstinate filter of L. □

Corollary 3.4. *Let L be a bounded semihoop and F be a proper filter of L. Then the following conditions are equivalent:*

(1) *F is an obstinate filter of L,*

(2) *$x \in F$ or $x^* \in F$, for any $x \in L$.*

Proof. (1) $\Rightarrow$ (2) Let F be an obstinate filter and $x \notin F$. By Theorem 3.3, we have $(x^*)^n \in F$ for some $n \geq 1$. Further by Proposition 2.2(8), one can obtain that $(x^*)^n \leq x^*$ and hence $x^* \in F$.

(2) $\Rightarrow$ (1) Suppose that $x \notin F$. To prove that F is an obstinate filter of L, we need only to show that $(x^*)^n \in F$ for some $n \geq 1$. By hypothesis, we get that $x^* \in F$. Therefore F is an obstinate filter of L. □

In the following, we discuss the relationship between obstinate filters and other filters in semihoops.

Proposition 3.5. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is a maximal filter of L.*

Proof. Suppose that F is an obstinate filter but it is not a maximal filter of L. So there exists a proper filter G strictly greater than F (with respect to set-inclusion). Let $a \in G\backslash F$, then $(a^*)^n \in F$, for some $n \geq 1$. We know that $(a^*)^n \leq a^*$. By the filter property $a^* \in F$ and also $a^* \in G$. So $a \odot a^* = 0 \in G$ and which is a contradiction to G is a proper filter of L. □

The following example shows that the converse of Proposition 3.5 is not true in general.

Example 3.6. Let $L = \{0, a, b, c, 1\}$, where $0 \leq a \leq b \leq c \leq 1$. Define $\odot$ and $\rightarrow$ as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	b	b
c	0	0	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Then $(L, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop and $F = \{c, 1\}$ is a maximal filter. However, F is not an obstinate filter of L since $a, b \notin F$ and $b \rightarrow a = b \notin F$.

Theorem 3.7. *Let L be a bounded semihoop and $\{1\}$ be an obstinate filter of L. Then L is a simple bounded semihoop.*

Proof. If $\{1\}$ is an obstinate filter of L, by Proposition 3.5, $\{1\}$ is a maximal filter of L and hence $\{1\}$ is the only maximal filter of L. Therefore L is a simple bounded semihoop. □

Theorem 3.8. *The two element chain is the only simple bounded semihoop which has an obstinate filter.*

Proof. It is easy to check that, in the two-element chain, the trivial filter is obstinate filter. Now let L be a simple bounded semihoop which is not the two-element chain. Then L has at least one element x which is different from 0 and 1. Assume by absurdum that L has an obstinate filter, F . Then F is a proper filter of L, hence F is the trivial filter: $F = \{1\}$. So $x \neq 1$, thus $x \notin F$, and F is an obstinate filter, hence $x^* \in F = \{1\}$ by Corollary 3.4. Thus $x \rightarrow 0 = x^* = 1$, that is $x \leq 0$, which means that $x = 0$. This is a contradiction to the choice of x. Thus, L has no obstinate filter. □

Proposition 3.9. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is a positive implicative filter of L.*

Proof. Suppose that F is not a positive implicative filter of L. By the definition of positive implicative filter, there exist $x, y \in L$ such that $(x \to y) \to x \in F$ but $x \notin L$. We have $y \in F$ or $y \notin L$. Consider the following cases:

Case 1: let $y \in F$, we have $y \leq x \to y$. Then $x \to y \in F$. Since $(x \to y) \to x \in F$ and F is a filter, then $x \in F$, which is a contradiction to $x \notin L$.

Case 2: let $y \notin F$. Since F is an obstinate filter of H, then $x \to y \in F$. We can obtain that $x \in F$, which is a contradiction to $x \notin L$. Hence F is a positive implicative filter of L. □

The following example shows that the converse of Proposition 3.9 is not true in general.

Example 3.10. Let $L = \{0, a, b, c, d, 1\}$, where $0 \leq a \leq b \leq 1$, $0 \leq a \leq d \leq 1$ and $0 \leq c \leq d \leq 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	d	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(L, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop and $F = \{1\}$ is a positive implicative filter. However, F is not an obstinate filter of L since $b \notin F$ and $c = (b)^* \notin F$.

Corollary 3.11. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is an implicative filter of L.*

Proof. The results follows from Propositions 2.6 and 3.9. □

The following example shows that the converse of Corollary 3.11 is not true in general.

Example 3.12. Let $L = \{0, a, b, c, 1\}$, where $0 \leq c \leq a \leq 1$ and $0 \leq c \leq b \leq 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\to$	0	c	a	c	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then $(L, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop and $F = \{b, 1\}$ is an implicative filter. However, F is not an obstinate filter of L.

Theorem 3.13. *Let L be a bounded semihoop and F be a proper filter of L. Then the following conditions are equivalent:*

(1) *F is an obstinate filter,*

(2) *F is a maximal and positive implicative filter,*

(3) *F is a maximal and implicative filter.*

Proof. (1) $\Rightarrow$ (2) The results follows from Propositions 3.5 and 3.9.

(2) $\Rightarrow$ (3) The results follows from Proposition 2.6.

(3) $\Rightarrow$ (1) Let $x, y \notin F$, we show that $L_y = \{u \in L \mid y \rightarrow u \in F\}$ is a filter containing F. It is easy to check that L_y is a filter. We have $x \leq y \rightarrow x$. If $x \in F$, we get $y \rightarrow x \in F$, then $x \in L_y$. Hence $F \subseteq L_y \subseteq L$. By hypothesis F is a maximal filter and since $y \notin F$, we get $L_y = L$ and so $y \rightarrow x \in F$. The case $x \rightarrow y \in F$ is similar. □

Proposition 3.14. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is a fantastic filter of L.*

Proof. The results follows from Propositions 2.6 and 3.9. □

The following example shows that the converse of Proposition 3.14 is not true in general.

Example 3.15. Let $L = \{0, a, b, c, d, 1\}$, where $0 \leq a \leq b \leq 1$, $0 \leq a \leq d \leq 1$ and $0 \leq c \leq d \leq 1$. Define $\odot$ and $\rightarrow$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	1	1	1	a
1	0	a	b	c	d	1

Then $(L, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop and $F = \{1, c\}$ is a fantastic filter. However, F is not an obstinate filter of L.

Proposition 3.16. *Let L be a bounded semihoop and F be an obstinate filter of L. Then F is a normal filter of L.*

Proof. By Proposition 2.5, to prove that F is a normal filter, we need only to show that $D(F) \subseteq F$. Let $D(F) \nsubseteq F$, then there exists a $x \in D(F)$ F, i.e. $x \in D(F)$ but $x \notin F$. So by Proposition 2.5, we have $x^{**} \in F$ and by Corollary 3.4, we obtain that $x^* \in F$. We know that $x^* \rightarrow 0 = (x \rightarrow 0) \rightarrow 0 = x^{**} \in F$. Therefore by filter property we get that $0 \in F$, which is a contradiction. Hence $D(F) \subseteq F$ and we conclude that F is a normal filter of L. □

Example 3.17. Let $L = \{0, a, b, c, d, 1\}$, where $0 \leq b \leq a \leq 1$, $0 \leq d \leq a \leq 1$ and $0 \leq d \leq c \leq 1$. Define $\odot$ and $\rightarrow$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
1	0	a	b	c	d	1

Then $(L, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop and $F = \{1, c\}$ is a normal filter. However, F is not an obstinate filter of L since $a \notin F$ and $d = (a)^* \notin F$.

Corollary 3.18. *Let F be an obstinate filter of L. Then L/F is a Boolean-algebra.*

Proof. It follows from Propositions 2.7 and 3.9. □

Proposition 3.19. *Let L be a bounded semihoop and F be an obstinate filter of L. Then every filter G containing F is a obstinate filter of L.*

Proof. Let $F \subseteq G$, and $x \notin F$. By Corollary 3.4, we have $x^* \in F$. Since $F \subseteq G$, then $x^* \in G$. Hence G is an obstinate filter. □

Corollary 3.20. *Let L be a bounded semihoop. Then $\{1\}$ is an obstinate filter of L if and only if every filter of L is an obstinate filter.*

Proof. It follows from Proposition 3.19. □

Theorem 3.21. *Let L be a bounded semihoop and F be a filter of L. Then F is an obstinate filter if and only if every filter of the quotient algebra L/F is an obstinate filter.*

Proof. Suppose that F is an obstinate filter of L and $x \in H$ such that $[x] \notin \{[1]\}$, i.e. $[x] \neq [1]$, then $x \notin F$. By Corollary 3.4, we have $x^* \in F$, then $[x^*] = [1]$, i.e. $[x^*] \in \{[1]\}$. Hence $\{[1]\}$ is an obstinate filter of L/F. By Corollary 3.20, we get every filter of the quotient algebra L/F is an obstinate filter.

Conversely, suppose that every filter of the quotient algebra L/F is an obstinate filter and $x \in L$ but $x \notin F$. By hypothesis, we get that $[x] \neq [1]$, i.e. $[x] \notin \{[1]\}$. Since $\{[1]\}$ is an obstinate filter of L/F, then $[x^*] \in \{[1]\}$, i.e. $[x^*] = [1]$. So $x^* \in F$. Hence F is an obstinate filter of L. □

Proposition 3.22. *Let L be a bounded semihoop, F, G and I be filter of L. If I is an obstinate filter and $F \cap G \subseteq I$, then $F \subseteq I$ or $G \subseteq I$.*

Proof. Let $F \cap G \subseteq I$, $F \nsubseteq I$ and $G \nsubseteq I$. We take $a \in F/I$ and $b \in G/I$, then $a \in F$, $a \notin I$ and $b \in G$, $b \notin I$. By Proposition 2.2, we have $a, b \leq a \vee b$. Since F and G are filter. We get $a \vee b \in F$ and $a \vee b \in G$. Therefore $a \vee b \in F \cap G \subseteq I$. Hence $a \vee b \in I$. Since I is an obstinate filter and $a \notin I$, $b \notin I$, we obtain $a^* \in I$ and $b^* \in I$. Since I is a filter, we get $a^* \odot b^* \in I$. By Proposition 2.2, we have $a^* \odot b^* \leq a^* \wedge b^*$. Hence $a^* \wedge b^* \in I$. By Proposition 2.2, we know that $a^* \wedge b^* = (a \vee b)^*$, hence $(a \vee b)^* \in I$. Therefore $(a \vee b) \to 0 = (a \vee b)^* \in I$. Since I is a filter and $a \vee b \in I$, we get that $0 \in I$, which is a contradiction. Therefore $F \subseteq I$ or $G \subseteq I$. □

Corollary 3.23. *Let L be a bounded semihoop, F, G and I be filter of L. If I is an obstinate filter and $I = F \cap G$, then $F = I$ or $G = I$.*

Proof. Suppose that $I = F \cap G$, hence $F \cap G \subseteq I$. By Theorem 3.22, we obtain $F \subseteq I$ or $G \subseteq I$. By hypothesis, we have $I = F \cap G \subseteq F, G$. Hence $F = I$ or $G = I$. □

Proposition 3.24. *Let L_1, L_2 be two bounded semihoops, f be homomorphism from L_1 to L_2 and G be an obstinate filter of L_2. Then the inverse image of G is an obstinate filter of L_1.*

Proof. Let G be an obstinate filter of L_2 and $x \in L_1$ such that $x \notin f^{-1}(G)$. Then $f(x) \notin G$, by hypothesis, $[f(x)]^* \in G$. By Definition 2.3, we have $f(x^*) \in G$. Then we get that $x^* \in f^{-1}(G)$. Hence $f^{-1}(G)$ is an obstinate filter of L_1. □

Theorem 3.25. *Let L be a semihoop and F is an obstinate filter of L. Then there exists a Brouwerian semilattice $G(L) = \{x \in L | x \odot x = x\}$ and a homomorphism from L to $G(L)$ such that $ker(f) = F$.*

Proof. Suppose that F is an obstinate filter of L. We define f as follows

$$f(x) = \begin{cases} 1, & x \in F \\ a, & x \in L \backslash F \end{cases}.$$

Where a is any fixed element of $G(L)$ and $a \neq 1$ and take $f(0) = 0$. In order to verify f is a homomorphism from L to $G(L)$, we will divide our investigation into our four cases.

Case 1: If $x, y \in F$, then by filter property we get that $x \odot y \in F$. Hence $f(x) = 1 = f(y)$ and $f(x \odot y) = 1$. On the other hand, $f(x) \odot f(y) = 1 \odot 1 = 1$. Therefore $f(x \odot y) = f(x) \odot f(y)$. Moreover, since $x, y \in F$, by Proposition 2.2, we have $y \leq x \to y$. We get that $x \to y \in F$. Hence $f(x \to y) = 1$. On the other hand, $f(x) \to f(y) = 1 \to 1 = 1$. Therefore $f(x \to y) = f(x) \to f(y)$.

Case 2: If $x, y \notin F$, then $x \odot y \notin F$. So $f(x \odot y) = a$. On the other hand $f(x) \odot f(y) = a \odot a = a$, since $a \in G(L)$. It follows that $f(x \odot y) = f(x) \odot f(y)$. Moreover, if $x, y \notin F$, then $x \to y \in F$ because F is an obstinate filter, and so $f(x) \to f(y) = a \to a = 1$. It follows that $f(x \to y) = f(x) \to f(y)$.

Case 3: If $x \notin F$ and $y \in F$, then $x \odot y \notin F$. So $f(x \odot y) = a$. On the other hand $f(x \odot y) = a \odot 1 = 1$. It follows that $f(x \odot y) = f(x) \odot f(y)$. Moreover, if $x \notin F$ and $y \in F$, then we have $y \leq x \to y$, hence $x \to y \in F$. Then $f(x \to y) = 1$. On the other hand $f(x) \to f(y) = a \to 1 = 1$. It follows that $f(x \to y) = f(x) \to f(y)$.

Case 4: If $x \in F$ and $y \notin F$, then $x \odot y \notin F$. So $f(x \odot y) = a$. On the other hand $f(x) \odot f(y) = 1 \odot a = a$. It follows that $f(x \odot y) = f(x) \odot f(y)$. Moreover, if we have $x \in F$ and $y \notin F$, then $x \to y \notin F$. So $f(x \to y) = a$. On the other hand $f(x) \to f(y) = 1 \to a = a$. It follows that $f(x \to y) = f(x) \to f(y)$.

Summarizing all the above we know f is a homomorphism from L to $G(L)$. It is clear that $\mathrm{Ker}(f) = f^{-1}(1) = F$. □

4. Conclusion

In this paper, motivated by the previous research of obstinate filters in BL-algebras, we extended the concept of obstinate filters to a more generally algebraic structure semihoops. We have also presented several different characterizations and many important properties of obstinate filters in semihoops. We also proved that if F is an obstinate filter, then L/F is a Boolean algebra. Finally, we give a characterization of simple semihoop by obstinate filter and prove that the homomorphic image of obstinate filters are also obstinate filters. These results will provide a more general algebraic foundation for inference rule in fuzzy logic based on left continuous t-norms. In our future work, we will consider the notion of rough obstinate filters in semihoops.

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