Projective curvature tensor on generalized (k, μ)-space forms

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Abstract. In this paper, we study the projective curvature tensor on generalized (k, μ)-space forms. Here we study the projectively flat, ξ-projectively flat, pseudoprojectvely flat, h-projectively semisymmetric, ϕ-projectively semisymmetric, and $P \cdot S$ on generalized (k, μ)-space forms.
Keywords: generalized (k, μ)-space form, projective curvature tensor, η-Einstein manifold.

1. Introduction

An odd-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exists a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η on M satisfying [4],[5]

$$\phi^2 = -I + \eta \circ \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0, \;\; g(X, \xi) = \eta(X), \tag{1.3}$$

for any vector fields X, Y on M.

The fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y. It is well known that contact metric manifolds are almost contact metric manifolds such that $\Phi = d\eta$.

The sectional curvature of a manifold determines its curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known

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as a real space form and its curvature tensor is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}.$$

Model for these spaces are Euclidean spaces ($c = 0$), spheres ($c > 0$) and hyperbolic spaces ($c < 0$).

When we come to the case of complex manifolds, a Kaehlerian manifold (M, J, g) with constant holomorphic sectional curvature c is called a complex space form and its curvature tensor satisfies,

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ\}.$$

Many authors studied complex space forms and their submanifolds in different points of view. On the basis of this, Sasakian space forms were introduced in contact geometry (for more details refer [4]). For such a manifold, the curvature tensor is given by [1, 2];

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y \\ &- g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}, \end{aligned} \tag{1.4}$$

where f_1, f_2 and f_3 are constants. If f_1, f_2 and f_3 are differentiable functions then we call such a space as generalized Sasakian space form.

Alegre and Carriazo in [1, 2] studied generalized Sasakian space forms with contact metric and trans-Sasakian structure, its submanifolds and effect of conformal changes of metric on them. Many authors worked on its product submanifolds [3, 20, 31], Legendrian warped product submanifolds [22] and parallel submanifolds [17]. Different curvature tensors and their flatness, symmetries and recurrence properties on this space forms were studied by the geometers in [23, 24, 26].

An almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is a generalized (k, μ) space form if there exists differential functions $f_1, f_2, f_3, f_4, f_5, f_6$ on $M^{2n+1}(f_1, \ldots, f_6)$, whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{1.5}$$

where $R_1, R_2, R_3, R_4, R_5, R_6$ are the following tensors;

$$\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi, \\ R_4(X, Y)Z &= g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y, \\ R_5(X, Y)Z &= g(hY, Z)hX - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX, \\ R_6(X, Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX, Z)\eta(Y)\xi - g(hY, Z)\eta(X)\xi, \end{aligned}$$

where the tensor h is defined by $2h = L_\xi\phi$ is symmetric and satisfies the following conditions

$$h\xi = 0, \quad h\phi = -\phi h, \quad tr(h) = 0, \quad \eta \circ h = 0, \quad h^2 = (k-1)\phi^2, \quad k \leq 1 \tag{1.6}$$

and where L is the usual Lie derivative and X, Y, Z are vector fields on $M^{2n+1}(f_1, \ldots, f_6)$. In particular if $f_4 = f_5 = f_6 = 0$, then generalized (k,μ)-space form $M^{2n+1}(f_1, \ldots, f_6)$ reduces to generalized Sasakian space forms. Moreover, it was proved in [19] that (k,μ)-space forms are natural examples of generalized (k,μ)-space forms for constant functions $f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - k, f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1 - \mu$.

In [8], after introducing a generalized (k,μ) space forms, some basic identities for generalized (k,μ)-space forms were obtained in an analogous way to those satisfied by Sasakian manifold and they examined in depth the contact metric case with some examples for all possible dimensions. Further they proved that the curvature tensor of a generalized (k,μ)-space form is not unique in 3-dimensional case.

In a generalized (k,μ) space forms, the following relations hold [8];

$$\begin{aligned} QX = {} & (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi \\ & + ((2n-1)f_4 - f_6)hX, \end{aligned} \tag{1.7}$$

$$\begin{aligned} S(X,Y) = {} & (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y) \\ & + ((2n-1)f_4 - f_6)g(hX,Y), \end{aligned} \tag{1.8}$$

$$r = 2n\{(2n+1)f_1 + 3f_2 - 2f_3\}, \tag{1.9}$$

$$\begin{aligned} R(\xi,X)Y = {} & (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\} \\ & + (f_4 - f_6)\{g(hX,Y)\xi - \eta(Y)hX\}, \end{aligned} \tag{1.10}$$

$$\begin{aligned} R(X,Y)\xi = {} & (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \\ & + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}, \end{aligned} \tag{1.11}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y). \tag{1.12}$$

After the conformal curvature tensor, the projective curvature tensor is another important curvature tensor in differential geometry. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n > 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes and it is given by [14]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}. \tag{1.13}$$

Definition 1.1. *An almost contact metric manifold is said to be*

- *Projectively flat if* $P(X,Y)Z = 0$,

- *ξ-Projectively if $P(X,Y)\xi = 0$,*
- *Pseudoprojectively flat if $g(P(\phi X,Y)Z,\phi W) = 0$,*

for all vector fields X, Y, Z.

Definition 1.2. *An almost contact metric manifold is said to be [12]*

- *h-projectively semisymmetric if $P(X,Y)\cdot hZ = 0$,*
- *ϕ-projectively semisymmetric if $P(X,Y)\cdot \phi Z = 0$,*

for all vector fields X, Y, Z.

Definition 1.3. *An almost contact metric manifold M is said to be*

- *Einstein manifold if $S(X,Y) = \lambda_1 g(X,Y)$,*
- *η-Einstein manifold if $S(X,Y) = \lambda_1 g(X,Y) + \lambda_2\eta(X)\eta(Y)$,*
- *Special type of η-Einstein manifold if $S(X,Y) = \lambda_1\eta(X)\eta(Y)$,*

where S is the Ricci tensor and λ_1 and λ_2 are constants.

2. Projectively flat generalized (k,μ)-space form

Definition 2.1. *A generalized (k,μ)-space forms is said to be projectvely flat if it satisfies*

$$P(X,Y)Z = 0$$

for all vector fields X, Y, Z.

A projectively flat generalized (k,μ)-space forms can be written from (1.13), we have

$$R(X,Y)Z = \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\}. \tag{2.1}$$

Inserting $X = \xi$ and using (1.8) in (2.1), we have

$$R(\xi,Y)Z = \frac{1}{2n}\{S(Y,Z)\xi - S(\xi,Z)Y\}. \tag{2.2}$$

Replace Z by ϕZ in (2.2), we get

$$\{(1-2n)f_3 - 3f_2\}g(Y,\phi Z)\xi + \{f_4 - (2n-1)f_6\}g(hY,\phi Z)\xi = 0. \tag{2.3}$$

Since $g(Y,\phi Z) \neq 0$ and $g(hY,\phi Z) \neq 0$, from (2.3) we have

$$f_3 = \frac{3f_2}{(1-2n)} \quad \textit{and} \quad f_6 = \frac{f_4}{(2n-1)}.$$

The generalized Sasakian space form is a projectively flat if and only if $f_3 = \frac{3f_2}{(1-2n)}$ (Theorem 3.1 of [16]). But for the case of generalized (k,μ)-space form $M^{2n+1}(f_1,\ldots,f_6)$, following theorem states;

Theorem 2.1. *If a $(2n+1)$-dimensional generalized (k,μ)-space form is projectively flat if $f_3 = \frac{3f_2}{1-2n}$ and $f_6 = \frac{f_4}{(2n-1)}$.*

3. ξ-projectively flat generalized (k, μ)-space forms

Definition 3.1. *A generalized (k, μ) space forms is said to be ξ-projectvely flat if it satisfies*

$$P(X, Y)\xi = 0$$

for all vector field X, Y

A ξ-projectively flat generalized (k, μ)-space forms can be written from (1.13), we have

$$R(X, Y)\xi = \frac{1}{2n}\{S(Y, \xi)X - S(X, \xi)Y\}. \tag{3.1}$$

Using (1.8) and (1.11) in (3.1), we get

$$(f_4 - f_6)\{\eta(X)hY - \eta(Y)hX\} = 0. \tag{3.2}$$

From (3.2), we can conclude that

$$(f_4 - f_6) = 0 \Rightarrow f_4 = f_6,$$

If $f_4 = f_6$, then it is reduces to generalized Sasakian space forms and observing the above results we can state the following;

Theorem 3.1. *If a ξ-projectively flat generalized (k, μ)-space form is ξ-projectively flat generalized Sasakian space forms if and only if $f_4 = f_6$.*

Remark 3.1. In article [25] author's proved 3-dimensional contact metric generalized (k, μ)-space forms is ξ-projectively flat. The obtained our result of ξ-projectively flat generalized (k, μ)-space forms is stronger compare with the results of paper [25]. Because of generalized (k, μ)-space forms weaker than the contact metric generalized (k, μ)-space forms.

4. Pseudoprojectively flat generalized (k, μ)-space forms

Definition 4.1. *A generalized (k, μ)-space forms is called pseudoprojectively flat if it satisfies the condition*

$$g(P(\phi X, Y)Z, \phi W) = 0$$

for all vector fields X, Y, Z, W.

In view of (1.13), pseudoprojectively flat generalized (k, μ)-space form can be written as

$$g(R(\phi X, Y)Z, \phi W) = \frac{1}{2n}\{S(Y, Z)g(\phi X, \phi W) - S(\phi X, Z)g(Y, \phi W)\}. \tag{4.1}$$

Now we take orthogonal basis $X = W = e_i$ in (4.1), we have

$$g(R(\phi e_i, Y)Z, \phi e_i) = \frac{1}{2n}\{S(Y,Z)g(\phi e_i, \phi e_i) - S(\phi e_i, Z)g(Y, \phi e_i)\}, \tag{4.2}$$

We know that almost contact metric manifold of $(2n+1)$-dimensional, if $\{e_1, e_2, \dots e_{2n}, \xi\}$ is a local orthonormal basis of a vector fields in M, then $\{\phi e_1, \phi e_2, \dots, \phi e_{2n}, \xi\}$ is a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \tag{4.3}$$

$$\begin{aligned}\sum_{i=1}^{2n} g(e_i, Y)S(e_i, Z) &= \sum_{i=1}^{2n} g(\phi e_i, Y)S(\phi e_i, Z)\\ &= S(Y,Z) - S(\xi, Y)\eta(Z),\end{aligned} \tag{4.4}$$

$$\begin{aligned}\sum_{i=1}^{2n} g(R(e_i, Y)Z, W) &= \sum_{i=1}^{2n} g(R(\phi e_i, Y)Z, \phi e_i) = S(Y,Z) - g(R(\xi, Y)Z, \xi)\\ &= S(Y,Z) - 2n(f_1 - f_3)g(\phi Y, \phi Z) + (f_4 - f_6)g(hY, Z).\end{aligned} \tag{4.5}$$

Using (4.3)-(4.5) in (4.2), we have

$$S(Y,Z) = 2n(f_1 - f_3)g(Y,Z) - 2n(f_4 - f_6)g(hY,Z). \tag{4.6}$$

Calculate the value of $g(hY,Z)$ from (1.8) and then inserting value of $g(hY,Z)$ in (4.6), we get

$$S(Y,Z) = Ag(Y,Z) + B\eta(Y)\eta(Z).$$

Where

$$A = \left\{\frac{2n(f_1 - f_3) + (2n(f_4 - f_6))((2n-1)f_4 - f_6) + (2nf_1 + 3f_2 - f_3)}{2n(f_4 - f_6) - ((2n-1)f_4 - f_6)}\right\}$$

and

$$B = \left\{\frac{2n(f_4 - f_6)(3f_2 + (2n-1)f_3)}{2n(f_4 - f_6) - ((2n-1)f_4 - f_6)}\right\}.$$

Now we can state the following statement

Theorem 4.1. *If a generalized (k,μ)-space forms is pseudoprojectively flat, then it is an η-Einstein manifold.*

5. h-projectively semisymmetric generalized (k,μ)-space forms

Definition 5.1. *A generalized (k,μ)-space forms is said to be h-projectively semisymmetric if it satisfies*

$$P(X,Y) \cdot hZ = 0,$$

for all vector fields X, Y, Z.

Consider $M^{2n+1}(f_1,\ldots,f_6)$ be a h-projectively semisymmetric i.e. $P(X,Y)\cdot hZ=0$, then we have

$$P(X,Y)hZ-h(P(X,Y)Z)=0, \tag{5.1}$$

for all vector fields X,Y,Z.
Now we consider

$$P(X,Y)hZ=R(X,Y)hZ-\frac{1}{2n}\{S(Y,hZ)X-S(X,hZ)Y\}, \tag{5.2}$$

$$h(P(X,Y)Z)=h(R(X,Y)Z)-\frac{1}{2n}\{S(Y,Z)hX-S(X,Z)hY\}. \tag{5.3}$$

Substitute (5.3) and (5.2) in (5.1) and then using (1.5), (5.1) reduces to

$$\begin{aligned}(f_1-f_3)&\{g(Y,hZ)X-g(X,hZ)Y\}+(f_4-f_6)\{g(Y,hZ)hX-g(X,hZ)hY\}\\&-(f_1-f_3)\{g(Y,Z)hX-g(X,Z)hY\}\\&-(f_4-f_6)\{g(Y,Z)h^2X-g(X,Z)h^2Y\}\\&-\frac{1}{2n}\{S(Y,hZ)X-S(X,hZ)Y-S(Y,Z)hX+S(X,Z)hY\}=0.\end{aligned} \tag{5.4}$$

Replace X by hX in (5.4) and then taking inner product with respect to ξ on both side, we have

$$-(f_1-f_3)g(hX,hZ)\eta(Y)=0. \tag{5.5}$$

Again replace $Y=\xi$ in (5.5) and then replace X by hX, Z by hZ, we get

$$(f_1-f_3)(k-1)^2\{g(X,Z)+\eta(X)\eta(Z)\}=0.$$

Now we can state the following theorem;

Theorem 5.1. *If a generalized (k,μ)-space forms is h-projectively semisymmetric if and only if either $f_1=f_2$ or $k=1$.*

6. ϕ-projectively semisymmetric generalized (k,μ)-space forms

Definition 6.1. *A generalized (k,μ)-space forms is said to be ϕ-projectively semisymmetric if it satisfies*

$$P(X,Y)\cdot\phi Z=0,$$

for all vector field X,Y,Z.

Let $M^{2n+1}(f_1,\ldots,f_6)$ be a ϕ-projectively semisymmetric, i.e. $P(X,Y)\cdot\phi Z=0$, then we have

$$P(X,Y)\phi Z-\phi(P(X,Y)Z)=0, \tag{6.1}$$

for all vector field X, Y, Z.
Now we consider

$$P(X,Y)\cdot\phi Z = R(X,Y)\phi Z - \frac{1}{2n}\{S(Y,\phi Z)X - S(X,\phi Z)Y\}, \tag{6.2}$$

and

$$\phi(P(X,Y)Z) = \phi(R(X,Y)Z) - \frac{1}{2n}\{S(Y,Z)\phi X - S(X,Z)\phi Y\}. \tag{6.3}$$

In view of (6.2) and (6.3) in (6.1), we get

$$\begin{aligned} P(X,Y)\cdot\phi Z &= R(X,Y)\phi Z - \frac{1}{2n}\{S(Y,\phi Z)X - S(X,\phi Z)Y\} \\ &- \phi(R(X,Y)Z) + \frac{1}{2n}\{S(Y,Z)\phi X - S(X,Z)\phi Y\}. \end{aligned} \tag{6.4}$$

Inserting $Y = \xi$ in (5.4) and with the help of (1.11), one can get

$$\begin{aligned} &- (f_1 - f_3)g(X,\phi Z)\xi - (f_4 - f_6)g(hX,\phi Z)\xi \\ &+ \frac{1}{2n}S(X,\phi Z)\xi + 2(f_1 - f_3)\eta(Z)\phi X = 0. \end{aligned} \tag{6.5}$$

Again inserting $Z = \xi$ in (6.5), we have

$$(f_4 - f_6)\phi X \quad \Rightarrow \quad f_4 - f_6 = 0 \quad \textit{or} \quad \phi X = 0. \tag{6.6}$$

Theorem 6.1. *If a generalized (k, μ)-space forms is ϕ-projectively semisymmetric if and only if either $f_4 = f_6$ or $\phi X = 0$.*

If a generalized (k, μ)-space forms is ϕ-projectively semisymmetric, then it is reduces to generalized Sasakian space forms. De and et.al proved in [12], a generalized Sasakian space forms is ϕ-projectvely semisymmetric if and only if $f_3 = \frac{3f_2}{1-2n}$.

Corollary 6.1. *A generalized (k, μ)-space forms is ϕ-projectively semisymmetric if and only if $f_3 = \frac{3f_2}{1-2n}$.*

Carriazo et.al shown in [8], If $h = 0$, then it is K-contact manifold and generalized (k, μ)-space forms reduce to generalized Sasakian space forms.

Corollary 6.2. *Let $M^{2n+1}(f_1, ..., f_6)$ be a generalized (k, μ)-space form. If M^{2n+1} is a K-contact manifold, then $f_3 = f_1 - 1$. Moreover, M^{2n+1} is Sasakian.*

Corollary 6.3. *Let $M^{2n+1}(f_1, ..., f_6)$ be a generalized (k, μ)-space form. If M^{2n+1} is a Sasakian manifold, then $f_2 = f_3 = f_1 - 1$.*

7. Generalized (k,μ)-space forms satisfying $P\cdot S=0$

Let $M(f_1,\ldots,f_6)$ generalized (k,μ) space form satisfying $P\cdot S=0$, then

$$P(U,V)S(Y,Z)=S(P(U,V)Y,Z)+S(Y,P(U,V)Z). \tag{7.1}$$

Setting $U=Z=\xi$ in (7.1), we have

$$S(Y,Z)=Ag(Y,Z)+B\eta(Y)\eta(Z),$$

where

$$A=-\left\{\frac{\begin{array}{c}[2n(f_1-f_3)-2n(f_4-f_6)(k-1)((2n-1)f_4-f_6)\\+(-(f_4-f_6)(1+(2nf_1+3f_2-f_3))(2nf_1+3f_2-f_3))]\end{array}}{((2n-1)f_4-f_6)(-1-2n(f_1-f_3)+2n(f_4-f_6)(1+(2nf_1+3f_2-f_3)}\right\}$$

$$B=-\left\{\frac{\begin{array}{c}[2n(f_4-f_6)((2n-)f_4-f_6)(1+(2nf_1+3f_2-f_3))(3f_2+(2n-1)f_3\\+(4n^2(f_1-f_3)(k-1)((2n-1)f_4-f_6)(2n-1)f_4-f_6)]\end{array}}{((2n-1)f_4-f_6)(-1-2n(f_1-f_3)+2n(f_4-f_6)(1+(2nf_1+3f_2-f_3)}\right\}$$

Theorem 7.1. *A generalized (k,μ) space form is satisfying $P\cdot S=0$, then it is an η-Einstein manifold.*

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