Geometric results of Han-Banach theorem for functionals on weak hypervector spaces

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Abstract. In this paper we prove some basic results of Han-Banach theorem for functionals on normed normal weak hypervector spaces.
Keywords: normal hypervector space, normed weak hypervector space, weak linear functional.

1. Introduction

The concept of hyperstructure was first introduced by Marty [3] in 1934 and has attracted attention of many authors in the last decades and has constructed some other structures such as hypergroups, hypermodules, hyperfields and hypervector spacses. These constructions have been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatories, codes, cryptography, probability and etc. Some of applications of this concepts are given in [1, 3] and [16].

In 1988 , the concept of hypervector space was first introduced by Scafati-Tallini. She later considered more properties of such spaces. Authors, in [6]-[12] considered hypervector spaces in viewpoint of analysis. In mentioned papers, authors introduce concepts such as dimension of hypervectore spaces, normed hypervector spaces, operator on these spaces and another important concepts.

Authors in [7] proved the Han-Banach Theorem for normed normal weak hypervector spaces. In this paper we prove the basic geometric results of Han-Banach theorem for functionals on normed normal weak hypervector spaces.

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2. Preliminaries

Definition 2.1 ([15])**.** A weak or weakly distributive hypervector space over a field F ($\mathbb{C}$ *or* $\mathbb{R}$) is a quadruple $(X, +, o, F)$ such that $(X, +)$ is an abelian group and $o : F \times X \to P*(X)$ is a multivalued product such that

1. $\forall a \in F,\ \forall x, y \in X,\ [ao(x + y)] \cap [aox + aoy] \neq \emptyset$;
2. $\forall a, b \in F,\ \forall x \in X,\ [(a + b)ox] \cap [aox + box] \neq \emptyset$;
3. $\forall a, b \in F,\ \forall x \in X,\ ao(box) = (ab)ox$;
4. $\forall a \in F,\ \forall x \in X,\ ao(-x) = (-a)ox = -(aox)$;
5. $\forall x \in X,\ x \in 1ox$.

The properties 1 and 2 are called weak right and left distributive laws, respectively. Note that the set $ao(box)$ in 3 is of the form $\cup_{y \in box} aoy$.

Definition 2.2. Let X be a weak hypervector space over F, $a \in F$ and $x \in X$. Essential point of $a \circ x$, that we denote it by $e_{a \circ x}$, for $a \neq 0$ is the element of $a \circ x$ such that $x \in a^{-1} \circ e_{a \circ x}$. For $a = 0$, we define $e_{a \circ x} = 0$.

Remark 2.3. Note that $e_{a \circ x}$ is not unique, necessarily. Hence we denote the set of all essential points by $E_{a \circ x}$. When in this note we use $e_{a \circ x}$ in an equation, we mean any element of $E_{a \circ x}$.

Definition 2.4. A weak hypervector space X over the field F is said to be normal if for every $a, b \in F$ and $x, y \in X$ the followings hold:

1. $(E_{a \circ x} + E_{a \circ y}) \cap E_{a \circ (x+y)} \neq \emptyset$
2. $E_{a \circ x} + E_{b \circ x} \subseteq E_{(a+b) \circ x}$

Lemma 2.5 ([6])**.** *Let X be a weak hypervector space over F, $a, b \in F$ and $x \in X$. Then the following properties hold:*

1. $x \in E_{1 \circ x}$
2. *If* $b \neq 0$ *, then* $a \circ e_{b \circ x} = ab \circ x$
3. $E_{-a \circ x} = -E_{a \circ x}$
4. *If* $a \neq 0$ *, then there exists an* $y \in X$ *such that* $x \in E_{a \circ y}$.
5. *If* X *is normal, then* $E_{a \circ x}$ *is singleton.*

Definition 2.6. A subset E of a weak hypervector space X is said to be hyperconvex if

$$E_{t \circ x} + E_{(1-t) \circ y} \subseteq E$$

for every $x, y \in E$ and $t \in [0, 1]$.

Definition 2.7 ([6]). An additive subgroup M of a weak hypervector space X is called a weak hypervector subspace of X, when $E_{a\circ x} \subseteq M$ for every $a \in F$ and $x \in M$.

Definition 2.8 ([15]). A norm on a weak hypervector space X is a mapping $\|\cdot\| : X \to \mathbb{R}$ such that for every $a \in F$ and $x, y \in X$ we have:

1. $\|x\| = 0 \Leftrightarrow x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $\sup \|a \circ x\| = |a| \cdot \|x\|$

X with a norm is called a normed weak hypervector space.

Definition 2.9 ([7]). Let X be a weak hypervector space over F. A map $f : X \to F$ is called weakly linear functional if f is additive and satisfies $f(e_{a\circ x}) = af(x)$, for every $a \in F$ and $x \in X$.
If X is normed weak hypervector space over F, we denote the set of all bounded (see [8]) weakly linear functionals on X by X^*_w.

Lemma 2.10 ([7]). *Suppose X is a normal weak hypervector space over $\mathbb{C}$.*
1. *If u is the real part of a complex weakly linear functional f, then u is real weakly linear and*

$$f(x) = u(x) - iu(e_{i\circ x}), \quad x \in X. \tag{2.1}$$

2. *If u is a real weakly linear functional on X and f to be defined by* (2.1)*, then f is a complex weakly linear functional on X.*
3. *If, X in addition, is normed and f to be defined by* (2.1) *, then $\|f\| = \|u\|$.*

3. Main results

Now we prepare to prove some results of Han-Banach Theorem for weakly linear functionals on normed normal hypervector spaces.

Throughout of this section, X is a normed normal weak hypervector space over the field $\mathbb{C}$.

Lemma 3.1. *Let $C \subseteq X$ be an open hyperconvex set with $0 \in C$. For every $x \in C$, set*

$$p(x) = \inf\{\alpha > 0 |\ e_{\alpha^{-1}\circ x} \in C\}$$

the p satisfies

1. *$p(x + y) \leq p(x) + p(y)$, $\forall x, y \in X$*
2. *$p(e_{\lambda\circ x}) = \lambda p(x)$, $\quad x \in X$, $\lambda > 0$*
3. *There is a constant M such that $0 \leq p(x) \leq M\|x\|$, for every $x \in M$.*

4. $C = \{x \in X|\, p(x) < 1\}$.

Proof. For 2, Let $\lambda > 0$ and $x \in X$ be given, then

$$\begin{aligned} p(e_{\lambda\circ x}) &= \inf\{\alpha > 0|\, e_{\alpha^{-1}\cdot(\lambda\circ x)} \in C\} \\ &= \inf\{\alpha > 0|e_{\alpha^{-1}\lambda\circ x} \in C\} \\ &= \lambda \inf\{\alpha\lambda^{-1}|e_{\alpha^{-1}\lambda\circ x} \in C\} = \lambda p(x). \end{aligned}$$

4. First suppose $x \in C$, since C is open, the continuity of mapping $t \to e_{(1+t)\circ x}$ for each $x \in X$ and $t \in F$, implies that $e_{(1+\epsilon)\circ x} \in C$ for some $\varepsilon > 0$, and therefore $p(x) \leq \frac{1}{1+\varepsilon} < 1$. Conversely, if $p(x) < 1$, there exists $\alpha \in (0,1)$ such that $e_{\alpha^{-1}\circ x} \in C$ and thus,

$$x = e_{\alpha\circ e_{\alpha^{-1}\circ x}} + e_{(1-\alpha)\circ o} \in C.$$

3. Let $r > 0$ be such that $N_r(0) \subseteq C$, clearly for $x \neq 0$,

$$\|e_{\frac{r}{2\|x\|}\circ x}\| = \frac{r}{2\|x\|}\|x\| = \frac{r}{2} < r.$$

Therefore $e_{\frac{r}{2\|x\|}\circ x} \in C$ and hence $p(x) \leq \frac{2}{r}\|x\|$.
1- Let $x, y \in X$ be given, for $\varepsilon > 0$ we have

$$p(e_{\frac{1}{p(x)+\varepsilon}\circ p(x)}) = \frac{1}{p(x)+\varepsilon}p(x) < 1.$$

Then 4 implies $e_{\frac{1}{p(x)+\varepsilon}\circ p(x)} \in C$, similarly $e_{\frac{1}{p(x)+\varepsilon}\circ p(y)} \in C$ thus hyperconvexity of C with $t = \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon} \in [0,1]$ implies that

$$e_{\frac{1}{p(x)+p(y)+2\varepsilon}} \circ (x+y) \in C$$

therefore $p\left(e_{\frac{1}{p(x)+p(y)+2\varepsilon}\circ(x+y)}\right) < 1$, and 2 implies
$p(x+y) < p(x) + p(y) + 2\varepsilon$. □

Lemma 3.2. *Let $C \subseteq X$ be a nonempty open hyperconvex set and let $x_0 \in X$ with $x_0 \notin C$, then there exists $f \in X_w^*$ such that*

$$\forall x \in C, \qquad f(x) < f(x_0).$$

Proof. Without loss of generality we may assume that C contains 0.

We may thus introduce the function p of Lemma 3.1 for C. Consider the weak hypervector subspace $M = \{e_{t\circ x}|\, t \in \mathbb{R}\}$, and weakly linear functional

$$\begin{cases} g : M \to \mathbb{R} \\ g(e_{t\circ x}) = t, \quad t \in \mathbb{R}. \end{cases}$$

It is clear that $g \leq p$ on M, for, if $t > 0$, then since $x_0 \notin C$, $p(x_0) \geq 1$, therefore

$$g(e_{t_0 \circ x_0}) = t \geq tp(x_0) = p(e_{t_0 \circ x_0}),$$

and for $t < 0$,

$$g(e_{t_0 \circ x_0}) = t < 0 \leq p(e_{t_0 \circ x_0}).$$

Now it follows from Theorem 3.8 of [7] that there exists a weakly linear functional f on X that extends g and satisfies $f \leq p$ on X.
In particular, we have $f(x_0) = g(x_0) = 1$ and that f is continuous by Lemma 2.3-3, finally we deduce from lemma 2.3-4 that $\forall x \in C, f(x) < 1 = f(x_0)$. □

Theorem 3.3. *Let $A \subseteq X$ and $B \subseteq X$ be two nonempty, hyperconvex subsets such that $A \cap B = \emptyset$. Assume that one of them is open, then there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that*

$$Re\ f(x) \leq \alpha \leq Re\ f(y),$$

for every $x \in A$ and $y \in B$.

Proof. Lemma 3.2 implies that it is enough to prove this for real scalars.

Set $C = A - B$, so that C is hyperconvex. C is open and $0 \notin C$ by Lemma 3.2 there exists $f \in X_w^*$ such that $f(z) < 0$ for every $z \in C$, that is $f(x) < f(y)$ for every $x \in A$ and $y \in B$.

Fix a constant α satisfying

$$\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y),$$

this complete the proof. □

Theorem 3.4. *Let $A \subseteq X$ and $B \subseteq X$ be two nonempty, hyperconvex subsets such that $A \cap B = \emptyset$. Assume that A is closed and B is compact, then there exist $f \in X_w^*$ and $B \in \mathbb{R}$ such that*

$$Re\ f(x) < \beta < Re\ f(y)$$

for every $x \in A$ and $y \in B$.

Proof. As in proof of Theorem 3.3 , it is enough to use real scalars.

Set $C = A - B$, so that C is hyperconvex and closed and $0 \notin C$. Hence there is some $r > 0$ such that $N_r(0) \cap C = \emptyset$. By Theorem 3.3 there is some $f \in X_w^*$ such that

$$\forall x \in A, y \in B, z \in N_1(0), \qquad f(x - y) \leq f(e_{r \circ z}).$$

It follows $f(x - y) \leq -r\|f\|$ for every $x \in A$ and $y \in B$, with $\varepsilon = \frac{1}{2} r\|f\| > 0$ we have

$$f(x) + \varepsilon < f(y) - \varepsilon, \qquad x \in A, y \in B.$$

Choosing β s.t

$$\sup_{x \in A} f(x) + \varepsilon \leq \beta \leq \inf_{y \in B} f(y) - \varepsilon,$$

this complete the proof. □

Corollary 3.5. *X_w^* separates points on X.*

Proof. Apply Theorem 3.4 for $A = \{x\}$, $B = \{y\}$ with $x, y \in X$, $x \neq y$. □

Corollary 3.6. *Suppose that E is a weak hypervector subspace of X. If $x_0 \in X$ and $x_0 \notin \bar{E}$, then there exists $f \in X_w^*$ such that $f(x_0) = 1$ and $f = 0$ on E.*

Proof. Using Theorem 3.4 with $A = \bar{E}$ and $B = \{x_0\}$, thus we have $f(x) < \alpha < f(x_0)$ for every $x \in E$ and some $\alpha \in \mathbb{R}$. Now suppose $x \in E$ be given, then

$$\forall \lambda \in \mathbb{R} \qquad \lambda f(x) = f(e_{\lambda \circ x}) < \alpha.$$

Therefore $f(x) < \frac{\alpha}{\lambda}$ for $\lambda > 0$ and $f(x) > \frac{\alpha}{\lambda}$ for $\lambda < 0$, letting $\lambda \to \pm\infty$ we have $f(x) = 0$. This completes the proof. □

Corollary 3.7. *If f is a continuous weakly linear functional on a weak hypervector subspace M of X, then there exists $g \in X_w^*$ such that $g = f$ on M.*

Proof. Assume without loss of generality that $f \neq 0$ on M, put $M_0 = \{x \in M | \, f(x) = 0\}$ and pick $x_0 \in M$ such that $f(x_0) = 1$. Since f is continuous, $x_0 \notin \bar{M_0}^M$ (the closure of M_0 with respect of M), therefore $x_0 \notin \bar{M_0}^X$, Corollary 3.6 therefore assures the existence of a $g \in X_w^*$ such that $g(x_0) = 1$ and $g(M_0) = \{0\}$. If $x \in M$ then $x - e_{f(x) \circ x_0} \in M_0$.

Hence

$$g(x) - f(x) = g(x) - f(x)g(x_0) = g(x - e_{f(x) \circ x_0}) = 0,$$

thus $g = f$ on M. □

References

[1] P. Corsini, *Prolegomena of hypergroup theory*, Aviani editore, 1993.

[2] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory*, Kluwer Academic Publishers, Advances in Mathematics (Dordrecht), 2003.

[3] F. Marty, *Sur nue generalizeation de la notion de group*, 8^{th} congress of the Scandinavic Mathematics, Stockholm, 1934, 45–49.

[4] P. Raja and S. M. Vaezpour, *Normed hypervector spaces*, Iranian Journal of Mathematical Sciencesand Informatics (IJMSI), 2 (2007), 35-44.

[5] S. Roy and T. K. Samanta, *Innerproduct hyperspaces*, Accepted in Italian J. of Pure and Appl. Math., 2010.

[6] A. Taghavi and R. Hosseinzadeh, *A note on dimension of weak hypervector spaces*, Italian J. of Pure and Appl. Math., 33 (2014), 7-14.

[7] A. Taghavi and R. Hosseinzadeh, *Hahn-Banach theorem for functionals on hypervector spaces*, The Journal of Mathematics and Computer Science (JMCS), 2 (2011), 682-690.

[8] A. Taghavi and R. Hosseinzadeh, *Operators on normed hypervector spaces*, Southeast Asian Bulletin of Mathematics (SABM), 35 (2011), 367-372.

[9] A. Taghavi and R. Hosseinzadeh, *Operators on weak hypervector spaces*, Ratio Mathematica (RM), 22 (2012), 37-43.

[10] A. Taghavi and R. Hosseinzadeh, *Uniform boundedness principle for operators on hypervector spaces*, Iranian Journal of Mathematical Sciences and Informatics (IJMSI), 7 (2012), 9-16.

[11] A. Taghavi and R. Hosseinzadeh and H. Rohi, *Hyperinner product spacess*, Journal of Algebraic Hyperstructures and it's Applications (JAHA), 1 (2014), 95-100.

[12] A. Taghavi and R. Parvinianzadeh, *Hyperalgebras and quotient hyperalgebras*, Italian J. of Pure and Appl. Math. (JPAM), 26 (2009), 17-24.

[13] A. Taghavi and T. Vougiouklis and R. Hosseinzadeh, *A note on operators on normed finite dimensional weak hypervector spaces*, Scientific bulletin, Series A, 74 (2012), 103-108.

[14] M.Scafati-Tallini, *Characterization of remarkable hypervector space*, Proc. 8^{th} congress on "Algebraic Hyperstructures and Aplications", Samotraki, Greece, (2002), Spanidis Press, Xanthi, (2003), 231-237.

[15] M.Scafati-Tallini, *Weak hypervector space and norms in such spaces*, Algebraic Hyperstructures and Applications Hadronic Press., 1994, 199-206.

[16] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield. Algebraic hyperstructures and applications (Xanthi, 1990)*, World Sci. Publishing, Teaneck, NJ, (1991), 203–211.

[17] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, 1994.

[18] M. M. Zahedi, *A review on hyper k-algebras*, Iranian Journal of Mathematical Sciences and Informatics (IJMSI), 1 (2006), 55-112.

Accepted: 18.01.2018