Unique fixed point results for pairs of mappings on complete metric spaces

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Abstract. In this article, we prove a coincidence point results for four pairs of mappings on four complete metric spaces. The obtained result generalize some well known fixed point results of literature such as Namdeo et al.[Thai Journal of Mathematics, 7(1) (2009), 129 – 135] and Gupta et al. [International Journal of Applied Physics and Mathematics, 2 (3), (2012), 169 – 171].

Keywords: complete metric spaces, fixed point, Cauchy sequence.

1. Introduction

Banach contraction mapping principle [1] is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed point of certain self maps of metric spaces and provides a constructive method to find fixed points. This well-known classical principle has been studied and generalized in many different directions such as generalizing the used metric spaces, for different type of contractions, and for number of mappings. For some more related results on the theory of fixed point, authors refer to see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

Some authors also generalized Banach results for two and three metric spaces such as Fisher [12], Jain [13], Namdeo *et al.* [14, 15], Nung [16], Kikina *et al.* ([17, 18]), Mishra *et al* [19] and Gupta *et al.* ([20, 21, 22]).

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This manuscript is devoted to the investigation of the existence of fixed points for four pair of mapping in four metric spaces. We generalize the result of three metric spaces to obtain a new fixed point theorem for four pairs of mappings on four metric spaces.

2. Preliminaries

Namdeo *et al* [15] proved following related fixed result for three pairs of mappings on three complete metric spaces.

Theorem 2.1. *Let* (X,d), (Y,ρ) *and* (Z,σ) *be complete metric spaces. Let* $A,B : X \mapsto Y$, $C,D : Y \mapsto Z$ *and* $E,F : Z \mapsto X$ *satisfying the inequalities,*

$$d\left(ECAx, FDBx'\right) \leqslant c\frac{f_1\left(y,y',z,z'\right)}{g_1\left(x,x'\right)} \tag{2.1}$$

$$\rho\left(BECy, AFDy'\right) \leqslant c\frac{f_2\left(z,z',x,x'\right)}{g_2\left(y,y'\right)} \tag{2.2}$$

$$\sigma\left(DBEz, CAFz'\right) \leqslant c\frac{f_3\left(x,x',y,y'\right)}{g_3\left(z,z'\right)}, \tag{2.3}$$

$\forall\ x,x' \in X$, $y,y' \in Y$ *and* $z,z' \in Z$ *for which* $g_1(x,x') \neq 0$, $g_2(y,y') \neq 0$, $g_3(z,z') \neq 0$, *where* $0 \leqslant c < 1$ *and*

$$\begin{aligned}
f_1\left(y,y',z,z'\right) = &\max\left\{\rho\left(y,y'\right) d\left(Ez,Fz'\right), \sigma\left(Cy,Dy'\right)\ \rho\left(BEz,AFz'\right),\right.\\
&\left. d\left(ECy,FDy'\right)\ \sigma\left(DBEz,CAFz'\right)\right\}\\
f_2\left(z,z',x,x'\right) = &\max\left\{\sigma\left(z,z'\right)\rho\left(Ax,Bx'\right), d\left(Ez,Fz'\right)\ \sigma\left(CAx,DBx'\right),\right.\\
&\left.\rho\left(BEz,AFz'\right)\ d\left(ECAx,FDBx'\right)\right\}\\
f_3\left(x,x',y,y'\right) = &\max\left\{d\left(x,x'\right)\sigma\left(Cy,Dy'\right), \rho\left(Ax,Bx'\right)\ d\left(ECy,FDy'\right),\right.\\
&\left.\sigma\left(CAx,DBx'\right)\ \rho\left(BECy,AFDy'\right)\right\}
\end{aligned}$$

$$\begin{aligned}
g_1\left(x,x'\right) = &\max\left\{d\left(x,x'\right), \rho\left(Ax,Bx'\right),\ \sigma\left(CAx,DBx'\right), d\left(ECAx,FDBx'\right)\right\}\\
g_2\left(y,y'\right) = &\max\left\{\rho\left(y,y'\right), \sigma\left(Cy,Dy'\right),\ d\left(ECy,FDy'\right), \rho\left(BECy,AFDy'\right)\right\}\\
g_3\left(z,z'\right) = &\max\left\{\sigma\left(z,z'\right), d\left(Ez,Fz'\right),\ \rho\left(BEz,AFz'\right), \sigma\left(DBEz,CAFz'\right)\right\}.
\end{aligned}$$

If A *and* C *or* B *and* D *are continuous, then* ECA *and* FDB *have a unique common fixed point* $u \in X$, BEC *and* AFD *have a unique fixed point* $v \in Y$, *and* DBE *and* CAF *have a unique common fixed point* $w \in Z$. *Further,* $Au = Bu = v$, $Cv = Dv = w$ *and* $Ew = Fw = u$.

We prove our main result for four pairs of mappings on four complete metric spaces.

3. Main result

Theorem 3.1. *Let (X, d_1), (Y, d_2), (Z, d_3) and (Q, d_4) be complete metric spaces. Let $A, B : X \mapsto Y$, $C, D : Y \mapsto Z$, $E, F : Z \mapsto Q$ and $G, H : Q \mapsto X$, satisfying the inequalities,*

$$d_1\left(GECAx, HFDBx'\right) \leqslant c\frac{f_1\left(z, z', q, q'\right)}{g_1\left(x, x'\right)}, \tag{3.1}$$

$$d_2\left(BGECy, AHFDy'\right) \leqslant c\frac{f_2\left(q, q', x, x'\right)}{g_2\left(y, y'\right)}, \tag{3.2}$$

$$d_3\left(DBGEz, CAHFz'\right) \leqslant c\frac{f_3\left(x, x', y, y'\right)}{g_3\left(z, z'\right)}, \tag{3.3}$$

$$d_4\left(FDBGq, ECAHq'\right) \leqslant c\frac{f_4\left(y, y', z, z'\right)}{g_4\left(q, q'\right)}, \tag{3.4}$$

$\forall\, x, x' \in X$; $y, y' \in Y$; $z, z' \in Z$ and $q, q' \in Q$ for which $g_1(x, x') \neq 0$; $g_2(y, y') \neq 0$; $g_3(z, z') \neq 0$ and $g_4(q, q') \neq 0$ where $0 \leqslant c < 1$ and

$$\begin{aligned}
f_1\left(z, z', q, q'\right) &= \max\left\{d_3\left(z, z'\right) d_1\left(Gq, Hq'\right), d_4\left(Fz, Fz'\right) d_2\left(BGq, AHq'\right),\right. \\
&\quad \left. d_1\left(GEz, HFz'\right) d_4\left(FDBGq, ECAHq'\right)\right\} \\
f_2\left(q, q', x, x'\right) &= \max\left\{d_4\left(q, q'\right) d_2\left(Ax, Bx'\right), d_1\left(Gq, Hq'\right) d_3\left(CAx, DBx'\right)\right. \\
&\quad \left., d_2\left(BGq, AHq'\right) d_1\left(GECAx, HFDBx'\right)\right\} \\
f_3\left(x, x', y, y'\right) &= \max\left\{d_1\left(x, x'\right) d_3\left(Cy, Dy'\right), d_2\left(Ax, Bx'\right) d_4\left(ECy, FDy'\right),\right. \\
&\quad \left. d_3\left(CAx, DBx'\right) d_2\left(BGECy, AHFDy'\right)\right\} \\
f_4\left(y, y', z, z'\right) &= \max\left\{d_2\left(y, y'\right) d_4\left(Ez, Fz'\right), d_3\left(Cy, Dy'\right) d_1\left(GEz, HFz'\right),\right. \\
&\quad \left. d_4\left(ECy, FDy'\right) d_3\left(DBGEz, CAHFz'\right)\right\} \\
g_1\left(x, x'\right) &= \max\left\{d_1\left(x, x'\right), d_2\left(Ax, Bx'\right), d_3\left(CAx, DBx'\right),\right. \\
&\quad \left. d_4\left(ECAx, FDBx'\right), d_1\left(GECAx, HFDBx'\right)\right\} \\
g_2\left(y, y'\right) &= \max\left\{d_2\left(y, y'\right), d_3\left(Cy, Dy'\right), d_4\left(ECy, FDy'\right),\right. \\
&\quad \left. d_1\left(GECy, HFDy'\right), d_2\left(BGECy, AHFDy'\right)\right\} \\
g_3\left(z, z'\right) &= \max\left\{d_3\left(z, z'\right), d_4\left(Ez, Fz'\right), d_1\left(GEz, HFz'\right),\right. \\
&\quad \left. d_2\left(BGEz, AHFz'\right), d_3\left(DBGEz, CAHFz'\right)\right\} \\
g_4\left(q, q'\right) &= \max\left\{d_4\left(q, q'\right), d_1\left(Gq, Hq'\right), d_2\left(BGq, AHq'\right),\right. \\
&\quad \left. d_3\left(DBGq, CAHq'\right), d_4\left(FDBGq, ECAHq'\right)\right\}.
\end{aligned}$$

If A, C and E or B, D and F are continuous, then $GECA$ and $HFDB$ have a unique common fixed point $u \in X$. $BGEC$ and $AHFD$ have a unique common fixed point $v \in Y$. $DBGE$ and $CAHF$ have a unique common fixed point $w \in Z$ and $FDBG$ and $ECAH$ have a unique common fixed point $t \in Q$. Further, $Au = Bu = v$, $Cv = Dv = w$, $Ew = Fw = t$, $Gt = Ht = u$.

Proof. Let $x = x_0$ be an arbitrary point in X. We define the sequence $\{x_n\}$ in X,$\{y_n\}$ in Y, $\{z_n\}$ in Z and $\{q_n\}$ in Q inductively for $n = 1, 2, \ldots$ by,

$$Ax_{2n-2} = y_{2n-1}, \quad Cy_{2n-1} = z_{2n-1}, \quad Ez_{2n-1} = q_{2n-1}, \quad Gq_{2n-1} = x_{2n-1};$$
$$Bx_{2n-1} = y_{2n}, \quad Dy_{2n} = z_{2n}, \quad Fz_{2n} = q_{2n}, \quad Hq_{2n} = x_{2n};$$

First we suppose that for some n, $g(x_{2n}, x_{2n-1})$ is zero then we obtain fixed point of mapping directly. For this suppose that,

$$\begin{aligned} g_1(x_{2n}, x_{2n-1}) = & \max\{d_1(x_{2n}, x_{2n-1}), d_2(Ax_{2n}, Bx_{2n-1}), d_3(CAx_{2n}, DBx_{2n-1}), \\ & d_4(ECAx_{2n}, FDBx_{2n-1}), d_1(GECAx_{2n}, HFDBx_{2n-1})\} \\ = & \max\{d_1(x_{2n}, x_{2n-1}), d_2(y_{2n+1}, y_{2n}), d_3(z_{2n+1}, z_{2n}), \\ & d_4(q_{2n+1}, q_{2n}), d_1(x_{2n+1}, x_{2n})\} \\ = & \ 0 \qquad\qquad \text{for some } n. \end{aligned}$$

Then on putting,

$$x_{2n-1} = x_{2n} = x_{2n+1} = u, y_{2n+1} = y_{2n} = v, z_{2n+1} = z_{2n} = w, q_{2n+1} = q_{2n} = t,$$

we get that,

$$\begin{aligned} GECAu = HFDBu = u = Gt = Ht, \qquad & AHFDv = v = Au = Bu, \\ CAHFw = w = Cv = Dv, \qquad & ECAHt = t = Ew = Fw. \end{aligned}$$

From which it follows that,

$$BGECv = v, \quad DBGEw = w, \quad FDBGt = t.$$

Similarly, $g_1(x_{2n}, x_{2n+1}) = 0$ for some n implies that there exist point $u \in X$, $v \in Y$, $w \in Z$ and $t \in Q$ such that,

$$\begin{aligned} GECAu &= HFDBu = u = Gt = Ht, \\ AHFDv &= BGECv = v = Au = Bu, \\ CAHFw &= DBGEw = w = Cv = Dv, \\ ECAHt &= FDBGt = t = Ew = Fw. \end{aligned} \tag{3.5}$$

Similarly, if one of $g_2(y_{2n-1}, y_{2n})$, $g_2(y_{2n+1}, y_{2n})$, $g_3(z_{2n-1}, z_{2n})$, $g_3(z_{2n+1}, z_{2n})$, $g_4(q_{2n-1}, q_{2n})$, $g_4(q_{2n+1}, q_{2n})$ is equal to zero for some n, then inequality (3.4) follows.

Next we suppose that $g_1(x_{2n-1}, x_{2n})$, $g_1(x_{2n}, x_{2n+1})$, $g_2(y_{2n-1}, y_{2n})$, $g_2(y_{2n+1}, y_{2n})$, $g_3(z_{2n-1}, z_{2n})$, $g_3(z_{2n+1}, z_{2n})$, $g_4(q_{2n-1}, q_{2n})$, $g_4(q_{2n+1}, q_{2n})$ are all non-zero for all n, then we also get fixed point of mappings.

Thus,

$$\begin{aligned} f_1(z, z', q, q') = & \max\{d_3(z, z')d_1(Gq, Hq'), d_4(Ez, Fz')d_2(BGq, AHq'), \\ & d_1(GEz, HFz')d_4(FDBGq, ECAHq')\} \end{aligned}$$

$$\begin{aligned}f_1(z_{2n-1}, z_{2n}, q_{2n-1}, q_{2n}) = \ &\max\{d_3(z_{n-1}, z_{2n})d_1(x_{2n-1}, x_{2n}),\\ d_4(q_{2n-1}, q_{2n})d_2(y_{2n}, y_{2n+1}), &d_1(x_{2n-1}, x_{2n})d_4(q_{2n}, q_{2n+1})\}\end{aligned} \tag{3.6}$$

$$\begin{aligned}f_2(q_{2n-1}, q_{2n}, x_{2n}, x_{2n-1}) = \ &\max\{d_4(q_{2n-1}, q_{2n})d_2(y_{2n+1}, y_{2n}),\\ d_1(x_{2n-1}, x_{2n})d_3(z_{2n+1}, z_{2n}), &d_2(y_{2n}, y_{2n+1})d_1(x_{2n+1}, x_{2n})\}\end{aligned} \tag{3.7}$$

$$\begin{aligned}f_3(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = \ &\max\{d_1(x_{2n}, x_{2n-1})d_3(z_{2n-1}, z_{2n}),\\ d_2(y_{2n+1}, y_{2n})d_4(q_{2n-1}, q_{2n}), &d_3(z_{2n+1}, z_{2n})d_2(y_{2n}, y_{2n+1})\}\end{aligned} \tag{3.8}$$

$$\begin{aligned}f_4(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}) = \ &\max\{d_2(y_{2n-1}, y_{2n})d_4(q_{2n-1}, q_{2n}),\\ d_3(z_{2n-1}, z_{2n})d_1(x_{2n-1}, x_{2n}), &d_4(q_{2n-1}, q_{2n})d_3(z_{2n}, z_{2n+1})\}\end{aligned} \tag{3.9}$$

$$\begin{aligned}g_1(x_{2n}, x_{2n-1}) = \ &\max\{d_1(x, x'), d_2(Ax, Bx'), d_3(CAx, DBx'),\\ &d_4(ECAx, FDBx')d_1(GECAx, HFDBx')\}\\ = \ &\max\{d_1(x_{2n}, x_{2n-1}), d_2(y_{2n+1}, y_{2n}), d_3(z_{2n+1}, z_{2n}),\\ &d_4(q_{2n+1}, q_{2n}), d_1(x_{2n+1}, x_{2n})\}\end{aligned} \tag{3.10}$$

$$\begin{aligned}g_2(y_{2n-1}, y_{2n}) = \ &\max\{d_2(y_{2n-1}, y_{2n}), d_3(z_{2n-1}, z_{2n}), d_4(q_{2n-1}, q_{2n}),\\ &d_1(x_{2n-1}, x_{2n}), d_2(y_{2n}, y_{2n+1})\}\end{aligned} \tag{3.11}$$

$$\begin{aligned}g_3(z_{2n-1}, z_{2n}) = \ &\max\{d_3(z_{2n-1}, z_{2n}), d_4(q_{2n-1}, q_{2n}), d_1(x_{2n-1}, x_{2n}),\\ &d_2(y_{2n}, y_{2n+1}), d_3(z_{2n}, z_{2n+1})\}\end{aligned} \tag{3.12}$$

$$\begin{aligned}g_4(q_{2n-1}, q_{2n}) = \ &\max\{d_4(q_{2n-1}, q_{2n}), d_1(x_{2n-1}, x_{2n}), d_2(y_{2n}, y_{2n+1}),\\ &d_3(z_{2n}, z_{2n+1}), d_4(q_{2n}, q_{2n+1})\}.\end{aligned} \tag{3.13}$$

Applying (3.1), we obtain

$$\begin{aligned}d_1\left(x_{2n+1}, x_{2n}\right) &= d_1\left(GECAx_{2n}, HFDBx_{2n-1}\right)\\ &\leqslant \frac{cf_1\left(z_{2n-1}, z_{2n}, q_{2n-1}, q_{2n}\right)}{g_1\left(x_{2n}, x_{2n-1}\right)}.\end{aligned} \tag{3.14}$$

It follows from (3.6), (3.10) and (3.14),

$$d_1(x_{2n}, x_{2n+1}) \leqslant c\max\{d_1(x_{2n-1}, x_{2n}), d_3(z_{2n-1}, z_{2n}), d_4(q_{2n-1}, q_{2n})\}. \tag{3.15}$$

Applying inequality (3.2),

$$\begin{aligned}d_2\left(y_{2n}, y_{2n+1}\right) &= d_2\left(BGECy_{2n-1}, AHFDy_{2n}\right)\\ &\leqslant \frac{cF_2\left(q_{2n-1}, q_{2n}, x_{2n}, x_{2n-1}\right)}{g_2\left(y_{2n-1}, y_{2n}\right)}.\end{aligned} \tag{3.16}$$

It follows from (3.7), (3.11) and (3.16), that

$$d_2(y_{2n}, y_{2n+1}) \leqslant c \max\{d_1(x_{2n+1}, x_{2n}), d_3(z_{2n+1}, z_{2n})\}. \tag{3.17}$$

From (3.3), we obtain

$$\begin{aligned} d_3(z_{2n}, z_{2n+1}) = {} & d_3(DBGEz_{2n-1}, CAHFz_{2n}) \\ \leqslant {} & c\frac{f_3(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}{g_3(z_{2n-1}, z_{2n})}. \end{aligned} \tag{3.18}$$

On Using (3.8), (3.12) and (3.18),

$$d_3(z_{2n}, z_{2n+1}) \leqslant c \max\{d_1(x_{2n-1}, x_{2n}), d_4(q_{2n-1}, q_{2n})\}. \tag{3.19}$$

Using (3.4), we get

$$\begin{aligned} d_4(q_{2n}, q_{2n+1}) = {} & d_4(FDBGq_{2n}, ECAHq_{2n}) \\ \leqslant {} & c\frac{f_4(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n})}{g_4(q_{2n-1}, q_{2n})}. \end{aligned} \tag{3.20}$$

Using (3.9), (3.13) and (3.20),

$$\begin{aligned} & d_4(q_{2n}, q_{2n+1}) \leqslant c \max\{d_2(y_{2n-1}, y_{2n}), d_3(z_{2n-1}, z_{2n})\} \\ & d_2(y_{2n}, y_{2n+1}) \leqslant c \max\{cd_1(x_{2n-1}, x_{2n}), cd_3(z_{2n-1}, z_{2n}), cd_4(q_{2n-1}, q_{2n})\}. \end{aligned} \tag{3.21}$$

On making use of (3.18) and (3.20), we get

$$d_2(y_{2n}, y_{2n+1}) \leqslant c \max\{d_1(x_{2n-1}, x_{2n}), d_3(z_{2n-1}, z_{2n}), d_4(q_{2n-1}, q_{2n})\}. \tag{3.22}$$

Applying inequality (3.1), again we get,

$$\begin{aligned} d_1(x_{2n-1}, x_{2n}) & = d_1(GECAx_{2n-2}, HFDBx_{2n-1}) \\ & \leqslant c\frac{f_1(z_{2n-1}, z_{2n-2}, q_{2n-1}, q_{2n-2})}{g_1(x_{2n-2}, x_{2n-1})}. \end{aligned}$$

For which it follows that

$$\begin{aligned} d_1(x_{2n-1}, x_{2n}) \leqslant c \max\{ & d_1(x_{2n-2}, x_{2n-1}), d_3(z_{n-2}, z_{2n-1}), \\ & d_4(q_{2n-2}, q_{2n-1})\} \end{aligned} \tag{3.23}$$

and similarly using inequality (3.2), (3.3) and (3.4),

$$d_2(y_{2n-1}, y_{2n}) \leqslant c \max\{d_1(x_{2n-1}, x_{2n}), d_3(z_{2n-1}, z_{2n})\} \tag{3.24}$$

$$d_3(z_{2n-1}, z_{2n}) \leqslant c \max\{d_1(x_{2n-2}, x_{2n-1}), d_4(q_{2n-2}, q_{2n-1})\} \tag{3.25}$$

$$d_4(q_{2n-1}, q_{2n}) \leqslant c \max\{d_2(y_{2n-2}, y_{2n-1}), d_3(z_{2n-2}, z_{2n-1})\}. \tag{3.26}$$

Using inequalities (3.23), (3.24), (3.25) and (3.26),

$$\begin{aligned} d_2(y_{2n-1}, y_{2n}) \leqslant c\max\{&d_1(x_{2n-2}, x_{2n-1}), \\ &d_3(z_{2n-2}, z_{2n-1}), d_4(q_{2n-2}, q_{2n-1})\}. \end{aligned} \tag{3.27}$$

It follows from inequalities (3.15) and (3.23) that

$$\begin{aligned} d_1(x_n, x_{n+1}) &\leqslant c\max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), d_4(z_{n-1}, z_n)\} \\ &\leqslant c^{n-1}\max\{d_1(x_1, x_2), d_3(z_1, z_2), d_4(q_1, q_2)\}. \end{aligned} \tag{3.28}$$

Similarly using inequalities (3.22), (3.27), (3.19), (3.25), (3.21) and (3.26),

$$d_2(y_n, y_{n+1}) \leqslant c^{n-1}\max\{d_1(x_1, x_2), d_3(z_1, z_2), d_4(q_1, q_2)\} \tag{3.29}$$

$$d_3(z_n, z_{n+1}) \leqslant c^{n-1}\max\{d_1(x_1, x_2), d_3(z_1, z_2), d_4(q_1, q_2)\} \tag{3.30}$$

$$d_4(q_n, q_{n+1}) \leqslant c^{n-1}\max\{d_1(x_1, x_2), d_3(z_1, z_2), d_4(q_1, q_2)\}. \tag{3.31}$$

Since $c < 1$, then it is clear that, $\{x_n\}$ is a Cauchy sequence in X with limit u, $\{y_n\}$ is a Cauchy sequence in Y with limit v, $\{z_n\}$ is a Cauchy sequence in Z with limit w, $\{q_n\}$ is a Cauchy sequence in Q with limit t.

Also, maps A, C and E are continuous then,

$$\begin{aligned} \lim_{n\to\infty} y_{2n+1} &= \lim_{n\to\infty} Ax_{2n} = Au = v. \\ \lim_{n\to\infty} z_{2n-1} &= \lim_{n\to\infty} Cy_{2n-1} = Cv = w. \\ \lim_{n\to\infty} q_{2n-1} &= \lim_{n\to\infty} Ez_{2n-1} = Ew = t. \end{aligned} \tag{3.32}$$

and hence,

$$\lim_{n\to\infty} f_1(w, Z_{2n}, t, q_{2n}) = d_1(Gt, u)\, d_4(FDBGt, t). \tag{3.33}$$

$$\lim_{n\to\infty} f_2(t, q_{2n}, u, x_{2n-1}) = d_2(BGt, v)\, d_1(Gt, u). \tag{3.34}$$

$$\lim_{n\to\infty} f_3(u, x_{2n-1}, v, y_{2n}) = 0. \tag{3.35}$$

$$\lim_{n\to\infty} f_4(v, y_{2n}, w, z_{2n}) = 0. \tag{3.36}$$

$$\lim_{n\to\infty} g_1(u, x_{2n-1}) = d_1(Gt, u) \tag{3.37}$$

$$\lim_{n\to\infty} g_2(v, y_n) = \max\{d_1(Gt, u), d_2(BGt, v)\} \tag{3.38}$$

$$\lim_{n\to\infty} g_3(w, z_n) = \max\{d_1(Gt, u), d_2(BGt, v),\ d_3(DBGt, w)\} \tag{3.39}$$

$$\begin{aligned} \lim_{n\to\infty} g_4(t, q_{2n}) = \max\{&d_1(Gt, u), d_2(BGt, v), \\ &d_3(DBGt, w), d_4(FDBGt, t)\}. \end{aligned} \tag{3.40}$$

If $\lim_{n\to\infty} g_1(u, x_{2n}) = 0$ then, $Gt = u, GEw = u, GECv = u, GECAu = u$. And if, $\lim_{n\to\infty} g_1(u, x_{2n}) = d_1(Gt, u) \neq 0$ then applying inequalities (3.1), (3.32), (3.33) and (3.37),

$$d_1(Gt, u) = \lim_{n\to\infty} d_1(GECAu, HFDBx_{2n-1}) \leqslant cd_4(FDBGt, t) \tag{3.41}$$

Using (3.4),(3.36) and (3.40),

$$d_4(FDBGt, t) = \lim_{n\to\infty} d_4(FDBGt, ECAHq_{2n}) = 0$$

Thus by using (3.2), (3.32), (3.34) and (3.38)

$$d_2(BGt, v) = \lim_{n\to\infty} d_2(BGECv, AHFDy_{2n}) \leqslant cd_1(Gt, u) = 0$$

This implies that, $Bu = v, DBu = w, Dv = w$. Therefore from (3.3),

$$d_3(DBGt, w) = \lim_{n\to\infty} d_3(DBGEw, CAHFz_{2n}) = 0.$$

and $DBGEw = w, Bu = v, FDBGt = t, Dv = w, GECAu = u, GT = u$. Suppose that $Ht \neq u$. Now Applying inequality (3.1),

$$\begin{aligned} d_1(u, Ht) &= \lim_{n\to\infty} d_1(GECAx_{2n}, HFDBu) \\ &\leqslant c\frac{\lim_{n\to\infty} f_1(z_{2n-1}, w, q_{2n-1}, t)}{\lim_{n\to\infty} g_1(x_{2n}, u)} = cd_4(t, ECAHt). \end{aligned} \tag{3.42}$$

Applying inequality (3.4),

$$\begin{aligned} d_4(t, ECAHt) &= \lim_{n\to\infty} d_4(FDBGq_{2n}, ECAHt) \\ &\leqslant c\frac{\lim_{n\to\infty} f_4(y_{2n-1}, v, z_{2n-1}, w)}{\lim_{n\to\infty} g_4(q_{2n-1}, t)} = 0. \end{aligned}$$

This implies that $ECAHt = t$ and hence from inequality (3.42) we must have $Ht = u$. Equation (3.5) follows similarly if B, D, F are continuous.

Uniqueness

Let $GECA$ and $HFDB$ have another fixed point u', then

$$\begin{aligned} d_1(u, u') = d_1(GECAu, HFDBu') &\leqslant \frac{cF_1(CAu, DBu', ECAu, FDBu')}{g_1(u, u')} \\ &\quad c\max\{d_3(CAu, DBu')\, d_1(GECAu, HFDBu'), \\ &\quad d_4(ECAu, FDBu')\, d_2(BGECAu, AHFDBu'), \\ &\quad d_1(GECAu, HFDBu') \\ &\leqslant \frac{d_4(FDBGECAu, ECAHFDBu')\}}{g_1(u, u')} \end{aligned}$$

$$\leqslant \frac{\begin{array}{l} c\max\left\{d_3\left(CAu, DBu'\right)d_1\left(u,u'\right),\right. \\ d_4\left(ECAu, FDBu'\right)d_2\left(Bu, Au'\right), d_1\left(u,u'\right) \\ \left. d_4\left(FDBu, ECAu'\right)\right\} \end{array}}{\begin{array}{l} \max\left\{d_1\left(u,u'\right), d_2\left(Au, Bu'\right), d_3\left(CAu, DBu'\right),\right. \\ \left. d_4\left(ECAu, FDBu'\right), d_1\left(GECAu, HFDBu'\right)\right\} \end{array}}$$

$$\leqslant \frac{\begin{array}{l} c\max\left\{d_3\left(w, DBu'\right)d_1\left(u,u'\right), d_4\left(t, FDBu'\right)\right. \\ \left. d_2\left(v, Au'\right), d_1\left(u,u'\right)d_4\left(t, ECAu'\right)\right\} \end{array}}{\begin{array}{l} \max\left\{d_1\left(u,u'\right), d_2\left(v, Bu'\right), d_3\left(w, DBu'\right),\right. \\ \left. d_4\left(t, FDBu'\right), d_1\left(u,u'\right)\right\}. \end{array}}$$

This implies that,

$$d_1\left(u,u'\right) \leqslant c\max\left\{d_3\left(w, DBu'\right), d_2\left(v, Au'\right), d_4\left(t, ECAu'\right)\right\}. \tag{3.43}$$

Using inequality (3.2), we get

$$d_2\left(v, Au'\right) = d_2\left(BGECAu, AHFDBu'\right) \leqslant c\frac{F_2\left(ECAu, FDBu', u, u'\right)}{g_2\left(Au, Bu'\right)}$$

$$\leqslant c\max\left\{d_2\left(v, Bu'\right), d_1\left(u,u'\right)\right\}. \tag{3.44}$$

Inequality (3.4) implies that

$$d_4\left(t, ECAu'\right) = d_4\left(FDBGECAu, ECAHFDBu'\right) \leqslant c\frac{f_4\left(Au, Bu', CAu, DBu'\right)}{g_4\left(ECAu, FDBu'\right)}$$

$$d_4\left(t, ECAu'\right) \leqslant c\max\left\{d_2\left(v, Bu'\right), d_3\left(w, DBu'\right),\ d_3\left(w, CAu'\right)\right\}. \tag{3.45}$$

On using inequality (3.2) again,

$$d_2\left(Bu', v\right) = d_2\left(BGECAu', AHFDBu\right) \leqslant c\frac{f_2\left(ECAu', FDBu, u, u'\right)}{g_2\left(Bu, Au'\right)}$$

$$\leqslant c\max\left\{d_2\left(v, Au'\right), d_1\left(u,u'\right)\right\}. \tag{3.46}$$

From inequality (3.3),

$$d_3\left(w, DBu'\right) = d_3\left(DBu', w\right) = d_3\left(DBGECAu', CAHFDBu\right) \tag{3.47}$$

$$\leqslant c\frac{f_3\left(u, u', Au', Bu\right)}{g_3\left(CAu', DBu\right)}$$

$$\leqslant c\max\left\{d_1\left(u,u'\right), d_2\left(v, Au'\right), d_3\left(w, CAu'\right)\right\}. \tag{3.48}$$

Using inequality (3.3) again, we obtain

$$d_3\left(w, CAu'\right) = d_3\left(DBGECAu, CAHFDBu'\right) \leqslant c\frac{f_3\left(u, u', Au, Bu'\right)}{g_3\left(CAu, DBu'\right)}$$

$$\leqslant d_1\left(u,u'\right), d_2\left(v, Bu'\right), d_2\left(v, Au'\right). \tag{3.49}$$

Now from inequalities (3.43)–(3.49)

$$d_1\left(u,u'\right)\leqslant c\max\left\{d_2\left(v,Au'\right),d_3\left(v,Bu'\right)\right\} \tag{3.50}$$

and then inequality (3.44), (3.46) and (3.50) implies that $u=u'$, this proves the uniqueness of u. Similarly we can prove that v is the unique common fixed point of $BGEC$, $AHFD$ and w is the unique common fixed point of $DBGE$, $CAHF$ and t is the unique common fixed point of $FDBG$, $ECAH$. □

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