Generalization of T-small submodules

Sahira M. Yaseen
Mathematics Department
College of Science
University of Baghdad
Iraq
Sahira.mahmood@gmail.com

Abstract. Let R be associative ring with identity and let M be unitary left R - module. A submodule N of M is called, T - small in M denoted by $N \ll_T M$, in case for any submodule $X \subseteq M$, $T \subseteq N + X$ implies that $T \subseteq X$ .In this paper ,we introduce the concept of GT - small submodule in M. A submodule N of an R-module M is called GT-small submoduleo in M, denoted by $N \ll_{GT} M$, in case for every essential submodule X of M, $T \subseteq N + X$ implies that $T \subseteq X$. We introduce and study the concepts GT-hollow module, GT-lifiting modules and GT-supplement submodules as a generalization of T-hollow module, T-lifiting moules and T-supplement submodules respectively we supply some examples and properties of these modules.

Keywords: GT-hollow module, GT-lifting module, T-small submodule, GT-supplement submodules.

1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a small submodule of M and denoted by $N \ll M$ [1]. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, N is called a direct summand of M and it denoted by $M = N \oplus K$. A submodule N of an R-module M is called an essential submodule and denoted by $N \leq_e M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$. Let M be an R-module and K be a submodule of M. K is called a G-small submodule of $M (K \leq_G M)$ if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$. There are some important properties of G-small submodules in [6], [8]. The concept of small submodule has been generalized by some researchers, for this see [7, 2, 8].

In [3] the authors introduced the concept of small submodule with respect to an arbitrary submodule. Recall that a submodule N of M is called, T-small in M denoted by $N \ll_T M$, in case for any submodule $X \leq M$, $T \subseteq N + X$ implies that $T \subseteq X$.

In this paper, we introduce the concept of GT-small submodule in M as generalization of T-small submodule. A submodule N of an R-module M is called GT-small submodule in M, denoted by $N \ll_{GT} M$, in case for essential submodule X of M, $T \subseteq N + X$ implies that $T \subseteq X$. It is clear that every T-small submodule is GT-small. We show by example that GT-small submodule of M need not be T-small submodule see(1.2). Let M be a non-zero module and T be a submodule of M. M is a T-hollow module if every submodule K of M such that $T \nsubseteq K$ is a T-small submodule of M [3]. We introduce and study the concept of GT-hollow module as a generalization of T-hollow module. M is called; T-lifting module if for; any submodule N of M there exists a direct summand D of M and $H \ll_T M$ such that $N = D + H$. In section three we introduce the notion of GT-lifting modules and discus some properties of this kind of modules. In section four we introduce the notion of GT-supplement submodules, we supply some examples and properties of these submodules.

2. GT-small submodule

In this section we introduce the concept of Generalized T-small submodules (GT-small submodule) and discuss some of basic properties.

Definition 2.1. Let T be a submodule of an R-module M. A submodule N of an R-module M is called GT-small submodule in M, denoted by $N \ll_{GT} M$, in case for essential submodule X of M, $T \subseteq N + X$ implies that $T \subseteq X$.

Examples and remarks 2.2.

1. If $T = 0$, then every submodule of M is GT-small in M. and If $T = M$, then $N \ll_{GM} M$ if and only if $N \ll_G M$.

2. It is clear that if N is T-small submodule of M then N is GT-small submodule in M, but the converse is not true in general. For example, in the Z-module Z_{24}, let $T = \{\bar{0}, \bar{8}, \bar{16}\}$ and the only essential submodules in Z_{24} are Z_{24}, $2Z_{24}$ and $4Z_{24}$, let $N = 6Z_{24}$ then $T \subseteq 6Z_{24} + 2Z_{24}$ and $T \subseteq 2Z_{24}$ also $T \subseteq 6Z_{24} + 4Z_{24}$ and $T \subseteq 4Z_{24}$. Then the submodule $6Z_{24}$ is GT-small submodule. But is not T- small, since if $X = 3Z_{24}$, $T \subseteq 6Z_{24} + 3Z_{24}$ but T is not submodule of $3Z_{24}$.

3. Let Z be the ring of integers. It is easy to see that (0) is the only small submodule of Z and also for any nonzero integer m, the submodule (0) is the only GmZ-small submodule of Z.

Proposition 2.3. *Let M be an R-module, $K \leq L \leq M$ and $L \leq_e M$ if $K \ll_{GT} M$, then $K \ll_{GT} L$.*

Proof. Let $T \subseteq K + X$, $X \leq_e L$ and $L \leq_e M$ then $X \leq_e M$ [9], $K \ll_{GT} M$, then $T \subseteq X$ so $K \ll_{GT} L$. □

Proposition 2.4. *Let M be an R-module with submodules $N \leq K \leq M$ and $T \leq K$. If $N \ll_{GT} K$, then $N \ll_{GT} M$.*

Proof. Suppose that $T \subseteq N+X$, for some $X \leq_e M$. Then $T \subseteq (N+X) \cap K = N+(X \cap K)$. Since $N \ll_{GT} K$, $X \leq_e M$ and $K \leq_e K$ then $(X \cap K) \leq_e K$, we have $T \subseteq X \cap K \subseteq X$ so $N \ll_{GT} M$. □

Proposition 2.5. *Let M be an R-module with submodules N_1 ,N_2 and T. Then $N_1 \ll_{GT} M$ and $N_2 \ll_{GT} M$ if and only if $N_1 + N_2 \ll_{GT} M$.*

Proof. Clear. □

Proposition 2.6. *Let M be an R-module with submodules $K \leq N \leq M$ and $K \subseteq T$. If $N \ll_{GT} M$, then $K \ll_{GT} M$ and $\frac{N}{K}$ is $G\frac{T}{K}$ - small in $\frac{M}{K}$.*

Proof. Suppose that $N \ll_{GT} M$ and $T \subseteq K+X$ for some $X \leq_e M$. Then $T \subseteq N+X$ and by our assumption $T \subseteq X$. Thus $K \ll_{GT} M$. Now assume that $\frac{T}{K} \subseteq \frac{N}{K} + \frac{X}{K} = \frac{(N+X)}{K}$ for some $K \subseteq X \subseteq M$ and $\frac{X}{K} \leq_e \frac{M}{K}$. Then $T \subseteq N+X$ and $X \leq_e M$ [9], so $T \subseteq X$ and $\frac{T}{K} \subseteq \frac{X}{K}$. □

Proposition 2.7. *Let M be an R=module with $K_1 \leq M_1 \leq M$ and $K_2 \leq M_2 \leq M$ such that $T \subseteq M_1 \cap M_2$. Then $K_1 \ll_{GT} M_1$ and $K_2 \ll_{GT} M_2$ if and only if $K_1 + K_2 \ll_{GT} M_1 + M_2$.*

Proof. Assume that $K_1 \ll_{GT} M_1$ and $K_2 \ll_{GT} M_2$. Then By Proposition 2.4 $K_1 \ll_{GT} M_1+M_2$ and $K_2 \ll_{GT} M_1+M_2$. And by Proposition 2.5, $K_1+K_2 \ll_{GT} M_1+M_2$. The other direction is clear. □

Proposition 2.8. *Let M and N be an R-modules and $f : M \longrightarrow N$ be an R-homomorphism. If K and T are submodules of M such that, $K \ll_{GT} M$, then $f(K) \ll_{Gf(T)} N$. In particular, if $K \ll_{GT} M$, $M \subseteq N$, then $K \ll_{GT} N$.*

Proof. Let $f(T) \neq 0$ and $f(T) \subseteq f(K)+X$, for some $X \leq_e N$. It is clear that $T \subseteq K + f^{-1}(X)$ and $f^{-1}(X) \leq_e M$. But Since $K \ll_{GT} M$, then $T \subseteq f^{-1}(X)$ and hence $f(T) \subseteq X$. □

Proposition 2.9. *Let T_1 and T_2 be submodules of an R-module M and K be a submodule of M. If $K \ll_{GT_1} M$, and $K \ll_{GT_2} M$, then $K \ll_{G(T_1+T_2)} M$.*

Proof. Since $K \ll_{GT_1} M$, then if $T_1 \subseteq N+X$ for some $X \leq_e M$,then $T_1 \subseteq X$ and $K \ll_{GT_2} M$, then if $T_2 \subseteq N+X$ for some $X \leq_e M$, then $T_2 \subseteq X$. Thus $T_1+T_2 \subseteq N+X$ and $T_1+T_2 \subseteq X$ So $K \ll_{G(T_1+T_2)} M$. □

Proposition 2.10. *Let $M = H_1 \oplus H_2$ be a module with $R = ann(H_1)+ann(H_2)$. If $H_1 \ll_{GT_1} M$, and $H_2 \ll_{GT_2} M$, then $H_1 \oplus H_2 \ll_{G(T_1 \oplus T_2)} M$.*

Proof. Let $T_1 \oplus T_2 \subseteq H_1 \oplus H_2 + X$, for some $X \leq_e M$ Since $R = ann(H_1) + ann(H_2)$ then $X = X_1 \oplus X_2$. By [10] $X_1 \leq_e H_1$ and $X_2 \leq_e H_2$ and $T_1 \oplus T_2 \subseteq H_1 \oplus H_2 + X_1 \oplus X_2 = (H_1 + X_1) \oplus (H_2 + X_2)$ it is clear that $T_1 \subseteq H_1 + X_1$ and $T_2 \subseteq H_2 + X_2$. Since $H_1 \ll_{GT_1} M$ and $H_2 \ll_{GT_2} M$, then $T_1 \subseteq X_1$ and $T_2 \subseteq X_2$. Thus $T_1 \oplus T_2 \subseteq X_1 \oplus X_2 \subseteq X$ and $H_1 \oplus H_2 \ll_{G(T_1 \oplus T_2)} M$. □

Proposition 2.11. *Let M be finitely generated, faithful and multiplication module, and let I, J be ideals in R. Then $I \ll_{GJ} R$ if and only if $IM \ll_{GJM} M$.*

Proof. Assume; that $I \ll_{GJ} R$. Let I be an ideal of R. Then IM; is a submodule of M, Let $JM \subseteq IM + X$ for some essential submodule X of M, M is multiplication module then $X = KM$ for some ideal K of R by. Then $JM \subseteq IM + KM = (I + K)M$. Since M is finitely generated, faithful and multiplication module then by [4], $J \subseteq (I + K)$, since $KM \leq_e M$ then by [4, th.2.13] $K \leq_e R$. Since $I \ll_{GJ} R$ then $J \subseteq K$ thus $JM \subseteq KM = X$. Then $IM \ll_{GJM} M$.

Conversely, assume; that $IM \ll_{GJM} M$. Let J be an ideal of R such that $J \subseteq I + K$, $K \leq_e R$, M is multiplication module then $JM \subseteq IM + KM$ and by [4, th.2.13] $KM \leq_e M$, $IM \ll_{GJM} M$ thus $JM \subseteq KM$ so $J \subseteq K$. Then $I \ll_{GJ} R$. □

3. The GT-hollow module

Let M be a non-zero module and T be a submodule of M. M is a T-hollow module if every submodule K of M such that $T \nsubseteq K$ is a T-small submodule of M. And that M is a G-hollow module if every submodule of M a G-small submodule of M.

Definition 3.1. Let M be a non-zero module and T be a submodule of M. We say that M is a GT-hollow module if every submodule K of M such that $T \nsubseteq K$ is a GT-small submodule of M.

Remark 3.2. (a) Let M be a non-zero module. Then M is GM-hollow module if and only if M is G-hollow module. Z as Z-module is not Z-hollow module and not GZ-hollow module.

(b) A GT-hollow module need not to be hollow module as the following example shows : Consider the module Z_6 as Z-module. If $T = \{\bar{0}, \bar{3}\}$, then one can easily show Z_6 is GT-hollow module. But Z_6 is not hollow module.

(c) If M is uniform R-module. Then M is GM-hollow module if and only if M is hollow module.

(d) Every T-hollow module is GT-hollow module.

(e) The Z-module Z_{24} is not GT-hollow module.

Proposition 3.3. *Let M be a GT-hollow module then every essential submodule N of M such that $T \subseteq N$ is a GT-hollow module.*

Proof. Let M be a GT-hollow module and N any essential submodule of M, $T \subseteq N$. To show that N is GT-hollow module, let L be a proper Submodule of N such that $T \nsubseteq L$. Since M be a GT-hollow module, then $L \ll_{GT} M$. By proposition 2.3, then $L \ll_{GT} N$. Thus N is GT-hollow module. □

Proposition 3.4. *Let M be a GT-hollow module and let $f : M \longrightarrow N$ be an epimorphism, where N is a non-zero module. Then N is $Gf(T)$-hollow module.*

Proof. Suppose that M is a GT-hollow module and let $f : M \longrightarrow N$ be an epimorphism. To show that N is $Gf(T)$-hollow. Let $K \not\leq N$ such that $f(T) \nsubseteq K$. To show that $K \ll f(T)N$. Let $f(T) \subseteq K + X$, for some $X \leq_e N$. Then $f^{-1}(f(T)) \subseteq f^{-1}(K+X)$. Therefore $T + \ker f \subseteq f^{-1}(K) + f^{-1}(X)$. Thus $T \subseteq f^{-1}(K) + f^{-1}(X)$. To show that $T \nsubseteq f^{-1}(K)$. Assume $T \subseteq f^{-1}(K)$. Then $f(T) \subseteq K$ which is a contradiction. Thus $T \nsubseteq f^{-1}(K)$. Since M is GT-hollow module, then $f^{-1}(K) \ll_{GT} M$ and hence $T \subseteq f^{-1}(X)$ Therefore $f(T) \subseteq X$. Thus N is $f(T)$-hollow module. □

Proposition 3.5. *Let T and K be submodules of a module M such that $K \subseteq T$. If K is GT-small submodule of M and $\frac{M}{K}$ is $\frac{GT}{K}$-hollow module, then M is GT-hollow.*

Proof. Assume that $K \ll_{GT} M$ and $\frac{M}{K}$ is $\frac{GT}{K}$-hollow module. Let $N \leq M$ such that $T \nsubseteq N$ and let $T \subseteq N + X$ for some $X \leq_e M$. Then $\frac{T}{K} \subseteq \frac{(N+X)}{K}$ and hence $\frac{T}{K} \subseteq \frac{(N+K)}{K} + \frac{(X+K)}{K}$. To show that $\frac{T}{K} \nsubseteq \frac{(N+K)}{K}$. Assume that $T/K = (N+K)/K$. Then $T = N + K$ and hence $T \subseteq N + K$. Since $K \ll_{GT} M$, then $T \subseteq N$ which is a contradiction. Thus $T/K \nsubseteq (N+K)/K$. Since M/K is a GT/K-hollow module, then $(N+K)/K \ll_{GT/K} M/K$. Therefore $T/K \subseteq (X+K)/K$. Thus $T \subseteq X + K$. Since $K \ll_{GT} M$, then $T \subseteq X$. Thus M is GT-hollow module. □

Proposition 3.6. *Let T be a non-zero submodule of a module M. If M is GT-hollow module. Then T is indecomposable.*

Proof. Suppose that there are proper submodules K and L of T such that $T = K \oplus L$. Therefore $T \nsubseteq K$. Since M is GT-hollow module, then $K \ll_{GT} M$. But $T \subseteq K \oplus L$, therefore $T \subseteq L$ and hence $T = L$. This is a contradiction. Thus T is indecomposable. □

4. GT-lifting module

M is G-lifting; module if for any submodule N of M there exist; submodules L, K of M such that $N = L \oplus K$ with $L \leq N$ where L is direct summand of M; and $K \ll_G N$ [5]. M is called; T-lifting module if for; any submodule N of M

there exists a direct summand D of M and $H \ll_T M$ such that $N = D + H$. In this section we introduce the notion of GT-lifting modules and discus some properties of this kind of modules.

Definition 4.1. Let T be a submodule of a module M. M is called; GT-lifting module if for; any submodule N of M there exists a direct summand D of M and $H \ll_{GT} M$ such that $N = D + H$.

Examples and remarks 4.2.

1. Let M be a module. M is GM-lifting module if and only if M is G-lifting module.

   Proof. let M be GM-lifting module. Let N submodule of a module M. Then there exists a direct summand D of M and $H \ll_{GT} M$ such that $N = D + H$. Thus $N = N \cap (D \oplus L) = D \oplus (N \cap L)$. Let $H = N \cap L$ then $H \ll_G M$ by (2.2) thus M is G-lifting module. Other direction is clear. □

2. Let M be a module. If M is T-lifting module then M is GT-lifting module.
3. Let Z_8 as Z-module, $T = \{\bar{0}, \bar{4}\}$ and, $N = \{\bar{0}, \bar{4}\}$ then Z_8 is not GT-lifting module.
4. If M is indecomposable module. then M is not GT-lifting module for every non trivial submodule T of M.

   Proof. Let T be non trivial submodule of M. If M is GT-lifting module then $T = D + H$ where D is direct summand D of M and $H \ll_{GT} M$ but M is indecomposable module, then $D = 0$. Thus $T = H \ll_{GT} M$ which is a contradiction then M is not GT-lifting module. □

5. Le M be a GT-lifting module then every essential submodule N of M such that $T \subseteq N$ is also GT-lifting.

   Proof. Let M be GT-lifting module and N a essential submodule of M such that $T \subseteq N$ and $X \subseteq N$ then $X = D + H$ where D is direct summand D of M and $H \ll_{GT} M$. It is clear that D is direct summand D of N, $T \subseteq N$ and $N \leq_e M$ then $H \ll_{GT} N$ by (prop 2.3). Thus N is GT-lifting. □

Let H_1 be GT_1-lifting and H_2 is GT_2-lifting modules, then $M = H_1 \oplus H_2$ need not be $GT_1 \oplus GT_2$-lifting module as the following example:

Let $H_1 = Z_8$, $H_2 = Z_2$, each of H_1, H_2 is GH_i-lifting module but $M = Z_8 \oplus Z_2$ as Z- module, M is not GM-lifting module by (Ex.4.2 (1)).

Now we give a sufficient condition under which $M = H_1 \oplus H_2$ is $GT_1 \oplus GT_2$-lifting module.

Proposition 4.3. *Let $M = H_1 \oplus H_2$ be a module with $R = ann(H_1) + ann(H_2)$. If H_1 is GT_1-lifting and H_2 is GT_2-lifting modules, then M is $GT_1 \oplus GT_2$-lifting module.*

Proof. Let N submodule of M. Since $R = ann(H_1) + ann(H_2)$. then $N = N_1 \oplus N_2$ where $N_1 \subseteq H_1$ and $N_2 \subseteq H_2$. H_1 is GT_1- lifting and H_2 is GT_2-lifting modules, then for each $i \in \{1, 2\}$,there exists a direct summund D_i of H_i, such that $N_i = D_i \oplus L_i$ with $D_i \leq N_i$ and $L_i \ll_{GT} H_i$ then , $N = N_1 \oplus N_2 = (D_1 \oplus L_1) \oplus (D_2 \oplus L_2) = (D_1 \oplus D_2) \oplus (L_1 \oplus L_2)$, we have$(D_1 \oplus D_2) \leq N$, then $(D_1 \oplus D_2)$ is direct summund of M by(Prop:2.10) then $(L_1 \oplus L_2) \ll_{G(T1+T2)} M$. Thus M is $GT_1 \oplus GT_2$-lifting module. □

Proposition 4.4. *Let M be finitely generated, faithful and multiplication module. Then M is GT-lifting module if and only if R is $[GT : M]$-lifting.*

Proof. Assume that M is GT-lifting module. Let I be an ideal of R. M is GT-lifting hence there exist submodules $D \leq \oplus M$ and $H \ll_{GT} M$ such that $N = D + H$. But M is a multiplication R-module, so there are ideals J and K of R such that $D = JM$ and $H = KM$. Then $IM = JM + KM = (J + K)M$. But M is finitely generated, faithful and multiplication module then by [4] $I = J + K$, Let $M = D + L$ and $L = J'M$ for some J' of R. Then $RM = M = JM \oplus J'M = (J + J')M$ Then $R = J + J'$. Since M is finitely generated, faithful and multiplication module then $0 = JM \cap J'M = (J \cap J')M$ thus $JJ' = 0$,and $J \leq \oplus R$ by (prop. 2.11) $K \ll_{G[T:M]} R$. Thus R is $[GT : M]$-lifting. Conversely, let R be $[GT : M]$-lifting and N submodule of M. Since M is finitely generated, faithful and multiplication module then there exist I an ideal of R such that $N = IM$ and exist $J \leq \oplus R$ and $K \ll_{G[T:M]} R$ with $I = J + K$. Then $IM = JM + KM = (J + K)M$. Thus $N = JM + KM$, let $R = J \oplus J'$ for some J' of R then $M = RM = (J + J')M = JM \oplus J'M$. Since M is finitely generated, faithful and multiplication module then $JM \cap J'M = (J \cap J')M = 0M = 0$. Then $JM \leq \oplus M$ by (prop.2.11), $k \ll_{GT} M$. Then M is GT-lifting module. □

5. GT-supplemente submodule

Definition 5.1. Let M be an R-module and T, X, $Y \leq M$. Y is called a GT-supplement of X in M, if $T \subseteq X + Y$ and $X \cap Y \ll_{GT} Y$. If every submodule of M has a GT-supplement in M, then M is called a GT-supplemented module.

Examples and remarks 5.2.

1. If $T = 0$, then every submodule of M is GT-supplement in M.

2. and If $T = M$, then M is GM-supplement in M if and only if M is G-supplement in M.

3. Let Z be the ring of integers. It is easy to see that (0) is the only GmZ-small submodule of Z. Now let $T = 0$, $X = 2Z$ and $Y = 3Z$ then

$T \subseteq 2Z + 3Z$ and $2Z \cap 3Z = 6Z \ll_{GT} 2Z$. Then Y is GT-supplement in M.

4. Let Z_6 as Z-module, $T = \{\bar{0}, \bar{3}\}$, $X = \{\bar{0}, \bar{2}, \bar{4}\}$, and $Y = \{\bar{0}, \bar{3}\}$. It is clear that $T \subseteq X + Y$ and $X \cap Y = 0 \ll_{GT} Z_6$, so Y is GT-supplement in Z_6.

Proposition 5.3. *Let M be an R-module, T, X and $Y \leq M$ such that Y is GT-supplement of X in M if $T \subseteq K + Y$, for some submodule K of M. Then Y is a GT-supplement of K in M.*

Proof. Let Y be is GT-supplement of X in M, K submodule of M such that $T \subseteq K + Y$. Since $K \cap Y \subseteq X \cap Y \ll_{GT} Y$ by(Prop:2.8). Then Y is a GT-supplement of K in M. □

Proposition 5.4. *Let M be an R-module, T, X and $Y \leq M$ and Y be a GT-supplement of X in M, $L \leq Y$ and $L \ll_{GT} Y$. Then Y is a GT-supplement of $X + L$ in M.*

Proof. Let Y be a GT-supplement of X in M and $L \leq Y$ and $L \ll_{GT} Y$. Then $T \subseteq X+Y \subseteq X+Y+L$, $X \cap Y \ll_{GT} Y$. Then Y. To show that $Y \cap (X+L) \ll_{GT} Y$. Let K be essential submodule in M such that $T \subseteq Y \cap (X + L) + K$. Then $T \subseteq (X \cap Y) + L + K$, $K \subseteq L + K$ is essential submodule in M hence $T \subseteq L + K$. Since $L \ll_{GT} Y$ thus $T \subseteq K$. Then Y is a GT-supplement of $X + L$ in M. □

Proposition 5.5. *Let M and N be R-modules, and let $f : M \longrightarrow N$ be an epimorphism. If M is GT-supplemented module. Then N is $Gf(T)$-supplemented module.*

Proof. Suppose that M is a GT-supplemented module and let $f : M \longrightarrow N$ be an epimorphism. Let K be submodule of N, M is a GT-supplemented module then $T \subseteq L + f^{-1}(K)$ and $f^{-1}(K) \cap L \ll_{GT} Y$. Then $f(T) \subseteq f(L + f^{-1}(K))$. Then $f(T) \subseteq f(L) + K$. Since $f^{-1}(K) \cap L \ll_{GT} Y$ then $K \cap f(L) = f(f^{-1}(K)) \cap L \ll_{Gf(T)} f(Y)$. Therefore by(Prop:2.8) $f(L)$ is $Gf(T)$-supplement submodule of K in M. □

Proposition 5.6. *Let M be GT-lifting module and Y be a GT-supplement of X in M. Then Y contains a GT-supplement of X which is direct summand of M.*

Proof. Suppose that M is GT-lifting module and Y be a GT-supplement of X in M, Then $T \subseteq X + Y$, $X \cap Y \ll_{GT} Y$. M is GT-lifting then $Y = D + H$, where $D \leq \oplus M$ and $H \ll_{GT} M$. Since $T \subseteq X + Y$, then $T \subseteq X + D + H$ thus $T \subseteq X + D$, now $X \cap D \subseteq X \cap Y \ll_{GT} Y$ by(Prop: 2.6) $X \cap D \ll_{GT} Y$ then D is a GT-supplemente of X in M. □

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Accepted: 14.04.2018