Existence of tube solution for a fractional initial value problem

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Abstract. In this paper, we show existence of tube solution (generalization of lower and upper solution) for a new initial value problem of a fractional nonlinear differential equation of order $0 < \alpha < 1$ with the conformable fractional derivative. By applying a variety of tools including the Schauder fixed-point theorem, some new existence results are achieved.
Keywords: fractional differential equations, existence of solutions, conformable fractional derivative, initial value problem.

1. Introduction

Fractional calculus is one of the most interesting that attracts many investigators, specially mathematics and engineering sciences. Many of natural phenomena can be present by fractional differential equations that implies importance and location of the fractional calculus in the sciences and engineering. Also, factional differential equation is applicable in various fields such as chemical physics, fluid flows, electrical networks, visco-elasticity (see [1, 2, 3, 4]).

In 2014, Khalil *et al.* [7] presented a new well-behaved definition of fractional derivative called the conformable fractional derivative. For $0 < \alpha \leq 1$, this new definition coincides by the classical definitions on polynomials and if $\alpha = 1$, the definition coincides by the classical definition of first order derivative. In 2015, Abdeljawad [8] proceed on the conformable fractional derivative and some properties its such as chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions, Laplace transforms are proposed.

Recently, some authors searched existence and uniqueness of solutions for fractional differential equation with conformable fractional derivative. For example, in 2015, Prasad and Krushna [12] discussed, by utilizing fixed point theorems on cone and under suitable conditions, the existence of multiple positive solutions for a coupled system of iterative type boundary value problems,

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involving new conformable fractional order derivative. Moreover, existence and uniqueness theorem for sequential linear conformable fractional differential equations is proved in [11]. Asawasamrit *et al.* [13] showed the existence of solutions for periodic boundary value problems for impulsive fractional integro-differential equations by concept of conformable fractional derivative and the existence of solutions verified by using the method of lower and upper solutions. In 2015, Batarfi *et al.* [10] studied the existence and uniqueness of solutions for a class of differential equations with three-point boundary conditions as follow:

$$\begin{cases} D^{\alpha}\left(D+\lambda\right)x(t)=f(t,x(t)), & t\in\left[0,1\right], \\ x(0)=x'(0)=0,\ x(1)=\beta x(\eta), & \end{cases}$$

where D^{α} is the conformable fractional derivative, D is the ordinary derivative, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a known continuous function, λ and β are real numbers, $\lambda>0$ and $\eta\in(0,1)$. In 2017, Bayour and Torres in [6], by using the notion of tube solution and Schauder's fixed- point theorem, established existence of tube solution the following problem:

$$\begin{cases} x^{(\alpha)}(t)=f(t,x(t)), & t\in[a,b],\ a>0, \\ x(a)=x_0, & \end{cases}$$

where $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order $\alpha\in(0,1)$. Notice that the tube solution method generalizes the method of lower and upper solution.

In this article, we verified existence of tube solution for a new nonlocal Cauchy problem of the form:

$$\begin{cases} z^{(\alpha)}(s)=h(s,z(s)), & s\in[0,K],\ K\in\mathbb{R}, \\ z(0)+g(z)=z_0, & \alpha\in(0,1), \end{cases} \tag{1.1}$$

where $z^{(\alpha)}$ is conformable fractional derivative of z of order $\alpha\in(0,1)$, $h\in C\left([0,K]\times\mathbb{R},\mathbb{R}\right)$ and $z:[0,K]\to\mathbb{R}$ is a function, also suppose that $g:\mathbb{R}\to\mathbb{R}$ is a continuous function. By utilizing the tube solution method and Schauder's fixed-point theorem, existence results are obtained.

The paper is organized as follows: In section 2, we give the initial definitions and preliminary theorems of conformable fractional calculus. Our main results involving existence of tube solution problem (1.1) is presented in section 3. Finally, some conclusions are added.

2. Preliminaries

In this section, we present some basic definitions, lemmas and theorems that will be used to prove our new results.

Definition 2.1 ([7]). *Given a function, $h : [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of h of order α is defined by*

$$T_\alpha(h)(t) = \lim_{\varepsilon \to 0} \frac{h(t + \varepsilon t^{1-\alpha}) - h(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. If h is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} h^{(\alpha)}(t)$ exists, then define

$$h^{(\alpha)}(0) = \lim_{t \to 0^+} h^{(\alpha)}(t).$$

We will, sometimes, write $h^{(\alpha)}(t)$ for $T_\alpha(h)(t)$, to denote the conformable fractional derivatives of h of order α.

Definition 2.2 ([8]). *The(left) conformable fractional derivative from a of a function $h : [a, \infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined by*

$$\left(T_\alpha^a h\right)(t) = \lim_{\varepsilon \to 0} \frac{h(t + \varepsilon (t-a)^{1-\alpha}) - h(t)}{\varepsilon}.$$

When $a = 0$, we write T_α. If $(T_\alpha h)(t)$ exists on (a, b) then $(T_\alpha^a h)(a) = \lim_{t \to a^+} (T_\alpha^a h)(t)$. The(right) conformable fractional derivative of order $0 < \alpha \leq 1$ at b of h is defined by

$$({}_\alpha^b T h)(t) = -\lim_{\varepsilon \to 0} \frac{h(t + \varepsilon (b-t)^{1-\alpha}) - h(t)}{\varepsilon}.$$

If $({}^b T_\alpha h)(t)$ exists on (a, b) then $({}^b T_\alpha h)(b) = \lim_{t \to b^-} ({}^b T_\alpha h)(t)$.

Note that if h is differentiable then $(T_\alpha^a h)(t) = (t-a)^{1-\alpha} h'(t)$ and $({}_\alpha^b T h)(t) = -(b-t)^{1-\alpha} h'(t)$.

Notation $(I_\alpha^a h)(t) = \int_a^t h(x) d\alpha(x, a) = \int_a^t (x-a)^{\alpha-1} h(x) dx$. When $a = 0$, we write $d\alpha(x)$. Similarly, in the right case we have $({}^b I_\alpha h)(t) = \int_t^b h(x) d\alpha(b, x) = \int_a^t (b-x)^{\alpha-1} h(x) dx$. The operators I_α^a and ${}^b I_\alpha$ are called conformable left and right fractional integrals of order $0 < \alpha \leq 1$.

Theorem 2.3 ([7]). *Let $\alpha \in (0, 1)$ and f, g be α-differentiable at a point $t > 0$. Then*

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$ *, for all $a, b \in \mathbb{R}$.*
2. $T_\alpha(t^p) = pt^{p-\alpha}$ *, for all $p \in \mathbb{R}$.*
3. $T_\alpha(\lambda) = 0$*, for all constant functions $f(t) = \lambda$.*
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
5. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

6. *If, in addition, f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Remark 2.4 ([6])**.** From theorem 2.3 it follows that if $f \in C^1$ (continuously differentiable functions set), then

1. $\lim_{\alpha\to 1} T_\alpha(f)(t) = f'(t)$.
2. $\lim_{\alpha\to 0} T_\alpha(f)(t) = tf'(t)$.

Example 2.5. Suppose $0 < \alpha \le 1$. Functions $f(t) = \sin\frac{1}{\alpha}t^\alpha$, $g(t) = \cos\frac{1}{\alpha}t^\alpha$, $p(t) = (ct+d)^n$, $c, d \in \mathbb{R}$, $n \in \mathbf{N}$, $h(t) = e^{\frac{1}{\alpha}t^\alpha}$, $m(t) = \tan(at)$, $a \in \mathbb{R}$, are α-differentiable, hence their conformable fractional derivatives of order α are equal with:

1. $T_\alpha(f)(t) = \cos\frac{1}{\alpha}t^\alpha$.
2. $T_\alpha(g)(t) = -\sin\frac{1}{\alpha}t^\alpha$.
3. $T_\alpha(p)(t) = nct^{1-\alpha}(ct+d)^{n-1}$, $c, d \in \mathbb{R}$, $n \in \mathbf{N}$.
4. $T_\alpha(m)(t) = at^{1-\alpha}(1+\tan^2(at))$, $a \in \mathbb{R}$.
5. $T_\alpha(h)(t) = e^{\frac{1}{\alpha}t^\alpha}$.

Remark 2.6 ([6])**.** Differentiability implies $\alpha-$differentiability but the contrary is not true: a non-differentiable function can be $\alpha-$differentiable.

Theorem 2.7 ([7])**.** *If a function $f : [0,\infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0,1]$, then f is continuous at t_0.*

Definition 2.8 ([7])**.** *(conformable fractional integral) Let $\alpha \in (0,1)$ and $h : [a,\infty) \to \mathbb{R}$. The conformable fractional integral of h of order α from a to t, denoted by $I_\alpha^a(h)(t)$, is defined by*

$$I_\alpha^a(h)(t) = \int_a^t \frac{h(s)}{s^{1-\alpha}}ds,$$

where the above integral is the usual improper Riemann integral.

Lemma 2.9 ([7])**.** *Assume that $f : [a,\infty) \to \mathbb{R}$ is continuous and $0 < \alpha \le 1$. Then, for all $t > a$ we have*

$$T_\alpha^a I_\alpha^a(f)(t) = f(t).$$

Lemma 2.10 ([8])**.** *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \le 1$. Then, for all $t > a$ we have*

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - f(a).$$

Proposition 2.11 ([8]). *Let $\alpha \in (n, n+1]$ and $f : [a, \infty) \to \mathbb{R}$ be $(n+1)$ times differentiable for $t > a$. Then, for all $t > a$ we have*

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}.$$

Note 2.12 ([6]). *Let $0 < a < b$ and ${}_\alpha\Im_a^b[f]$ be a value of the integral $\int_a^b \frac{f(s)}{s^{1-\alpha}} ds$, then*

$${}_\alpha\Im_a^b[f] = I_\alpha^a(f)(b).$$

Proposition 2.13 ([6]). *Assume $f \in L^1([a,b],\mathbb{R}), 0 < a < b$. Then $|{}_\alpha\Im_a^b[f]| \leq ({}_\alpha\Im_a^b[|f|])$.*

Note 2.14. We define by $C^{(\alpha)}([0,K],\mathbb{R}), K \in \mathbb{R}, \alpha > 0$, the set of all real-valued functions $h : [0,K] \to \mathbb{R}$ that are α-differentiable such that the α-derivative is continuous. We mostly abbreviate $C^{(\alpha)}([0,K],\mathbb{R})$ by $C^{(\alpha)}([0,K])$.

Lemma 2.15 ([6]). *Let $r \in C^{(\alpha)}([a,b]), 0 < a < b$, such that $r^{(\alpha)}(t) < 0$ on $\{t \in [a,b] : r(t) > 0\}$. If $r(a) \leq 0$, then $r(t) \leq 0$ for every $t \in [a,b]$.*

Theorem 2.16. *If $f \in L^1([0,K])$, then function $z : [0,K] \to \mathbb{R}$ as follows:*

$$z(s) = e^{-\frac{1}{\alpha}s^\alpha}\left(z_0 - g(z) +_\alpha \Im_0^s\left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right) \tag{2.1}$$

is a solution of problem

$$\begin{cases} z^{(\alpha)}(s) + z(s) = f(s), s \in [0,K], K \in \mathbb{R}, \\ z(0) + g(z) = z_0, \alpha \in (0,1). \end{cases} \tag{2.2}$$

Proof. Assume $z : [0,K] \to \mathbb{R}$, be a function that is as follow:

$$z(s) = e^{-\frac{1}{\alpha}s^\alpha}\left(z_0 - g(z) +_\alpha \Im_0^s\left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right).$$

By Theorem 2.3 and Lemma 2.9, we can write

$$\begin{aligned} z^{(\alpha)}(s) &= s^{1-\alpha}\left(-\frac{1}{\alpha}\alpha s^{\alpha-1}\right)e^{-\frac{1}{\alpha}s^\alpha}\left(z_0 - g(z) +_\alpha \Im_0^s\left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right) \\ &+ e^{-\frac{1}{\alpha}s^\alpha}\left[\frac{f(s)}{e^{-\frac{1}{\alpha}s^\alpha}}\right] \\ &= -e^{-\frac{1}{\alpha}s^\alpha}\left(z_0 - g(z) +_\alpha \Im_0^s\left[\frac{f(t)}{e^{\frac{-1}{\alpha}t^\alpha}}\right]\right) + f(s) \\ &= -z(s) + f(s). \end{aligned}$$

Also, we have

$$\begin{aligned} z(0) &= e^{-\frac{1}{\alpha}0^{\alpha}}\left(z_0 - g(z) +_{\alpha} \Im_0^0\left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^{\alpha}}}\right]\right) \\ &= e^0(z_0 - g(z) + 0) \\ &= z_0 - g(z). \end{aligned}$$

Multiply both side (2.2) at integral factor $e^{\frac{1}{\alpha}s^{\alpha}}$

$$z^{(\alpha)}(s)e^{\frac{1}{\alpha}s^{\alpha}} + z(s)e^{\frac{1}{\alpha}s^{\alpha}} = f(s)e^{\frac{1}{\alpha}s^{\alpha}}.$$

Then,

$$z^{(\alpha)}(s)e^{\frac{1}{\alpha}s^{\alpha}} + z(s)\left(e^{\frac{1}{\alpha}s^{\alpha}}\right)^{(\alpha)} = f(s)e^{\frac{1}{\alpha}s^{\alpha}}.$$

Now by theorem 2.3, we have

$$T_{\alpha}^0\left(z(s)e^{\frac{1}{\alpha}s^{\alpha}}\right) = f(s)e^{\frac{1}{\alpha}s^{\alpha}}.$$

By lemma 2.10, we have

$$\begin{aligned} I_{\alpha}^0 T_{\alpha}^0\left(z(s)e^{\frac{1}{\alpha}s^{\alpha}}\right) &= I_{\alpha}^0 f(s)e^{\frac{1}{\alpha}s^{\alpha}} \\ z(s)e^{\frac{1}{\alpha}s^{\alpha}} - z(0) &= I_{\alpha}^0 f(s)e^{\frac{1}{\alpha}s^{\alpha}} \\ z(s) &= e^{-\frac{1}{\alpha}s^{\alpha}}\left(\int_0^s \frac{f(t)e^{\frac{1}{\alpha}t^{\alpha}}}{t^{1-\alpha}}dt + z(0)\right) \\ z(s) &= e^{-\frac{1}{\alpha}z^{\alpha}}\left({}_{\alpha}\Im_0^s\left[\frac{f(t)}{e^{-\frac{1}{\alpha}t^{\alpha}}}\right] + z_0 - g(z)\right). \end{aligned}$$

Thus, relation (2.1) obtains. Therefore the proof of theorem is complete. □

Proposition 2.17 ([6])**.** *If $z : (0, \infty) \to \mathbb{R}$ is $\alpha-$differentiable at $s \in [a, b]$, then*

$$|z(s)|^{(\alpha)} = \frac{z(s)z^{(\alpha)}(s)}{|z(s)|}.$$

Definition 2.18. *Suppose $(\mu, N) \in C^{(\alpha)}([0, K], \mathbb{R})) \times C^{(\alpha)}([0, K], [0, \infty))$. Then (μ, N) is a tube solution of problem (1.1) if*

1. *$(x - \mu(s))(f(s, x) - \mu^{(\alpha)}) \leq N(s)N^{(\alpha)}(s)$ for each $s \in [0, K]$ and each $x \in \mathbb{R}$ such that $|x - \mu(s)| = N(s)$,*
2. *$\mu^{(\alpha)}(s) = f(s, \mu(s))$, $N^{(\alpha)}(s) = 0$ for each $s \in [0, K]$ such that $N(s) = 0$,*
3. *$|z_0 - g(z) - \mu(0)| \leq N(0)$.*

Now, present the following note that, next, we utilize in the proof of theorem 3.2.

Note 2.19. We introduce the following notation:

$$T(\mu, N) = \{z \in C^{(\alpha)}([0, K], \mathbb{R}) : |z(s) - \mu(s)| \leq N(s), s \in [0, K]\}.$$

Consider the following problem:

$$\begin{cases} z^{(\alpha)}(s) + z(s) = h(s, \tilde{z}(s)) + \tilde{z}(s), & s \in [0, K],\ K \in \mathbb{R} \\ z(0) + g(z) = z_0, \end{cases} \tag{2.3}$$

where

$$\tilde{z}(s) = \begin{cases} \frac{N(s)}{|z(s)-\mu(s)|}(z(s) - \mu(s)) + \mu(s), & |z(s) - \mu(s)| > N(s), \\ z(s), & otherwise \end{cases} \tag{2.4}$$

Let us define the operator $M : C([0, K]) \to C([0, K])$ is defined by

$$M(z(s)) = e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{h(t, \tilde{z}(t)) + \tilde{z}(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right). \tag{2.5}$$

3. Main result

In this section, we show existence of tube solution by Schauder fixed point theorem.

Proposition 3.1. *If $(\mu, N) \in C^{(\alpha)}([0, K], \mathbb{R}) \times C^{(\alpha)}([0, K], [0, \infty))$ is a tube solution of problem (1.1), then the operator $M : C([0, K]) \to C([0, K])$, is compact.*

Proof. Assume $\{z_m\}_{m\in\mathbf{N}}$, is a sequence of $C([0, K], \mathbb{R})$ converging to $z \in C([0, K], \mathbb{R})$. By proposition 2.13 and relation (2.5), we have

$$\begin{aligned}
&|M(z_m)(s) - M(z(s))| \\
&= \Bigg| e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) - \\
&e^{-\frac{1}{\alpha}s^\alpha} \left(z_0 - g(z) +_\alpha \Im_0^s \left[\frac{h(t, \tilde{z}(t)) + \tilde{z}(t)}{e^{-\frac{1}{\alpha}t^\alpha}} \right] \right) \Bigg| \\
&\leq \frac{A}{B} \left({}_\alpha\Im_0^s \left[|(h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)) - (h(t, \tilde{z}(t)) + \tilde{z}(t))| \right] \right) \\
&\leq \frac{A}{B} \left({}_\alpha\Im_0^s \left[|h(t, \tilde{z}_m(t)) - h(t, \tilde{z}(t))| \right] +_\alpha \Im_0^s \left[|\tilde{z}_m(t) - \tilde{z}(t)| \right] \right),
\end{aligned}$$

where $A = \max_{0\leq t\leq K}\{e^{-\frac{1}{\alpha}t^\alpha}\}$ and $B = \min_{0\leq t\leq K}\{e^{-\frac{1}{\alpha}t^\alpha}\}$. We need to show that the sequence $\{f_m\}_{m\in\mathbf{N}}$ presented by $f_m(t) = h(t, \tilde{z}_m(t)) + \tilde{z}_m(t)$ converges in $C^{(\alpha)}([0, K])$ to function $f(t) = h(t, \tilde{z}(t)) + \tilde{z}(t)$. Since there is a constant $D > 0$

such that $\|\tilde{z}\|_{C([0,K],\mathbb{R})} < D$, there is an index M such that $\|\tilde{z}_m\|_{C([0,K],\mathbb{R})} \le D$ for every $m > M$. Therefore, h is uniformly continuous on $[0,K] \times E_D(0)$. Hence, for $\varepsilon > 0$ given, there is a $\delta > 0$ such that

$$|y - z| < \delta < \frac{B\varepsilon\alpha}{2AK^\alpha}$$

for every $z, y \in \mathbb{R}$;

$$|h(t,y) - h(t,z)| < \frac{B\varepsilon\alpha}{2AK^\alpha}$$

for each $t \in [0,K]$. By hypothesis, there is an index $\tilde{M} > M$ such that $\|\tilde{z}_m - \tilde{z}\|_{C([0,K],\mathbb{R})} < \delta$ for $m > \tilde{M}$. Hence

$$\begin{aligned}
|M(z_m)(s) - M(z)(s)| &< \frac{A}{B}\left({}_\alpha\Im_0^K\left[\frac{B\varepsilon\alpha}{2AK^\alpha}\right] +_\alpha \Im_0^K\left[\frac{B\varepsilon\alpha}{2AK^\alpha}\right]\right)\\
&= \frac{2AB\varepsilon\alpha}{2ABK^\alpha}({}_\alpha\Im_0^K\,[1])\\
&= \frac{\varepsilon\alpha}{K^\alpha} \times \frac{K^\alpha}{\alpha}\\
&= \varepsilon.
\end{aligned}$$

This proves continuity of M. Also, we prove that the set $M(C[0,K])$ is relatively compact. Present a sequence $\{x_m\}_{m\in\mathbf{N}}$ of $M(C[0,K])$ for every $m \in \mathbf{N}$. There is $z_m \in C([0,K])$ such that $x_m = M(z_m)$. Now by proposition 2.13 we have

$$\begin{aligned}
|M(z_m)(s)| &= \left|e^{-\frac{1}{\alpha}s^\alpha}\left(z_0 - g(z) +_\alpha \Im_0^s\left[\frac{h(t,\tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right)\right|\\
&\le A\left(|z_0 - g(z)| + \frac{1}{B}\left({}_\alpha\Im_0^K\,[|h(s,\tilde{z}_m(t)) + \tilde{z}_m(t)|]\right)\right)\\
&\le A\left(|z_0 - g(z)| + \frac{1}{B}\left({}_\alpha\Im_0^K\,[|h(s,\tilde{z}_m(t))|]\right) + \frac{1}{B}\left({}_\alpha\Im_0^K\,[|\tilde{z}_m(t)|]\right)\right).
\end{aligned}$$

By definition, there is a $D > 0$ such that $|\tilde{z}_m(t)| \le D$ for every $t \in [0,K]$ and every $m \in \mathbf{N}$. The function h is compact on $[0,K] \times E_D(0)$ and we can understand the existence of a constant $R > 0$ such that

$$|h(t,\tilde{z}_m(t))| \le R$$

for each $t \in [0,K]$. The sequence $\{x_m\}_{m\in\mathbf{N}}$ is uniformly bounded for all $m \in \mathbf{N}$. Also see that for $s_1, s_2 \in [0,K]$ we can write

$$
\begin{aligned}
&|M(z_m)(s_2) - M(z_m)(s_1)| \\
&= \Bigg| e^{-\frac{1}{\alpha}s_2^\alpha}\left(z_0 - g(z) +_\alpha \Im_0^{s_2}\left[\frac{h(t,\tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right) \\
&-e^{-\frac{1}{\alpha}s_1^\alpha}\left(z_0 - g(z) +_\alpha \Im_0^{s_1}\left[\frac{h(t,\tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right)\Bigg| \\
&= \Bigg|\left(e^{-\frac{1}{\alpha}s_2^\alpha} - e^{-\frac{1}{\alpha}s_1^\alpha}\right)(z_0 - g(z)) + e^{-\frac{1}{\alpha}s_2^\alpha}\left({}_\alpha\Im_0^{s_2}\left[\frac{h(t,\tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right) \\
&-e^{-\frac{1}{\alpha}s_1^\alpha}\left({}_\alpha\Im_0^{s_1}\left[\frac{h(t,\tilde{z}_m(t)) + \tilde{z}_m(t)}{e^{-\frac{1}{\alpha}t^\alpha}}\right]\right)\Bigg| \\
&\le C\left|e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha}\right| + \frac{A}{B}\left|{}_\alpha\Im_0^{s_1}\left[h(t,\tilde{z}_m(t)) + \tilde{z}_m(t)\right] - ({}_\alpha\Im_0^{s_2}\left[h(t,\tilde{z}_m(t)) + \tilde{z}_m(t)\right])\right| \\
&= C\left|e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha}\right| + \frac{A}{B}\left|{}_\alpha\Im_{s_1}^{s_2}\left[h(t,\tilde{z}_m(t)) + \tilde{z}_m(t)\right]\right| \\
&\le C\left|e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha}\right| + \frac{A}{B}\left|{}_\alpha\Im_{s_1}^{s_2}\left[h(t,\tilde{z}_m(t))\right]\right| + \frac{A}{B}\left|{}_\alpha\Im_{s_1}^{s_2}\left[\tilde{z}_m(t)\right]\right| \\
&\le C\left|e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha}\right| + \frac{AR}{B}\left|{}_\alpha\Im_{s_1}^{s_2}[1]\right| + \frac{AD}{B}\left|{}_\alpha\Im_{s_1}^{s_2}[1]\right| \\
&= C\left|e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha}\right| + \frac{A}{B}(R+D)\left|{}_\alpha\Im_{s_1}^{s_2}[1]\right| \\
&< C\left|e^{-\frac{1}{\alpha}s_1^\alpha} - e^{-\frac{1}{\alpha}s_2^\alpha}\right| + \frac{A(R+D)}{B}\frac{|s_1^\alpha - s_2^\alpha|}{\alpha}.
\end{aligned}
$$

Where, $C = z_0 - g(z)$. This proves that the sequence $\{x_m\}_{m\in\mathbf{N}}$ is equicontinuous. By applying the Arzela-Ascoli theorem, $M(C([0,K]))$ is relatively compact and thus M is compact.□

Theorem 3.2. *If* $(\mu, N) \in C^{(\alpha)}([0,K],\mathbb{R}) \times C^{(\alpha)}([0,K],[0,\infty))$*, is a tube solution of the problem* (1.1)*. Then the problem* (1.1) *has a solution* $z \in C^{(\alpha)}([0,K],\mathbb{R}) \cap T(\mu, N)$*.*

Proof. According proposition 3.1, the operator M is compact. It has a fixed point by the Schauder fixed point theorem ([5],[9]). By theorem 2.16, a such fixed point is a solution of problem (2.3)-(2.4). Therefore, it is enough, we prove that for each solution z of problem (2.3)-(2.4) , $z \in T(\mu, N)$. By (2.3) we have :

$$z^{(\alpha)}(s) = h(s,\tilde{z}(s)) + \tilde{z}(s) - z(s). \tag{3.1}$$

By (2.4), we have :

$$\tilde{z}(s) - \mu(s) = \frac{N(s)}{|z(s)-\mu(s)|}(z(s)-\mu(s)) \to \frac{\tilde{z}(s)-\mu(s)}{N(s)} = \frac{z(s)-\mu(s)}{|z(s)-\mu(s)|}. \tag{3.2}$$

Set $E=\{s\in[0,K]:|z(s)-\mu(s)|>N(s)\}$. If $s\in E$, then by proposition 2.17, (3.1), (3.2) and definition 2.18, we give

$$\begin{aligned}
&(|z(s)-\mu(s)|-N(s))^{(\alpha)}\\
=\;&(|z(s)-\mu(s)|)^{(\alpha)}-N^{(\alpha)}(s)\\
=\;&\frac{(z(s)-\mu(s))(z(s)-\mu(s))^{(\alpha)}}{|z(s)-\mu(s)|}-N^{(\alpha)}(s)\\
=\;&\frac{(z(s)-\mu(s))(z^{(\alpha)}(s)-\mu^{(\alpha)}(s))}{|z(s)-\mu(s)|}-N^{(\alpha)}(s).
\end{aligned}$$

Since (μ,N) is a tube solution of (1.1), then on $\{s\in E:N(s)>0\}$

$$\begin{aligned}
&(|z(s)-\mu(s)|-N(s))^{(\alpha)}\\
&=\frac{(z(s)-\mu(s))(z^{(\alpha)}(s)-\mu^{(\alpha)}(s))}{|z(s)-\mu(s)|}-N^{(\alpha)}(s)\\
&=\frac{(z(s)-\mu(s))(h(s,\tilde{z}(s))+\tilde{z}(s)-z(s)-\mu^{(\alpha)}(s))}{|z(s)-\mu(s)|}-N^{(\alpha)}(s)\\
&=\frac{(z(s)-\mu(s))(h(s,\tilde{z}(s))-\mu^{(\alpha)}(s))}{|z(s)-\mu(s)|}+\frac{(z(s)-\mu(s))(\tilde{z}(s)-z(s))}{|z(s)-\mu(s)|}-N^{(\alpha)}(s)\\
&=\frac{(\tilde{z}(s)-\mu(s))(h(s,\tilde{z}(s))-\mu^{(\alpha)}(s))}{N(s)}+\frac{(\tilde{z}(s)-\mu(s))(\tilde{z}(s)-z(s))}{N(s)}-N^{(\alpha)}(s)\\
&=\frac{(\tilde{z}(s)-\mu(s))(h(s,\tilde{z}(s))-\mu^{(\alpha)}(s))}{N(s)}\\
&+\frac{(\frac{N(s)}{|z(s)-\mu(s)|}(z(s)-\mu(s))+\mu(s)-\mu(s))(\frac{N(s)}{|z(s)-\mu(s)|}(z(s)-\mu(s))+\mu(s)-z(s))}{N(s)}\\
&-N^{(\alpha)}(s)\\
&=\frac{(\tilde{z}(s)-\mu(s))(h(s,\tilde{z}(s))-\mu^{(\alpha)}(s))}{N(s)}+[N(s)-|z(s)-\mu(s)|]-N^{(\alpha)}(s)\\
&\le\frac{N(s)N^{(\alpha)}(s)}{N(s)}+[N(s)-|z(s)-\mu(s)|]-N^{(\alpha)}(s)<0.
\end{aligned}$$

Since $s\in\{t\in E,N(t)=0\}$, then $N(s)=0$

$$\begin{aligned}
&(|z(s)-\mu(s)|-N(s))^{(\alpha)}\\
=\;&\frac{(z(s)-\mu(s))(h(s,\tilde{z}(s))+\tilde{z}(s)-z(s)-\mu^{(\alpha)}(s))}{|z(s)-\mu(s)|}-N^{(\alpha)}(s)\\
=\;&\frac{(z(s)-\mu(s))(h(s,\tilde{z}(s))-\mu^{(\alpha)}(s))}{|z(s)-\mu(s)|}+[N(s)-|z(s)-\mu(s)|]-N^{(\alpha)}(s)\\
\le\;&\frac{N(s)N^{(\alpha)}(s)}{N(s)}-|z(s)-\mu(s)|-N^{(\alpha)}(s)\\
<\;&-N^{(\alpha)}(s)=0.
\end{aligned}$$

If $\gamma(s) = |z(s) - \mu(s)| - N(s)$, then $\gamma^{(\alpha)} < 0$ on $E = \{s \in [0, K] : \gamma(s) > 0\}$. On the other hand, since (μ, N) is a tube solution of problem (1.1) and z satisfies $|z_0 - g(z) - \mu(0)| \leq N(0)$, we have, $\gamma(0) \leq 0$, also lemma 2.15 implies that $E = \emptyset$. Hence, $z \in T(\mu, N)$, therefore, the proof of theorem is complete. □

Example 3.3. Consider the fractional initial value problem

$$\begin{cases} z^{(\frac{1}{2})} = \sqrt{s}\left(-\sin(s) - \frac{1}{8}\sin(2s) - \frac{s}{4} + \frac{1}{2}\int_0^s z^2(t)dt\right), & s \in [0,2], \\ z(0) = 1, \end{cases} \tag{3.3}$$

where, $g(z) = 0$. By definition 2.18, $(\mu, N) \equiv (0, 1)$ is a tube solution. Therefore, by theorem 3.2, the problem (3.3) has a solution z such that $|z(s)| \leq 1$ for all $s \in [0, 2]$.

4. Conclusions

In this paper, we have considered a new fractional initial value problem with the conformable fractional derivative. By the Schauder fixed point theorem and some new tools for fractional differential equations, existence of tube solution is proved. Also, we have presented an example to expression the results.

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