A note on a finite group with all non-nilpotent maximal subgroups being normal

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Abstract. In this paper we give an elementary proof to show that a finite group with all non-nilpotent maximal subgroups being normal is solvable.
Keywords: non-nilpotent maximal subgroup, normal, solvable.

1. Introduction

It is known that one of the important characterizations of a nilpotent group is that a finite group G is nilpotent if and only if all maximal subgroups of G are normal. As a generalization of this result, it is interesting to characterize the finite group with all non-nilpotent maximal subgroups being normal. The second author, C. Zhang and S. Guo [4, Lemma 4] used a theorem of Ballester-Bolinshe and Shemetkov [1, Theorem 2] about the p-nilpotent group to show that a finite group with all non-nilpotent maximal subgroups being normal is solvable. In [5] the second author used a theorem of Rose [3, Theorem 1] about the non-solvable group with a nilpotent maximal subgroup of even order to give two distinct proofs of the solvability of such a group.

In this paper, without using neither [1, Theorem 2] nor [3, Theorem 1], we give an elementary proof of the solvability of such a group.

Theorem 1.1. *A finite group G with all non-nilpotent maximal subgroups being normal is solvable.*

The following lemma is necessary in the proof of Theorem 1.1 which is given in Section 2.

Lemma 1.2 ([2, Theorem 9.1.10]). *Let the finite group G possess a nilpotent Hall $\pi-$subgroup H. Then all Hall $\pi-$subgroups of G are conjugate.*

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2. Proof of Theorem 1.1

Proof. Suppose that the theorem is not true. Let G be a counterexample of minimal order.

If all maximal subgroups of G are non-nilpotent. By the hypothesis of the theorem, one has that all maximal subgroups of G are normal. Then G is nilpotent, a contradiction. Thus G has nilpotent maximal subgroups. It is clear that G also must have non-nilpotent maximal subgroups. It follows that G is not a simple group.

Let N be a minimal normal subgroup of G. Since the hypothesis of the theorem holds for G/N and $|G/N| < |G|$, one has that G/N is solvable. It follows that N is non-solvable. In particular, G has no solvable non-trivial normal subgroup. Let L be a nilpotent maximal subgroup of G and P be any Sylow subgroup of L. If P is not a Sylow subgroup of G, then $L < N_G(P) \leq G$. One has $N_G(P) = G$ as L is maximal in G. It follows that G possesses a solvable non-trivial normal subgroup P, a contradiction. Then any Sylow subgroup of L is also a Sylow subgroup of G. Thus we have that every nilpotent maximal subgroup of G must be a nilpotent Hall subgroup of G.

Let L_1 be a nilpotent maximal subgroup of G which is a Hall π_1-subgroup of G for the set of prime divisors π_1 of $|G|$ and L_2 be a nilpotent maximal subgroup of G which is a Hall π_2-subgroup of G for the set of prime divisors π_2 of $|G|$. We will show that $\pi_1 = \pi_2$. Otherwise, assume $\pi_1 \neq \pi_2$. By [2, Theorem 10.4.2], one has $2 \in \pi_1$ and $2 \in \pi_2$. Let $R_1 \in \mathrm{Syl}_2(L_1)$ and $R_2 \in \mathrm{Syl}_2(L_2)$. Suppose $q \in \pi_1$ but $q \notin \pi_2$. Let $Q \in \mathrm{Syl}_q(L_1)$. Then $Q \leq N_G(R_1)$. One has $q \mid |N_G(R_1)|$. Since $L_2 \leq N_G(R_2)$ and G has no solvable non-trivial normal subgroup, one has $N_G(R_2) = L_2$. Note that $|N_G(R_1)| = |N_G(R_2)|$ since R_1 and R_2 are conjugate. It follows that $q \mid |L_2|$, a contradiction. Thus we have $\pi_1 = \pi_2$. That is, all nilpotent maximal subgroups of G are nilpotent Hall $\pi-$subgroups of G for a fixed π. By Lemma 1.2, we have that all nilpotent maximal subgroups of G are conjugate.

Let $\pi_e(N) = \{p_1, p_2, \cdots, p_s\}$ be the set of prime divisors of $|N|$. For every prime divisor p_i of $|N|$ with $1 \leq i \leq s$, let $P_i \in \mathrm{Syl}_{p_i}(N)$. By Frattini argument, one has $G = N_G(P_i)N$. Since G has no solvable non-trivial normal subgroup, one has $N_G(P_i) < G$. Note that every non-nilpotent maximal subgroup of G is normal in G and N is non-solvable. It follows that every non-nilpotent maximal subgroup of G must contain N. Thus $N_G(P_i)$ is contained in some nilpotent maximal subgroup of G.

Let M_i be a nilpotent maximal subgroup of G such that $N_G(P_i) \leq M_i$. Then $P_i \leq M_i$. Since all M_i are conjugate for $1 \leq i \leq s$, there exists a $g_i \in G$ such that ${M_i}^{g_i} = M_1$ for every $2 \leq i \leq s$. Thus ${P_i}^{g_i} \leq {M_i}^{g_i} = M_1$. Note that ${P_i}^{g_i} \leq N^{g_i} = N$ and N can be generated by all its Sylow subgroups, that is, $N =< P_1, {P_2}^{g_2}, \cdots, {P_s}^{g_s} >$. Then we have $N \leq M_1$. It follows that N is nilpotent, a contradiction.

So the counterexample of minimal order does not exist and then G is solvable. □

Acknowledgement

This research was supported in part by Shandong Provincial Natural Science Foundation, China (ZR2017MA022), NSFC (11561021 and 11761079) and Yantai University Graduate Education Innovation Project (120202). The authors are very grateful to the referee for his valuable comments.

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Accepted 10.10.2018