On discrete time schemes for the problem of fibre suspension flows

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Abstract. We present a time discrete stability results presented in [9], we use the tools presented in [3,6] to show that the results obtained in [9] can be generalized to obtain estimates not only for the velocity but also for the orientation tensor and a non zero body force, and without decomposing the orientation tensor into a sum of diagonal and a traceless tensors. As for the regular solutions, we prove the solutions of a discrete scheme are continuously dependent on the initial data and the same body force. Also, the restrictions imposed on the data con rm a well known result proven in [4, 8, 9], that for the rest state to be stable, the particle number Np must be less than 35/2.

Keywords: fibre suspension flows, continuously dependent, existence, stability energy. estimate.

1. Introduction

Fibre suspension flows are modelled by a coupled system of nonlinear partial differential equations, with the velocity, the pressure and orientation tensorial as primary variables variable (for more details, see [1, 2, 4, 7,8, 9]). Both the velocity and orientation tensor constitutive equations contain a fourth–order tensor, which in this paper, is approximated by the linear closure.

The constitutive equations investigated in the present work were the subject of theoretical studies by Galdi and Reddy [4], Munganga, Reddy and Diatezua [9], Munganga and Reddy [8]. Their work have helped to better understand the properties of the equations governing the flow of fibre suspensions. Galdi and Reddy, [4] showed that for the linear closure, there is a connection between stability and the particle number: in particular the rest state is unstable, in the sense of Liapounov, when the particle number exceeds 35/2.

In recent years, stability of fibre suspension flows, from thermodynamic and energetic perceptive were investigated by Munganga et al. [8,9] using linear closure and quadratic closure approximations. Under certain conditions, and when the rotary diffusivity is constant, with the use of the linear closure, Munganga et al. [8,9] proved locally and globally existence of a unique classical solution, in time. For the quadratic closure, the same authors also established the existence of a local solutions in time.

Galdi and Reddy [4] established a unique local solution in time and instability for the rest state flow for the constitutive equations for fibre suspension flows for the linear closure, if the particle number $N_p \geq 35/2$ and the rotary diffusivity D_r is proportional to $|\mathbf{D}|$,.

Munganga and Reddy [9] extended the stability analysis started in [4] by considering others closure rules like quadratic, orthotropic, hybrid. The question of consistency of the equations with respect to the second law of thermodynamic was also investigated in [9]. For D_r constant, the local in time solution constructed in [4], was shown by Munganga and Reddy [8] to be global and stable for the linear ($N_p < 35/2$) and quadratic closure in the absence of body force. Munganga [7] established existence, uniqueness, convergence and stability of solutions to the equations of steady flows of fibre suspension flow. It is important to point out that the stability results obtained in [9] only involve the velocity.

In this paper, we present a time discrete schemes similarly to the results obtained in [9] by estimating not only the velocity but also the orientation tensor where the external force is considered to be non zero. We also prove that the regular solutions of the equations of fibre suspension flows depend continuously on the initial data with the same non zero body force. The solution is proven to be bounded for all time in the absence of body force. The solutions of the discrete problem also satisfy the same properties as the solutions of continuous problem.

The rest of the paper is organised as follows. Section 2 reviews the mathematical model and carries out simplifications to the constitutive equations. Section 3 derives a priori estimates for the continuous problem, and shows the dependence of the solution with respect to the initial data. Section 4 establishes the continuous dependence of the solution with respect to the initial conditions of a time discrete scheme, results similar to continuous model are replicated in this section. Section 5 proves the instability of the rest state for a time discrete problem in the sense of Lyapounov if $N_p \geq 35/2$. We draw our conclusions in Section 6.

1.1 Notation and function spaces

Coordinate-free notation will be used wherever convenient, and vectors and tensors will be denoted by bold-face letters.

If $\mathbf{u}$ and $\mathbf{v}$ are two vectors, their scalar product will be denoted by $\mathbf{u} \cdot \mathbf{v}$ or by $(\mathbf{u}, \mathbf{v})$, while the scalar product of of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ will be denoted by $\mathbf{A} : \mathbf{B}$. In index form these expressions read $u_i v_i$ and $A_{ij} B_{ij}$, the summation convention on repeated indices being applied at all times. The magnitude of a vector $\mathbf{u}$ and a tensor $\mathbf{A}$ are then naturally defined by $|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2}$ and $|\mathbf{A}| = (\mathbf{A} : \mathbf{A})^{1/2}$.

We summarise here some notations that will occur throughout the paper:

- $\mathbf{x}$: location of fluid particle
- $\mathbf{u}$: velocity field of the flow
- $\mathbf{b}$: body force
- p: pressure
- μ: dynamic viscosity
- ρ: fluid mass density
- $\mathbf{A}$: second-order orientation tensor.
- $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ deformation tensor
- $\mathbf{W} = \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^T)$ spin tensor
- $\mathbf{T}$: symmetric Cauchy stress tensor
- $\mathcal{A}$: fourth-order orientation tensor

We will also need the following function spaces:

Here Ω denotes a bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3), with regular boundary Γ. We will assume that Ω is locally located on one side of Γ, and that Γ is at least locally Lipschitz, with more regularity to be specified whenever it is needed.

$\Omega_T = \Omega \times (0, T)$, for $T > 0$.

$L^p(\Omega), 1 \leq p \leq \infty$, the Lebesgue spaces with norms defined in the usual way and their norms denoted by $\| \cdot \|_{L^p}$.

For a multi-index $\alpha = (\alpha_1, ..., \alpha_d)$ where α_i are nonnegative integers, we define

$$\partial^\alpha (\cdot) := \frac{\partial^{\alpha_1 + \cdots + \alpha_d} (\cdot)}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d}.$$

The following classical function spaces will be used throughout

$$\begin{aligned} L^2(\Omega, \mathbb{R}) &= \left\{ v : \Omega \longrightarrow \mathbb{R}, \int_\Omega |v|^2 < \infty \right\} \\ H^1(\Omega) &= \left\{ v \in L^2(\Omega, \mathbb{R}) : \partial^\beta v \in L^2(\Omega, \mathbb{R}), |\beta| \leq 1 \right\}. \end{aligned}$$

We denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the L^2 -inner product and norm respectively. The notation will be used for scalar-, vector- and matrix-valued functions, with the norms for the latter two types of functions being defined in the usual, component-wise way. As usual, $\mathbf{u}(t)$ stands the function $\mathbf{x} \in \boldsymbol{\Omega} \longrightarrow \mathbf{u}(\mathbf{x}, t)$.

The double contraction $\tau, \sigma \in \mathbb{R}^{d \times d}$, two rank two tensors is denoted by $\tau : \sigma$, and given by

$$\tau : \sigma = tr\left[\tau\sigma^T\right] = tr\left[\tau^T\sigma\right] = \tau_{ij}\sigma_{ij}.$$

One should note that if τ is antisymmetric and σ is symmetric, then $\tau : \sigma = \mathbf{0}$.

The logarithm of a positive definite diagonal matrix is a diagonal matrix with, on its diagonal, the logarithm of each entry. If $\mathbf{B}$ is a symmetric matrix, the logarithm of $\mathbf{B}, ln\mathbf{B}$ is given by

$$\ln \mathbf{B} = \mathbf{Q}^T \ln \mathbf{C}\mathbf{Q}, \text{ where } \mathbf{B} = \mathbf{Q}^T \mathbf{C}\mathbf{Q},$$

with $\mathbf{Q}$ and $\mathbf{C}$ orthogonal and positive definite diagonal matrices respectively. We have the following two lemmas from [3]:

Lemma 1. *Let* $\mathbf{N}, \mathbf{M}$ *be two symmetric and positive definite matrices. Then*

(*a*) $tr \ln \mathbf{N} = \ln \det \mathbf{N}$,

(*b*) $\mathbf{N} - \ln \mathbf{N} - \mathbf{I}$ *is symmetric,* $tr\left(\mathbf{N} - \ln \mathbf{N} - \mathbf{I}\right) \geq 0$.

(*c*) $\mathbf{N}+\mathbf{N}^{-1}-2\mathbf{I}$ *is symmetric positive semi-definite, and* $tr\left(\mathbf{M} + \mathbf{M}^{-1} - 2\mathbf{I}\right) \geq 0$

(*d*) $tr\left[\left(\mathbf{M} - \mathbf{N}\right)\mathbf{N}^{-1}\right] = tr\left[\mathbf{M}\mathbf{N}^{-1} - \mathbf{I}\right] \geq \ln \det \left[\mathbf{M}\mathbf{N}^{-1}\right] = tr\left[\ln \mathbf{M} - \ln \mathbf{N}\right]$.

(*e*) $tr\left(\mathbf{N}\mathbf{M}\right) = tr\left(\mathbf{M}\mathbf{N}\right) \geq 0$

Lemma 2. *If* $\mathbf{A}(t) \in \left(C^1\left([0,T]\right)\right)^{\frac{d(d+1)}{2}}$, *then* $\forall t \in [0,T)$:

$$\left(\frac{d}{dt}\mathbf{A}\right) \cdot \mathbf{A}^{-1} = tr\left(\mathbf{A}^{-1}\frac{d}{dt}\mathbf{A}\right) = \frac{d}{dt}tr\left(\ln \mathbf{A}\right) \tag{1}$$

$$\left(\frac{d}{dt}\ln \mathbf{A}\right) \cdot \mathbf{A} = tr\left(\mathbf{A}\frac{d}{dt}\ln \mathbf{A}\right) = \frac{d}{dt}tr\mathbf{A} \tag{2}$$

$H^k(\Omega)$, $k = 0, 1, ...$ the Sobolev spaces endowed with the inner product $(\cdot,\cdot)_{H^k}$ defined by

$$(\mathbf{u}, \mathbf{v})_{H^k} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha \mathbf{u} D^\alpha \mathbf{v} dx, \qquad \alpha = (\alpha_1, ..., \alpha_n).$$

$W^{k,\infty}(\Omega) = \{f \in L^\infty(\Omega) : D^\alpha f \in L^\infty(\Omega), 0 \leq \alpha \leq k\}, k = 1, 2, 3, ...$ will denotes the Sobolev spaces in $L^\infty(\Omega)$.

$H_0^k(\Omega) = \{f \in H^k(\Omega) : f_{/\Gamma} = 0 \text{ and } D^\alpha f = 0 \text{ on } \Gamma \text{ for } |\alpha| < k\}$

$H^{-k}(\Omega)$ the topological dual space of $H_0^k(\Omega)$.

We denote the space of vector- or tensor valued functions with components in one of the spaces introduced above as follows:

$\mathbb{L}^p(\Omega)$ will denote the space of vector-or tensor valued functions with components in $L^p(\Omega)$. The same will apply to $\mathbb{H}^k(\Omega)$,...

We will also require the following spaces:

$\mathbb{H} = \{\mathbf{u} : v_i \in L^2(\Omega), div\ \mathbf{u} = 0 \text{ in } \Omega, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$.

$\mathbb{V} = \{\mathbf{u} : u_i \in H_0^1(\Omega), div\ \mathbf{u} = 0 \text{ in } \Omega\}$.
$\mathbb{A} = \{\mathbf{A} : A_{ij} \in L^2(\Omega), A_{ij} = A_{ji},\ A_{ii} = 0 \quad \text{a.e in } \Omega\}$.
These spaces are equipped respectively with the norms

$$\begin{aligned} &\| \cdot \|_{\mathbb{H}} \equiv \| \cdot \|_{\mathbb{L}^2} \equiv | \cdot |; \\ &\| \cdot \|_{\mathbb{V}} \equiv \| \cdot \|_{\mathbb{H}^1} \quad and \\ &\| \cdot \|_{\mathbb{A}} \equiv \| \cdot \|_{\mathbb{L}^2}. \end{aligned}$$

$\mathbb{A}_k = \mathbb{A} \cap \mathbb{H}^k(\Omega)$.

We denotes by P the orthogonal projection of $\mathbb{L}^2(\Omega)$ onto $\mathbb{H}$, and the operator $\mathcal{L}$ by

$$\mathcal{L}(\mathbf{u}) = -P\Delta\mathbf{u};$$

where Δ is the Laplacian operator. The domain of $\mathcal{L}$ is given by $D(\mathcal{L}) = \mathbb{V} \cap \mathbb{H}^2(\Omega)$, and the norm $\|\mathbf{u}\|_{D(\mathcal{L})} = \|\mathcal{L}\mathbf{u}\|_{L^2}$ is equivalent to the natural $\mathbb{H}^2$-norm.

$\|\mathbf{u}\| = |\nabla\mathbf{u}|$ is the Dirichlet norm.
$\mathbb{L}_{sym} = \{\mathbf{A} : A_{ij} \in L^2(\Omega), A_{ij} = A_{ji}\}$.
$\mathbb{H}^k_{sym} = \{\mathbf{A} : A_{ij} \in H^k(\Omega), A_{ij} = A_{ji}\}$.
$\mathcal{H} = \mathbb{H} \times \mathbb{L}_{sym}$.
$\mathbb{X}_k = \mathbb{H}^k(\Omega) \cap D(\mathcal{L})$, $\mathbb{X}_1 = \mathbb{H}^1(\Omega) \cap \mathbb{H}$ and $\mathbb{H}^0 = \mathbb{H}$.
We set

$$\|\mathbf{u}\|_{\mathbb{L}^p(0,T,\mathbb{H}^k)} = \left(\int_0^T \|\mathbf{u}(t)\|^p_{\mathbb{H}^k} dt \right)^{\frac{1}{p}}.$$

$\mathbb{L}^p(0,T;\mathbb{H}^k) = \{\mathbf{u} : [0,T] \longrightarrow \mathbb{H}^k(\Omega) : \|\mathbf{u}\|_{\mathbb{L}^p(0,T;\mathbb{H}^k)} < \infty\}$.

2. Constitutive equations

We consider a fluid that occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with a regular boundary Γ, and a body force $\mathbf{b}$ per unit mass acting on the fluid. We will assume without loss of generality, that the fluid has mass density equal to unity.

The problem is then to find the velocity $\mathbf{u}(\mathbf{x},t)$, the pressure $p(\mathbf{x},t)$, and the orientation tensor $\mathbf{A}(\mathbf{x},t)$ which is solution of the set of constitutive equations (3)-(8), (for the linear closure, see [1,2,4,8,9] for more details).

(3) $$\rho(\mathbf{u}' + (\mathbf{u}\cdot\nabla)\mathbf{u}) - 2\mu_I \left(1 - \frac{2N_p}{35}\right)\Delta\mathbf{u} + \nabla p = \mathrm{div}\mathbf{T} + \rho\mathbf{b}, \text{ in } \Omega_T;$$

(4) $$\mathbf{T} = \frac{2\mu_I}{7}\left[N(\mathbf{AD} + \mathbf{DA}) + N_p(\mathbf{A}:\mathbf{D})\mathbf{I}\right];$$

$$\text{div}\mathbf{u} = 0 \ , \text{ in } \Omega_T; \tag{5}$$

$$\mathbf{A}' + (\mathbf{u} \cdot \nabla) \mathbf{A} = \mathbf{W}\mathbf{A} - \mathbf{A}\mathbf{W} + \frac{3\lambda}{7} (\mathbf{A}\mathbf{D} + \mathbf{D}\mathbf{A})$$

$$+ \frac{4\lambda}{7} \mathbf{D} - \frac{2\lambda}{7} (\mathbf{A} : \mathbf{D}) \mathbf{I} + D_r (\mathbf{I} - d\mathbf{A}) , \text{ in } \Omega_T; \tag{6}$$

$$\text{tr}\mathbf{A} = 1, \text{ and } \mathbf{A}^T = \mathbf{A}, \quad \text{in } \Omega_T; \tag{7}$$

$$\mathbf{u}|_\Gamma = 0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \mathbf{A}(t = 0) = \mathbf{A}_0, \mathbf{A}_0^T = \mathbf{A}_0, \text{ in } \Omega; \tag{8}$$

where for $T > 0$, $\Omega_T = \Omega \times [0, T]$, $p = p(\mathbf{x}, t)$, is the hydrodynamic pressure, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the fluid velocity, $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$ is the orientation tensor, $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ and $\mathbf{W} = \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^T)$, N_s and N_p are particle and shear number, respectively, $N = \frac{1}{7}(4N_p + 14N_s)$. $\lambda = \frac{r^2-1}{r^2+1}$, $r = \frac{l}{d}$, l and d, $(l \geq d)$ are the length and diameter of the particle respectively, λ, $0 \leq \lambda \leq 1$, describes the slenderness of the particle, if $\lambda = 0, (l = d)$, particle has the shape of a sphere, and if $\lambda = 1, (rarrow\infty)$, particle is an infinitely long cylinder. D_r is the rotary diffusivity due to Brownian motion. And $\mu_I = \mu(1 + hH)$ where μ is the solvent viscosity, h is the particle volume fraction, and H is a positive constant.

In lemma 3, we show that the constitutive equation (3)-(8) can be simplified further. It worth noting that lemma 3 is a generalisation of lemma 2.1 in [7].

Lemma 3. *If* $(\mathbf{u}, \mathbf{A})$ *satisfies* (3)-(8) *then* $\mathbf{A} : \mathbf{D} = 0$. *Moreover, if* $d = 2$, *then* $\mathbf{A}^{-1} : \mathbf{D} = 0$, *and if* $d = 3$, $\mathbf{A}^{-1}$: $\mathbf{D}$ *does not have a determined sign. For* $d = 2$, *if* $\mathbf{A}_0$ *is symmetric, and positive definite, so is* $\mathbf{A}(t)$.

Proof. We take the trace on both side of the evolution equation (6), to get

$$\begin{aligned} & \frac{d}{dt} tr\mathbf{A} + [\mathbf{u} \cdot \nabla] tr\mathbf{A} \\ = \ & tr(\mathbf{W}\mathbf{A}) - tr(\mathbf{A}\mathbf{W}) + \frac{3\lambda}{7} (tr(\mathbf{D}\mathbf{A}) + tr(\mathbf{A}\mathbf{D}) \\ & + \frac{2\lambda}{7} tr\mathbf{D} - \frac{2\lambda}{7} (\mathbf{A} : \mathbf{D}) + D_r d(1 - tr\mathbf{A}) \\ = \ & 2\mathbf{W} : \mathbf{A} + \frac{2\lambda}{7} tr\mathbf{D} + \frac{3\lambda}{7} \mathbf{A} : \mathbf{D} + D_r d(1 - tr\mathbf{A}). \end{aligned} \tag{9}$$

Since $\mathbf{A}$ is symmetric and $\mathbf{W}$ is antisymmetric so $\mathbf{W} : \mathbf{A} = 0$, and $tr\mathbf{D} = div\mathbf{u} = 0$. Hence returning to (9), one gets

$$\frac{d}{dt} tr\mathbf{A} + [\mathbf{u} \cdot \nabla] tr\mathbf{A} = \frac{3\lambda}{7} \mathbf{A} : \mathbf{D} + D_r d(1 - tr\mathbf{A}). \tag{10}$$

If $tr\mathbf{A} = 1$, then from (10) it is clear that $\mathbf{A} : \mathbf{D} = 0$.

Reciprocally, assume that $\mathbf{A} : \mathbf{D} = 0$, (10) implies that

$$\frac{d}{dt} tr\mathbf{A} + [\mathbf{u} \cdot \nabla] tr\mathbf{A} = dD_r(1 - tr\mathbf{A}).$$

From the initial condition on $\mathbf{A}$ (8), we obtain

$$tr\mathbf{A}(0) = tr\mathbf{A}_0 = 1 \ . \tag{11}$$

Thus, we need to solve for $z = tr\mathbf{A}$ that satisfies

$$\begin{aligned} z_t + [\mathbf{u} \cdot \nabla] z + dD_r(z-1) &= 0, && (12) \\ z(0) &= 1. && (13) \end{aligned}$$

One can easily check that $z(t) = tr\mathbf{A}(t) = 1$ solves (12)-(13).

If z, y are two solutions of (12)-(13) with $z(0) = y(0) = 1$, then $w = z - y$ satisfies

$$\begin{aligned} w_t + [\mathbf{u} \cdot \nabla] w + dD_r w &= 0 && (14) \\ w(0) &= 0. && (15) \end{aligned}$$

Applying the Green's formula, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 + d\int_\Omega D_r|w|^2 &= -\int_\Omega (\mathbf{u} \cdot \nabla) w \cdot w \\ &= -\frac{1}{2}\int_\Omega \mathbf{u} \cdot \nabla |w|^2 \\ &= \frac{1}{2}\int_\Omega \mathrm{div}\mathbf{u}|w|^2 + \frac{1}{2}\int_\Gamma \mathbf{u} \cdot \mathbf{n}|w|^2 = 0. \end{aligned}$$

Integration of the above equation gives

$$\|w(t)\|^2 + 2d\int_0^t ds \int_\Omega D_r|w(s)|^2 = \|w(0)\|^2 = 0,$$

thus $w(t) = z(t) - y(t) = 0$. Hence, (12)-(13) has a unique solution.

We then conclude that $z(t) = tr\mathbf{A}(t) = 1$ is the unique solution of (12)-(13).

Finally, we need to show that if $d = 2$, $\mathbf{A} : \mathbf{D} = 0$, and if $\mathbf{A}$ is reversible then $\mathbf{A}^{-1} : \mathbf{D} = 0$.

1. Assume that $d = 2$, we express $\mathbf{A}$ and $\mathbf{D}$ in component form relative to the principal basis of $\mathbf{A}$. Given the definition and properties of $\mathbf{A}$, and $\mathbf{D}$, we have

$$\mathbf{A} : \mathbf{D} = (A_1 + A_2 - 1)\, D_1 = 0, \quad \text{since } A_1 + A_2 = 1. \tag{16}$$

And $\mathbf{A}^{-1} : \mathbf{D} = \frac{D_1}{A_1} - \frac{D_1}{1-A_1} = \frac{1}{A_1(1-A_1)}\,(1 - A_1 - A_1)\, D_1 = 0.$

2. Assume that $d = 3$, once again we express $\mathbf{A}$ and $\mathbf{D}$ in component form relative to the principal basis and we suppose that $A_1 > A_2 > A_3 > 0$.

$$\mathbf{A} : \mathbf{D} = 0 \quad \text{is equivalent to} \quad (A_1 - A_3)\, D_1 + (A_2 - A_3)\, D_2 = 0. \tag{17}$$

Since $A_1 - A_3 > 0, A_2 - A_3 > 0$, thus D_1 and D_2 are of opposite signs. And

$$\begin{aligned} \mathbf{A}^{-1} : \mathbf{D} &= \frac{1}{A_1} D_1 + \frac{1}{A_2} D_2 + \frac{1}{A_3} (-D_1 - D_2) \\ &= -K \left[A_2 D_1 (A_1 - A_3) + A_1 D_2 (A_2 - A_3) \right] \end{aligned} \tag{18}$$

where $K = \frac{1}{A_1 A_2 A_3} > 0$. From (17) we have $D_2 = -\frac{A_1 - A_3}{A_2 - A_3} D_1$, substituting into equation (18) yields

$$\mathbf{A}^{-1} : \mathbf{D} = K D_1 (A_1 - A_3)(A_1 - A_2) \tag{19}$$

If we assume that $D_1 > 0$ and $D_2 < 0$, we have $\mathbf{A}^{-1} \colon \mathbf{D} = K D_1 (A_1 - A_3)(A_1 - A_2) > 0$.

And if $D_1 < 0$ and $D_2 > 0$, from (19), $\mathbf{A}^{-1} \colon \mathbf{D} = K D_1 (A_1 - A_3)(A_1 - A_2) < 0$.

Finally, we have to show that $\mathbf{A}(t)$ is symmetric and positive definite if $\mathbf{A}_0$ is symmetric, and positive definite, which can be done following Lemma 2.1 in [3], or lemma 1 in [6]. □

Remark 4. (1) From lemma 3, and the statement (b) in lemma 1, it is obvious that the quantity $\int_\Omega tr[\mathbf{A} - \ln \mathbf{A} - \mathbf{I}]$ is well defined and always positive if $\mathbf{A}(t = 0)$ is symmetric and positive definite.

(2) The mapping $L^2_{sym}(\Omega)^{d \times d} \cap \{\text{positive semi definite}\} \ni \mathbf{A} \longrightarrow \int_\Omega tr[\mathbf{A} - \ln \mathbf{A} - \mathbf{I}] \in \mathbb{R}_+$ is not a norm.

(3) In the rest of this work, unless specified otherwise, we will assume that $n = 2$, i.e. we will work in two space dimensions.

With help of lemma 3 and Remark 4 (3), the problem (3)-(8) can be rewritten as

$$\rho(\mathbf{u}' + (\mathbf{u} \cdot \nabla)\mathbf{u}) - 2\mu_I \left(1 - \frac{2N_p}{35}\right) \Delta \mathbf{u} + \nabla p = \text{div}\mathbf{T} + \rho \mathbf{b}, \text{ in } \Omega_T; \tag{20}$$

$$\mathbf{T} = \frac{2\mu_I}{7} [N(\mathbf{AD} + \mathbf{DA})]; \tag{21}$$

$$\text{div}\mathbf{u} = 0 \text{ , in } \Omega_T; \tag{22}$$

$$\begin{aligned} &\mathbf{A}' + (\mathbf{u} \cdot \nabla)\mathbf{A} = \mathbf{WA} - \mathbf{AW} + \frac{3\lambda}{7}(\mathbf{AD} + \mathbf{DA}) \\ &\quad + \frac{4\lambda}{7}\mathbf{D} + D_r(\mathbf{I} - 2\mathbf{A}), \text{ in } \Omega_T; \end{aligned} \tag{23}$$

$$\text{tr}\mathbf{A} = 1, \text{ and } \mathbf{A}^T = \mathbf{A}, \quad \text{in } \Omega_T; \tag{24}$$

$$\mathbf{u}|_\Gamma = 0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \mathbf{A}(t = 0) = \mathbf{A}_0, \mathbf{A}_0^T = \mathbf{A}_0, \text{ in } \Omega. \tag{25}$$

3. Continuous dependence of solutions

In this section, a priori estimates for the continuous problem (3)-(8) are derived to prove that the solutions (3)-(8) are bounded for all time $t \in \mathbb{R}_+$. We also studied the dependence of the solution $(\mathbf{u}, \mathbf{A})$ with respect to the initial data $(\mathbf{u}_0, \mathbf{A}_0)$.

3.1 Boundedness of solution in time

We state first an important result which will help us to show the boundedness of the solution.

Lemma 5. *Let f be a non negative, absolutely continuous function satisfying the inequality*

$$f' + kf \leq \alpha\left(f^2 + f^3 + f^4\right) + \beta \tag{26}$$

where $k > 0$, $\alpha > 0$ and $\beta \geq 0$ are some constants. Let $M_0 > 0$, be the unique real solution of $f^5 + f^3 + f^2 + f - \frac{k}{2\alpha} = 0$, and $0 < M < M_0$. If $f(0) \leq M$, and $\beta \leq \frac{kM}{2}$, then $f(t)$ is bounded by M for all $t > 0$.

Proof. See [5,9]. □

Theorem 6. *If $N_p \leq \frac{35}{2}$, then the solution $(\mathbf{u}, \mathbf{A})$ of the problem (20)-(25) remains bounded for all time $t \in \mathbb{R}_+$, provided that the data $\mathbf{b} \in L^\infty(\mathbb{R}_+; L^2(\Omega)^2)$, $(\mathbf{u}_0, \mathbf{A}_0) \in L^2(\Omega)^2 \times L^2_{sym}(\Omega)^{2\times 2}$, D_r, N_p and N_s are sufficiently small and we have*

$$E(t) \leq M, \quad \text{for all } t > 0, \tag{27}$$

where $Y = E(t) = \|\mathbf{u}\|^2 + \|\mathbf{A}\|^2$, and $0 < M < M_0$, and M_0 is the unique real solution of the equation

$$Y^5 + Y^3 + Y^2 + Y - \frac{\min\left\{\frac{1}{4N}\left(1 - \frac{2N_p}{35}\right), dD_r\right\}}{\frac{120\lambda N}{35 - 2N_p}} = 0 \tag{28}$$

Proof. First, we take the L^2 scalar product of the equation of (20) with $\mathbf{u}$, we obtain

$$\begin{aligned}\frac{7}{\mu_I N}\frac{d}{dt}\|\mathbf{u}\|^2 + \frac{7}{N}\left(1 - \frac{2N_p}{35}\right)\|\mathbf{D}\|^2 &= -\left(\mathbf{AD} + \mathbf{DA}, \mathbf{D}\right) + \frac{7\rho}{2\mu_I N}\left(\mathbf{b}, \mathbf{u}\right) \\ &= -2\left(\mathbf{A}, \mathbf{D}_2\right) + \frac{7\rho}{2\mu_I N}\left(\mathbf{b}, \mathbf{u}\right),\end{aligned}$$

Since $(\mathbf{A}, \mathbf{D}_2) \geq 0$, we get

$$\frac{d}{dt}\|\mathbf{u}\|^2 + \frac{2}{N}\left(1 - \frac{2N_p}{35}\right)\|\mathbf{D}\|^2 \leq 2\rho\left(\mathbf{b}, \mathbf{u}\right). \tag{29}$$

Now, we take the L^2 scalar pruduct of the equation of (23) with $\mathbf{A}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{A}\|^2 + 2D_r\|\mathbf{A}\|^2 &= \frac{6\lambda}{7}\left(\mathbf{A}_2 : \mathbf{D}\right) + \frac{4\lambda}{35}\left(\mathbf{A} : \mathbf{D}\right) + D_r\left(\mathbf{A} : \mathbf{I}\right) \\ &= \frac{6\lambda}{7}\left(\mathbf{A}_2 : \mathbf{D}\right) + \frac{4\lambda}{35}\left(\mathbf{A} : \mathbf{D}\right) + D_r\end{aligned} \tag{30}$$

We make use of Cauchy-Schwarz and Young's inequalities and we add (29) and (30), if $N_p \leq \frac{35}{2}$, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2+\|\mathbf{A}\|^2\right)+k\left(\|\mathbf{A}\|^2+\|\mathbf{u}\|^2\right)$$
$$\leq 2\rho\left(\mathbf{b},\mathbf{u}\right)+\frac{6\lambda}{7}\left(\mathbf{A}_2:\mathbf{D}\right)+\frac{4\lambda}{35}\left(\mathbf{A}:\mathbf{D}\right)+D_r$$

$$\frac{d}{dt}\left(\|\mathbf{u}\|^2+\|\mathbf{A}\|^2\right)+k\left(\|\mathbf{u}\|^2+\|\mathbf{A}\|^2\right)$$
$$\leq\left(\frac{60\lambda N}{35-2N_p}+1\right)\|\mathbf{A}\|^4+\frac{35N\|\mathbf{b}\|^2}{35-2N_p}+\left(\frac{16\lambda N}{35-2N_p}\right)^2+D_r \tag{31}$$

where $k=\min\{\frac{7}{4N}(1-\frac{2N_p}{35}),dD_r\}$. We see from (31) that $E(t)=\|\mathbf{u}\|^2+\|\mathbf{A}\|^2$ satisfies (26) with $Y=E(t)=\|\mathbf{u}\|^2+\|\mathbf{A}\|^2$, $k=\min\{\frac{7}{4N}(1-\frac{2N_p}{35}),dD_r\}$, $\alpha=\frac{60\lambda N}{35-2N_p}+1$ and $\beta=\frac{35N\|\mathbf{b}\|^2}{35-2N_p}+(\frac{16\lambda N}{35-2N_p})^2+D_r$. The result follows from (5), if the data are small enough, i.e. if $E(t)\leq M$, and $D_r\leq\frac{M}{2}\min\{\frac{1}{4N}(1-\frac{2N_p}{35}),dD_r\}-\frac{35N\|\mathbf{b}\|^2}{35-2N_p}-(\frac{16\lambda N}{35-2N_p})^2$. □

3.2 Continuous dependence of solutions on initial data $\mathbf{u}_0, \mathbf{A}_0$

In this section, we prove the regular solutions of (20)-(25) are continuous dependent with respect initial data $(\mathbf{u}_{1,0},\mathbf{A}_{1,0})$, $(\mathbf{u}_{2,0},\mathbf{A}_{2,0})$, and the same body force $\mathbf{b}$.

Theorem 7. *For $d=2,3$, let $\mathbf{b}\in L^\infty(\mathbb{R}_+;L^2(\Omega)^d)$. For $i=1,2$, let $(\mathbf{u}^i,\mathbf{A}^i,p^i)$ be the regular solution of (20)-(25), associated with initial conditions $(\mathbf{u}_0^i,\mathbf{A}_0^i)$ respectively, as constructed in [4].*

If D_r is constant, and $N_p\leq 35/2$, then the difference $(\mathbf{u},\mathbf{A})=(\mathbf{u}_2-\mathbf{u}_1,\mathbf{A}_2-\mathbf{A}_1)$, depends continuously in the L^2-norm on the initial data $(\mathbf{u}_{2,0}-\mathbf{u}_{1,0},\mathbf{A}_{2,0}-\mathbf{A}_{1,0})$ in the following sense

$$E\left(t\right)=\|\mathbf{u}(t)\|^2+\|\mathbf{A}(t)\|^2\leq\left(\|\mathbf{u}_0\|^2+\|\mathbf{A}_0\|^2\right)\exp\left(-Kt\right), \tag{32}$$

where $K>0$ is a positive constant independent of the differences $\mathbf{u}_1-\mathbf{u}_2$, $\mathbf{A}_1-\mathbf{A}_2$, but depending on the parameters of the problem, and the domain Ω.

Proof. The difference $(\mathbf{u},\mathbf{A},p)=(\mathbf{u}_2-\mathbf{u}_1,\mathbf{A}_2-\mathbf{A}_1,p_2-p_1)$ satisfies the problem

$$\begin{cases}\mathbf{u}_t+[\mathbf{u}_2\cdot\nabla]\mathbf{u}+[\mathbf{u}\cdot\nabla]\mathbf{u}_1-2\mu_I\left(1-\frac{2N_p}{35}\right)\mathbf{D}+p\mathbf{I}-\operatorname{div}\left(\mathbf{T}_2-\mathbf{T}_1\right)=\mathbf{0},\\ \mathbf{A}_t+[\mathbf{u}_1\cdot\nabla]\mathbf{A}+[\mathbf{u}\cdot\nabla]\mathbf{A}_2=\mathbf{W}\mathbf{A}_1-\mathbf{A}_1\mathbf{W}+\mathbf{W}^2\mathbf{A}-\mathbf{A}\mathbf{W}^2\\ +\frac{4\lambda}{7}\mathbf{D}+\frac{3\lambda}{7}\left(\mathbf{A}\mathbf{D}_2+\mathbf{D}_2\mathbf{A}+\mathbf{A}_1\mathbf{D}+\mathbf{D}\mathbf{A}_1\right)-2D_r\mathbf{A},\\ \mathbf{T}_2-\mathbf{T}_1=\frac{2\mu_I N}{7}\left(\mathbf{A}\mathbf{D}_2+\mathbf{D}_2\mathbf{A}-\mathbf{A}\mathbf{D}_1-\mathbf{D}_1\mathbf{A}\right),\\ \operatorname{div}\mathbf{u}=0,\\ \mathbf{u}\left(\mathbf{x},0\right)=\mathbf{u}_0=\mathbf{u}_{2,0}-\mathbf{u}_{1,0},\quad \mathbf{A}\left(\mathbf{x},0\right)=\mathbf{A}_0=\mathbf{A}_2-\mathbf{A}_{1,0},\ \mathbf{u}|_\Gamma=\mathbf{0}.\end{cases} \tag{33}$$

We take the $\mathbb{L}^2$–inner product of the $\mathbf{u}$–equation and $\mathbf{A}$–equation of (33) with $\mathbf{u}$ and $\mathbf{A}$ respectively, and integrate over Ω, after using Green's formula's, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + 2\mu_I\left(1-\frac{2N_p}{35}\right)\|\mathbf{D}\|^2 \\ &= -\frac{2\mu_I N}{7}\int_\Omega \mathbf{A}:(\mathbf{D}\mathbf{D}^2-\mathbf{D}_1\mathbf{D}) - \int_\Omega [\mathbf{u}\cdot\nabla]\mathbf{u}_1\cdot\mathbf{u}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{A}\|^2 + 2D_r\|\mathbf{A}\|^2 = &-\int_\Omega ((\mathbf{u}\cdot\nabla)\,\mathbf{A}_2:\mathbf{A}) + \int_\Omega ((\mathbf{W}\mathbf{A}_1-\mathbf{A}_1\mathbf{W}):\mathbf{A}) \\ &+\int_\Omega \left((\mathbf{W}^2\mathbf{A}-\mathbf{A}\mathbf{W}^2):\mathbf{A}\right) + \frac{4\lambda}{7}\int_\Omega \mathbf{D}:\mathbf{A} \\ &+\frac{6\lambda}{7}\int_\Omega \mathbf{D}_2:[\mathbf{A}]^2 + \frac{3\lambda}{7}\int_\Omega [\mathbf{A}_1\mathbf{D}+\mathbf{D}\mathbf{A}_1]:\mathbf{A}. \end{aligned} \tag{35}$$

Since

$$\int_\Omega [\mathbf{u}\cdot\nabla]\mathbf{A}_2:\mathbf{A} \le \left|\int_\Omega u_i\frac{\partial A^2_{ij}}{\partial x_k}A_{jk}\right| \le 2\|\nabla\mathbf{A}_2\|\left(\|\mathbf{A}_1\|_{L^\infty}+\|\mathbf{A}_2\|_{L^\infty}\right)\|\mathbf{u}\|, \tag{36}$$

$$\begin{aligned} \int_\Omega [\mathbf{W}\mathbf{A}_1-\mathbf{A}_1\mathbf{W}]:\mathbf{A} &\le \|\mathbf{W}\mathbf{A}_1+\mathbf{A}_1\mathbf{W}\|\|\mathbf{A}\| \\ &\le C_0(\|\mathbf{u}_1\|_2+\|\mathbf{u}_2\|_2)\|\mathbf{A}_1\|_2\|\mathbf{A}\|, \end{aligned} \tag{37}$$

$$\int_\Omega \mathbf{D}:\mathbf{A} \le (\|\mathbf{u}_2\|_2+\|\mathbf{u}_1\|_2)\|\mathbf{A}\|, \tag{38}$$

$$\int_\Omega \mathbf{D}_2:[\mathbf{A}]^2 \le \|\nabla\mathbf{u}_2\|_{L^\infty}\|\mathbf{A}\|^2, \tag{39}$$

$$\begin{aligned} &\int_\Omega [\mathbf{A}_1\mathbf{D}+\mathbf{D}\mathbf{A}_1]:\mathbf{A} \\ &\le \|\mathbf{A}_1\mathbf{D}+\mathbf{D}\mathbf{A}_1\|\|\mathbf{A}\| \le C_0(\|\mathbf{u}_1\|_2+\|\mathbf{u}_2\|_2)\|\mathbf{A}_1\|_2\|\mathbf{A}\|, \end{aligned} \tag{40}$$

$$\int_\Omega \mathbf{A}:(\mathbf{D}\mathbf{D}^2-\mathbf{D}_1\mathbf{D}) \le \|\mathbf{A}\|\|\mathbf{D}\mathbf{D}^2+\mathbf{D}_1\mathbf{D}\| \le C_0\|\mathbf{A}\|(\|\mathbf{u}_1\|_2+\|\mathbf{u}_2\|_2)^2, \tag{41}$$

$$\begin{aligned} \int_\Omega [\mathbf{u}\cdot\nabla]\mathbf{u}_1\cdot\mathbf{u} &\le \left|\int_\Omega u_iu_j\frac{\partial u^1_i}{\partial x_j}dx\right| \le \|\nabla\mathbf{u}_1\|_{L^\infty}\sum_{i,j=1}^d\int_\Omega |u_iu_j|dx \\ &\le 2\|\nabla\mathbf{u}_1\|_{L^\infty}\|\mathbf{u}\|^2. \end{aligned} \tag{42}$$

We use the estimates (36)-(42) in (34) and (35) to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2+\|\mathbf{A}\|^2\right) + \left(2D_r - \frac{6C_0\lambda}{7}\|\nabla\mathbf{u}_2\|_{L^\infty}\right)\|\mathbf{A}\|^2$$

$$+ 2\mu_I \left(1 - \frac{2N_p}{35}\right) \|\mathbf{D}\|^2 - 2\|\nabla \mathbf{u}_1\|_{L^\infty} \|\mathbf{u}\|^2$$

$$\leq \left(\frac{2\mu_I N}{7} C_0 \left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)^2 + \frac{3\lambda C_0}{7} \left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right) \|\mathbf{A}_1\|_2\right) \|\mathbf{A}\|$$

$$+ \left(\frac{4\lambda}{7} \left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)\right) \|\mathbf{A}\| + 2 \|\nabla \mathbf{A}_2\| \left(\|\mathbf{A}_1\|_{L^\infty} + \|\mathbf{A}_2\|_{L^\infty}\right) \|\mathbf{u}\| . \tag{43}$$

Now, applying young inequality and Korn inequalities and assuming $N_p \leq 35/2$, (43) becomes

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \|\mathbf{A}\|^2\right) + \left(2D_r - \frac{6C_0\lambda}{7}\|\nabla \mathbf{u}_2\|_{L^\infty} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2}\right)\|\mathbf{A}\|^2$$

$$+ \left(2\mu_I\left(1 - \frac{2N_p}{35}\right) + 2\|\nabla \mathbf{u}_1\|_{L^\infty} - \frac{\varepsilon_1}{2}\right)\|\mathbf{u}\|^2$$

$$\leq \frac{1}{2\varepsilon_2}\left(\frac{2\mu_I N}{7} C_0 \left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)^2 + \frac{3\lambda C_0}{7}\left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)\|\mathbf{A}_1\|_2\right)^2$$

$$+ \frac{1}{2\varepsilon_3}\left(\frac{4\lambda}{7}\left(\|\mathbf{u}_1\|_2 + \|\mathbf{u}_2\|_2\right)\right)^2 + \frac{1}{\varepsilon_1}\|\nabla \mathbf{A}_2\|^2\left(\|\mathbf{A}_1\|_{L^\infty} + \|\mathbf{A}_2\|_{L^\infty}\right)^2 . \tag{44}$$

So from Agmon inequality [10], one has

$$\|\nabla \mathbf{u}^i(t)\|_{L^\infty} \leq C\|\mathbf{u}^i(t)\|_3 \leq K_0 \quad , \quad \|\mathbf{A}^i(t)\|_{L^\infty} \leq C\|\mathbf{A}^i(t)\|_2 \leq K_0. \tag{45}$$

Next, we take $\epsilon_i > 0$ such that

$$2D_r - \frac{6C_0\lambda}{7}\|\nabla \mathbf{u}_2\|_{L^\infty} - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2} \geq 0,$$

$$2\mu_I\left(1 - \frac{2N_p}{35}\right) + 2\|\nabla \mathbf{u}_1\|_{L^\infty} - \frac{\varepsilon_1}{2} \geq 0, \tag{46}$$

Using (45) and (46), equation (44) becomes

$$\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \|\mathbf{A}\|^2\right) - K_1\left(\|\mathbf{u}\|^2 + \|\mathbf{A}\|^2\right) \leq K_2. \tag{47}$$

Finally, from Gronwall's lemma, (47) implies (32) thus the desired result. □

4. Time discrete scheme solutions

In this section, we present a time discrete scheme for the problem (20)-(25) that replicates the energy laws obtained in Theorem 6, and the continuous L^2–dependence of the solution with respect to the initial condition proven in theorem 7. It is worth to note that not all scheme can replicate these results.

In section 3, it was shown that solving (3)-(8) is equivalent to solve (20)-(25). We would like to know whether this equivalence holds for the discrete in time formulation we introduce next.

4.1 Preliminaries results

For some $k > 0$ and $n \in \mathbb{N}$, we let $t_n = nk$, and $u_k^n \approx u(t_n)$. Suppose that at a step $n \in \mathbb{N}$, $(\mathbf{u}_k^n, \mathbf{A}_k^n)$ is known, where $\mathbf{A}_k^n$ is symmetric, positive semi-definite with $\mathrm{tr}\mathbf{A}_k^n = 1$. Then we seek the triplet $(\mathbf{u}_k^{n+1}, \mathbf{A}_k^{n+1}, p_k^{n+1})$ such that

$$\rho\frac{\mathbf{u}_k^{n+1}-\mathbf{u}_k^n}{k} + [\mathbf{u}_k^{n+1}\cdot\nabla]\mathbf{u}_k^{n+1} - 2\mu_I\left(1-\frac{2N_p}{35}\right)\Delta\mathbf{u}^{n+1} + p_k^{n+1}I +$$

$$= \operatorname{div}\widetilde{\mathbf{T}} + \rho\mathbf{b}_k^{n+1} \tag{48}$$

$$\operatorname{div}\mathbf{u}_k^{n+1} = 0 \text{ in } \Omega_T, \tag{49}$$

$$\widetilde{\mathbf{T}} = \frac{2\mu_I}{7}\left(N\left(\mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1} + \mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1}\right) + N_p\left(\mathbf{A}_k^{n+1} : \mathbf{D}_k^{n+1}\right)\mathbf{I}\right), \tag{50}$$

$$\frac{\mathbf{A}_k^{n+1}-\mathbf{A}_k^n}{k} + (\mathbf{u}_k^{n+1}\cdot\nabla)\mathbf{A}_k^{n+1} = \mathbf{W}_k^{n+1}\mathbf{A}_k^{n+1} - \mathbf{A}_k^{n+1}\mathbf{W}_k^{n+1}$$

$$+\frac{3\lambda}{7}\left(\mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1} + \mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1}\right) + \frac{3\lambda}{7}\mathbf{D}_k^{n+1}$$

$$-\frac{2\lambda}{7}\left(\mathbf{A}_k^{n+1} : \mathbf{D}_k^{n+1}\right)\mathbf{I} + D_r(I - d\mathbf{A}_k^{n+1}), \text{ in } \Omega_T \tag{51}$$

$$[\mathbf{A}_k^{n+1}]^T = \mathbf{A}_k^{n+1}, \text{ and } \mathrm{tr}\mathbf{A}_k^{n+1} = 1 \tag{52}$$

$$\mathbf{u}_k^{n+1}|_\Gamma = 0\ , \quad \mathbf{u}^0 = \mathbf{u}_0, \quad \mathbf{A}^0 = \mathbf{A}_0\ , \text{ in } \Omega; \tag{53}$$

where $\mathbf{b}_k^{n+1} = \frac{1}{k}\int_{nk}^{(n+1)k}\mathbf{b}(t)\mathrm{dt}$.

As for the continuous model, one has

Lemma 8. *Let $(\mathbf{A}_k^{n+1}, \mathbf{u}_k^{n+1})$ be the solution of (48)-(53), with D_r constant. Then*

$$\mathbf{A}_k^{n+1} : \mathbf{D}_k^{n+1} = 0 \ \textit{ for all } n \quad \textit{if and only if} \quad tr\mathbf{A}_k^{n+1} = 1.$$

Moreover for $d = 2$, $[\mathbf{A}_k^{n+1}]^{-1} : \mathbf{D}_k^{n+1} = 0$ if $\mathbf{A}_k^{n+1} : \mathbf{D}_k^{n+1} = 0$.

Proof. First, we take the trace on both sides of the equation (51), we have

$$\frac{\mathrm{tr}\mathbf{A}_k^{n+1} - \mathrm{tr}\mathbf{A}_k^n}{k} + \mathbf{u}_k^{n+1}\cdot\nabla\mathrm{tr}\mathbf{A}_k^{n+1} = \frac{3\lambda}{7}\mathbf{D}^{n+1} : \mathbf{A}_k^{n+1} + D_r d(1 - \mathrm{tr}\mathbf{A}_k^{n+1})$$

Since $\mathrm{tr}\mathbf{A}_k^n = 1$, if $\mathrm{tr}\mathbf{A}_k^{n+1} = 1$, we obtain $\mathbf{D}_k^{n+1} : \mathbf{A}_k^{n+1} = 0$.

Conversely, if $\mathbf{D}_k^{n+1} : \mathbf{A}_k^{n+1} = 0$, and $\mathrm{tr}\mathbf{A}_k^m = 1$ for $m \leq n$, then

$$\frac{\mathrm{tr}\mathbf{A}_k^{n+1} - 1}{k} + \mathbf{u}_k^{n+1}\cdot\nabla\mathrm{tr}\mathbf{A}_k^{n+1} = dD_r(1 - \mathrm{tr}\mathbf{A}_k^{n+1}), \tag{54}$$

$$\mathrm{tr}\mathbf{A}^0 = \mathrm{tr}\mathbf{A}_0 = 1. \tag{55}$$

First, one can easily check that $\mathrm{tr}\mathbf{A}_k^{n+1} = 1$ is a solution of (54)-(55).

Next, with Lax-Milgram Theorem, one can observe that (54) has a unique solution.

Thus (54)-(55) admits a unique solution, that is $\mathrm{tr}\mathbf{A}^{n+1} = 1$.

As in Lemma 3, one can show that if $\mathbf{A}_k^n : \mathbf{D}_k^n = 0$, then $[\mathbf{A}_k^{n+1}]^{-1} : \mathbf{D}_k^{n+1} = 0$ for $d = 2$.

Thus we can reformulate the problem (48)-(53) as follows: suppose $d = 2$, and at step n, $(\mathbf{u}_k^n, \mathbf{A}_k^n)$ is known and $\mathbf{A}_k^n$ is symmetric, positive semi–definite with $\mathrm{tr}\mathbf{A}_k^n = 1$. We are looking for the triplet $(\mathbf{u}^{n+1}, \mathbf{A}^{n+1}, p^{n+1})$ such that

$$\frac{\mathbf{u}_k^{n+1} - \mathbf{u}_k^n}{k} + [\mathbf{u}_k^{n+1} \cdot \nabla]\mathbf{u}_k^{n+1} - 2\mu_I \left(1 - \frac{2N_p}{35}\right) \Delta \mathbf{u}_k^{n+1} + \nabla p_k^{n+1}$$

$$= \mathrm{div}\tilde{\mathbf{T}} + \mathbf{b}_k^{n+1} \tag{56}$$

$$\mathrm{div}\, \mathbf{u}_k^{n+1} = 0 \tag{57}$$

$$\widetilde{\mathbf{T}} = \frac{2\mu_I}{7} N \left(\mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1} + \mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1}\right); \tag{58}$$

$$\frac{\mathbf{A}_k^{n+1} - \mathbf{A}_k^n}{k} + (\mathbf{u}_k^{n+1} \cdot \nabla)\mathbf{A}_k^{n+1} = \mathbf{W}_k^{n+1}\mathbf{A}_k^{n+1} - \mathbf{A}_k^{n+1}\mathbf{W}_k^{n+1}$$

$$+ \frac{3\lambda}{7}\left(\mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1} + \mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1}\right) + \frac{4\lambda}{7}\mathbf{D}_k^{k+1} + D_r(I - d\mathbf{A}_k^{n+1}), \tag{59}$$

$$\left(\mathbf{A}_k^{n+1}\right)^T = \mathbf{A}_k^{n+1}, \quad \text{and} \quad \mathrm{tr}\mathbf{A}_k^{n+1} = 1; \tag{60}$$

$$\mathbf{u}_k^{n+1}|_\Gamma = 0 \text{ , and } \mathbf{u}^0 = \mathbf{u}_0, \quad \mathbf{A}^0 = \mathbf{A}_0 \ . \tag{61}$$

4.2 Discrete in time a priori estimates

The existence of solutions of (56)-(61)(respectively (48)-(53)) follows from [8, see Section 5], the reader can also consult [5]. However, it should be mentioned that the uniqueness of solutions is achieved only for the time step k very small.

The main result can be stated as follows

Theorem 9. *Let* $\mathbf{b} \in L^\infty(\mathbb{R}_+; L^2(\Omega)^2)$, $(\mathbf{u}_0, \mathbf{A}_0) \in L^2(\Omega)^2 \times L^2_{sym}(\Omega)^{2\times 2}$, *with* $tr\mathbf{A}_0 = 1$, *positive semi-definite. If moreover,* $n = 2, D_r$ *is constant, and* $N_p \leq 35/2$. *Then, the sequence of solutions* $(\mathbf{u}_k^n, \mathbf{A}_k^n)$ *of the system (56)-(61) remains bounded for all time in the following sense*

$$E_k^n \leq (1 + k\min(C_2, D_r))^{-n}E^0 + C_4\|\mathbf{b}\|^2_\infty \left[1 - (1 + k\min(C_2, D_r))^{-n}\right]$$

$$+ D_r C_5 \left[1 - (1 + k\min(C_2, D_r))^{-n}\right], \tag{62}$$

where C_2, C_4, C_5 *are positive constant, and*

$$E_k^n = \|\mathbf{u}_k^n\|^2 + \int_\Omega tr[\mathbf{A}_k^n - \ln \mathbf{A}_k^n - \mathbf{I}] \tag{63}$$

is the discrete counterpart of $E(t)$ *defined in (32).*

Proof. The existence of the solution $(\mathbf{u}^{n+1}, \mathbf{A}^{n+1}, p^{n+1})$ to the the problem (56)-(61) can be proven by following steps similar to [7, Theorem 3]. We take the L^2-scalar product of the equation of (56) with $2k\mathbf{u}_k^{n+1}$, and using the relation

$$2(\mathbf{u} - \mathbf{v}, \mathbf{u}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2, \quad \text{for all} \quad \mathbf{v}, \mathbf{u} \in L^2(\Omega) \tag{64}$$

the skew property of the convective term, we obtain

$$\|\mathbf{u}_k^{n+1}\|^2 - \|\mathbf{u}_k^n\|^2 + \|\mathbf{u}_k^{n+1} - \mathbf{u}_k^n\|^2 + 2\mu_I k \left(1 - \frac{2N_p}{35}\right) \|\nabla \mathbf{u}_k^{n+1}\|^2$$
$$+\frac{8\mu_I N}{7} k \underbrace{(\mathbf{A}_k^{n+1}, [\mathbf{D}_k^{n+1}]^2)}_{\geq 0} = 2k(\mathbf{b}_k^{n+1}, \mathbf{u}_k^{n+1}). \tag{65}$$

It follow that

$$\|\mathbf{u}_k^{n+1}\|^2 - \|\mathbf{u}_k^n\|^2 + 2\mu_I k \left(1 - \frac{2N_p}{35}\right) \|\nabla \mathbf{u}_k^{n+1}\|^2 \leq 2k(\mathbf{b}_k^{n+1}, \mathbf{u}_k^{n+1}).$$

Next, we take the L^2-scalar product of the equation of (59) with $k(\mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1})$, gives

$$\left(\mathbf{A}_k^{n+1} - \mathbf{A}_k^n, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1} \right) + k \left([\mathbf{u}_k^{n+1} \cdot \nabla] \mathbf{A}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1} \right)$$
$$= k \left(\mathbf{W}_k^{n+1} \mathbf{A}_k^{n+1} - \mathbf{A}_k^{n+1} \mathbf{W}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1} \right)$$
$$+ \frac{3k\lambda}{7} \left(\mathbf{D}_k^{n+1} \mathbf{A}_k^{n+1} + \mathbf{A}_k^{n+1} \mathbf{D}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1} \right)$$
$$- \frac{4\lambda k}{7} \left(\mathbf{D}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1} \right) + D_r k \left(I - d\mathbf{A}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1} \right). \tag{66}$$

We now need to analyze the terms appearing in (66). First,

$$(\mathbf{A}_k^{n+1} - \mathbf{A}_k^n, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) = \int_\Omega \left[\mathrm{tr} \mathbf{A}_k^{n+1} - \mathrm{tr} \mathbf{A}_k^n \right]$$
$$- \int_\Omega (\mathbf{A}_k^{n+1} - \mathbf{A}_k^n) : [\mathbf{A}_k^{n+1}]^{-1}. \tag{67}$$

The second term in the right hand side of (67) can be treated with Lemma 1, as follows

$$-\left(\mathbf{A}_k^{n+1} - \mathbf{A}_k^n, [\mathbf{A}_k^{n+1}]^{-1} \right) = \int_\Omega \mathrm{tr} \left((\mathbf{A}_k^n - \mathbf{A}_k^{n+1}) [\mathbf{A}_k^{n+1}]^{-1} \right)$$
$$= \int_\Omega \mathrm{tr} \left(\mathbf{A}_k^n [\mathbf{A}_k^{n+1}]^{-1} - \mathbf{I} \right)$$
$$\geq \int_\Omega \ln \det \left(\mathbf{A}_k^n \left[\mathbf{A}_k^{n+1} \right]^{-1} \right)$$
$$= \int_\Omega \mathrm{tr} \ln [\mathbf{A}_k^n [\mathbf{A}_k^{n+1}]^{-1}]$$
$$= \int_\Omega \mathrm{tr} \ln \mathbf{A}_k^n + \mathrm{tr} \ln [\mathbf{A}_k^{n+1}]^{-1}$$
$$= \int_\Omega \left[\mathrm{tr} \ln \mathbf{A}_k^n - \mathrm{tr} \ln \mathbf{A}_k^{n+1} \right]. \tag{68}$$

Thus combining (67) and (68), we obtain

$$(\mathbf{A}_k^{n+1} - \mathbf{A}_k^n, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) \geq \int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1}] - \int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^n].$$

From the divergence theorem and incompressibility condition, we get

$$\begin{aligned}
&\Big([\mathbf{u}_k^{n+1} \cdot \nabla]\mathbf{A}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}\Big) \\
&= \int_\Omega \mathbf{u}_k^{n+1} \cdot \nabla \mathrm{tr} \mathbf{A}_k^{n+1} - \Big([\mathbf{u}_k^{n+1} \cdot \nabla]\mathbf{A}_k^{n+1}, [\mathbf{A}_k^{n+1}]^{-1}\Big) \\
&= -\Big([\mathbf{u}_k^{n+1} \cdot \nabla]\mathbf{A}_k^{n+1}, [\mathbf{A}_k^{n+1}]^{-1}\Big) \\
&= -\int_\Omega \big(\mathbf{u}_k^{n+1} \cdot \nabla\big)\, \mathrm{tr} \ln \mathbf{A}_k^{n+1} \\
&= \int_\Omega \mathrm{div} \mathbf{u}_k^{n+1} \mathrm{tr} \ln \mathbf{A}_k^{n+1} - \int_\Gamma \mathbf{u}_k^{n+1} \cdot \mathbf{n} \mathrm{tr} \ln \mathbf{A}_k^{n+1} \\
&= 0. \qquad (69)
\end{aligned}$$

Next, a direct computation gives

$$(\mathbf{W}_k^{n+1}\mathbf{A}_k^{n+1} - \mathbf{A}_k^{n+1}\mathbf{W}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) = 0,$$

and from Lemma 8, and incompressibility condition

$$(\mathbf{D}_k^{n+1}\mathbf{A}_k^{n+1} + \mathbf{A}_k^{n+1}\mathbf{D}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) = 0, \qquad (70)$$

$$(\mathcal{A}\mathbf{D}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) = 0. \qquad (71)$$

Now, returning to (66) with (67)—(71), and adding and subtracting $\mathbf{I}$, we obtain

$$\begin{aligned}
&\int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1} - \mathbf{I}] - \int_\Omega \mathrm{tr}[\mathbf{A}_k^n - \ln \mathbf{A}_k^n - \mathbf{I}] \\
\leq\ & D_r k(I - 2\mathbf{A}_k^{n+1}, \mathbf{I} - [\mathbf{A}_k^{n+1}]^{-1}) \\
=\ & -D_r k \int_\Omega \mathrm{tr}[[\mathbf{A}_k^{n+1}]^{-1} + 2\mathbf{A}_k^{n+1} - 3\mathbf{I}] \\
=\ & -D_r k \int_\Omega \mathrm{tr}[[\mathbf{A}_k^{n+1}]^{-1} + \mathbf{A}_k^{n+1} - 2\mathbf{I}] - D_r k \int_\Omega \mathrm{tr}\mathbf{A}_k^{n+1} - D_r k \int_\Omega \mathrm{tr}\mathbf{I} \\
=\ & -D_r k \int_\Omega \mathrm{tr}\,\Big([\mathbf{A}_k^{n+1}]^{-1} + \mathbf{A}_k^{n+1} - 2\mathbf{I}\Big) - 3D_r k,
\end{aligned}$$

which with Lemma 1 part (c) gives

$$\begin{aligned}
&\int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1} - \mathbf{I}] + D_r k \int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1} - \mathbf{I}] \\
\leq\ & \int_\Omega \mathrm{tr}[\mathbf{A}_k^n - \ln \mathbf{A}_k^n - \mathbf{I}] - 3D_r k. \qquad (72)
\end{aligned}$$

Putting together (72) and (63), we obtain

$$E_k^{n+1} + 2\mu_I k \left(1 - \frac{2N_p}{35}\right) \|\nabla \mathbf{u}_k^{n+1}\|^2 + D_r k \int_\Omega \mathrm{tr}[\mathbf{A}_k^{n+1} - \ln \mathbf{A}_k^{n+1} - \mathbf{I}]$$
$$\le E_k^n + 2k(\mathbf{f}_k^{n+1}, \mathbf{u}_k^{n+1}) - 3D_r k, \tag{73}$$

which with Cauchy-Schwarz inequality, and assuming that $N_p \le 35/2$ yields

$$E_k^{n+1} \le \frac{1}{\alpha} E_k^n + \frac{kC_3}{\alpha} \|\mathbf{f}_k^{n+1}\|^2 + \frac{3D_r k}{\alpha}, \quad \text{with} \quad \alpha = 1 + k \min(C_2, D_r). \tag{74}$$

Using (74) recursively, we find

$$\begin{aligned} E_k^n &\le \frac{1}{\alpha^n} E^0 + kC_3 \sum_{i=1}^n \frac{1}{\alpha^i} \|\mathbf{f}_k^{n+1-i}\|^2 + 3D_r k \sum_{i=0}^n \frac{1}{\alpha^i} \\ &\le (1 + k\min(C_2, D_r))^{-n} E^0 + C_4 \|\mathbf{f}\|_\infty^2 \left[1 - (1 + k\min(C_2, D_r))^{-n}\right] \\ &\quad + D_r C_5 \left[1 - (1 + k\min(C_2, D_r))^{-n}\right], \end{aligned} \tag{75}$$

which is the desired result. □

Remark 10. (a) Theorem 9 is the discrete counterpart of theorem 6, and no additional conditions are imposed on the data.

(b) From the estimate obtained in theorem 9,

$$\limsup_{n\to\infty} E_k^n \le C_4 \|\mathbf{f}\|_\infty^2 + D_r C_5.$$

Corollary 11. *Under the assumptions of theorem 9, and* $0 < k < \frac{1}{\min(C_2, D_r)}$, *one one obtains*

$$E_k^n \le \exp\left(-\frac{n}{2}\, k \min(C_2, D_r)\right) E^0 + C_4 \|\mathbf{f}\|_\infty^2 + D_r C_5. \tag{76}$$

The inequality (76) is deduce from (75) by observing $\exp(x/2) \le 1 + x$ for $0 < x < 1$.

4.3 Continuous dependence

Let $(\mathbf{u}_k^{n+1}, \mathbf{A}_k^{n+1})$ be the regular solution of (56)-(61), with external force $\mathbf{f}_k^{n+1}$, and initial condition $(\mathbf{u}_0, \mathbf{A}_0)$.

Let $(\mathbf{v}_k^{n+1}, \mathbf{B}_k^{n+1})$ be the regular solution of (56)-(61), with external force $\mathbf{f}_k^{n+1}$, and initial condition $(\mathbf{v}_0, \mathbf{B}_0)$.

Theorem 12. *For* $d = 2, 3$. *If* $N_P \le 35/2$ *and* D_r *constant. Then one can find a positive constant* $K > 1$ *such that for all* $n \ge 0$

$$\|\mathbf{u}_k^{n+1} - \mathbf{v}_k^{n+1}\|^2 + \|\mathbf{A}_k^{n+1} - \mathbf{B}_k^{n+1}\|^2 \le K\left[\|\mathbf{u}_k^n - \mathbf{v}_k^n\|^2 + \|\mathbf{A}_k^n - \mathbf{B}_k^n\|^2\right]. \tag{77}$$

For the proof of theorem 12, it suffice to:

(a) write the equations satisfied by the difference $(\mathbf{u}_k^{n+1} - \mathbf{v}_k^{n+1}, \mathbf{A}_k^{n+1} - \mathbf{B}_k^{n+1})$.

(b) proceed as in the proof of theorem 7.

Remark 13. Using (77) recursively, we find

$$\|\mathbf{u}_k^n - \mathbf{v}_k^n\|^2 + \|\mathbf{A}_k^n - \mathbf{B}_k^n\|^2 \leq K^n \left[\|\mathbf{u}_0 - \mathbf{v}_0\|^2 + \|\mathbf{A}_0 - \mathbf{B}_0\|^2\right].$$

5. Instability of the rest state

We assume that $\mathbf{f} = \mathbf{0}$. In this section, we want to study the instability in the sense of Lyapounov of the rest state for the time discrete problem (56)-(53). We recall that in [4], the rest state $(\mathbf{0}, \overline{\mathbf{A}})$ has been shown to be unstable for the problem (20)-(25) provided that $N_p \geq 35/2$ and $\overline{\mathbf{A}} = e_1 \otimes e_1$, where e_i is the usual orthonormal basis. We want to show that the problem (56)-(61) inherited the later property.

Theorem 14. *Let $\mathbf{f} = \mathbf{0}, D_r = 0$, $\mathbf{u}_0 \in L^2(\Omega)^d$. If $N_p > 35/2$, with the time step k sufficiently small, then the rest state $\overline{\mathbf{A}} = \mathbf{e}_1 \otimes \mathbf{e}_1$ is unstable.*

Proof. We let $\mathbf{u} = u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, and take $\mathbf{A} = \mathbf{a} + \overline{\mathbf{A}}$, where $\mathbf{a}^T = \mathbf{a}$, $\mathrm{tr}\mathbf{a} = 0$, and satisfy (48)-(53). It is easy to see that $\overline{\mathbf{A}}\mathbf{D} = 0$.

Taking the scalar product of the system (56) with $2k\mathbf{u}_k^{n+1}$ and using (64), we obtain

$$\begin{aligned}
&\|\mathbf{u}_k^{n+1}\|^2 - \|\mathbf{u}_k^n\|^2 + \|\mathbf{u}_k^{n+1} - \mathbf{u}_k^n\|^2 + 4\mu_I k\|\mathbf{D}_k^{n+1}\|^2 + 4\mu_I k N_p(\mathcal{A}\mathbf{D}_k^{n+1}, \mathbf{D}_k^{n+1}) \\
&+8\mu_I N_s k(\mathbf{a}^{n+1}, [\mathbf{D}_k^{n+1}]^2) = \mathbf{0} \\
&\mathcal{A}\mathbf{D}_k^{n+1} = -\frac{2}{35}\mathbf{D}_k^{n+1} + \frac{2}{7}\left[\mathbf{a}^{n+1}\mathbf{D}_k^{n+1} + \mathbf{D}_k^{n+1}\mathbf{a}^{n+1}\right].
\end{aligned}$$

together with (56) gives

$$\begin{aligned}
&\|\mathbf{u}_k^{n+1}\|^2 + \|\mathbf{u}_k^{n+1} - \mathbf{u}_k^n\|^2 + 2\mu_I k\left(1 - \frac{2N_p}{35}\right)\|\nabla\mathbf{u}_k^{n+1}\|^2 \\
= \quad &-8\mu_I k\left(\frac{N_p}{7} + N_s\right)(\mathbf{a}^{n+1}, [\mathbf{D}_k^{n+1}]^2) + \|\mathbf{u}_k^n\|^2.
\end{aligned}$$

For each entry, $a_{i,j}^n$ we let

$$a_{i,j}^n = \frac{1}{k}\int_{(n-1)k}^{nk} a_{i,j}(t)dt.$$

Thus from Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}
&\|\mathbf{u}_k^{n+1}\|^2 + \|\mathbf{u}_k^{n+1} - \mathbf{u}_k^n\|^2 + 2\mu_I k\left(1 - \frac{2N_p}{35}\right)\|\nabla\mathbf{u}_k^{n+1}\|^2 \\
&+4\mu_I k\left(\frac{N_p}{7} + N_s\right)\|\mathbf{a}^{n+1}\|_\infty\|\nabla\mathbf{u}_k^{n+1}\|^2 \geq \|\mathbf{u}_k^n\|^2
\end{aligned}$$

which is re-written as

$$\|\mathbf{u}_k^{n+1}\|^2 + \|\mathbf{u}_k^{n+1} - \mathbf{u}_k^n\|^2 + 2\mu_I k\left[\left(1 - \frac{2N_p}{35}\right) + 2\left(\frac{N_p}{7} + N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right]\|\nabla \mathbf{u}_k^{n+1}\|^2 \geq \|\mathbf{u}_k^n\|^2. \tag{78}$$

Clearly, we want $(1 - \frac{2N_p}{35}) + 2(\frac{N_p}{7} + N_s)\|\mathbf{a}^{n+1}\|_\infty \leq 0$, which possible if we take **a** such that

$$2\left(\frac{N_p}{7} + N_s\right)\|\mathbf{a}^{n+1}\|_\infty \leq -\left(1 - \frac{2N_p}{35}\right)$$

because $\|\mathbf{a}\|_\infty$ is always positive, we should take $1 - \frac{2N_p}{35} < 0$, that is $N_p > 35/2$. With the above restrictions, and dropping some positive terms, one obtains

$$\left[1 + 2\mu_I k\left[\left(1 - \frac{2N_p}{35}\right) + 2\left(\frac{N_p}{7} + N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right] C_p\right]\|\mathbf{u}_k^{n+1}\|^2 \geq \|\mathbf{u}_k^n\|^2. \tag{79}$$

We now take the step size k such[1] that

$$1 + 2\mu_I k\left[\left(1 - \frac{2N_p}{35}\right) + 2\left(\frac{N_p}{7} + N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right] C_p \geq 0.$$

Combining (78) and (79), one gets

$$\|\mathbf{u}_k^{n+1}\|^2 \geq \left[1 + 2\mu_I k\left[\left(1 - \frac{2N_p}{35}\right) + 2\left(\frac{N_p}{7} + N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right] C_p\right]^{-1}\|\mathbf{u}_k^n\|^2. \tag{80}$$

Using (80) recursively, one obtains

$$\|\mathbf{u}_k^n\|^2 \geq \left[1 + 2\mu_I k\left[\left(1 - \frac{2N_p}{35}\right) + 2\left(\frac{N_p}{7} + N_s\right)\|\mathbf{a}^{n+1}\|_\infty\right] C_p\right]^{-n}\|\mathbf{u}^0\|^2, n = 1, 2, 3... \tag{81}$$

From (81), one sees that $[1 + 2\mu_I k[(1 - \frac{2N_p}{35}) + 2(\frac{N_p}{7} + N_s)\|\mathbf{a}^{n+1}\|_\infty]C_p] < 1$, therefore (81) implies that

$$\limsup_{n \longrightarrow \infty}\|\mathbf{u}_k^n\|^2 = \infty,$$

which ends the proof. □

1. this choice is not restrictive, since for the existence and uniqueness, we need k to be small enough

6. Concluding remarks

This study generalises the results obtained in [4,8,9], by proving instability results for the rest state not only for the velocity but also for the orientation tensor and assuming no zero body forces. It was proved that both the regular solutions and a time discrete solution dependent continuously on the initial data, furthermore, restrictions inposed on the data confirm the results found elsewhere: the rest state is unstable if particle number $N_p > 35/2$. It is important to mention that not all schemes can replicate the results obtained here For $N_p > 35/2$, the system is not mechanically acceptable, thus it is not surprising that instability results are obtained for N_p in that range. The instability of the rest state is shown irrespective of dimension which was not the case for the stability for the moment.

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