Some classes of pure sub-acts over semi-groups

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Abstract. In this paper the concepts of approximately pure subacts , approximately pure intersection subacts, and pure sub-acts relative to sub-acts have been studied. Some properties and some characterization of these notions are established and study approximately pure sub-acts with intersection property and pure sub-acts relative to sub-acts with intersection property.

Keywords: pure sub-acts, approximately pure sub-acts, approximately pure intersection sub-acts, pure sub-acts relative to sub-acts.

1. Introduction

"Let S be a monoid. A right S-act M_S is a nonempty set M together with a map (written multiplicatively) from $M \times S$ into M satisfying $m.1 = m$ and $m(st) = (ms)t$, for all $m \in M$ and $s, t \in S$. A nonempty subset N of an S-act M_S is S-subact if $ns \in N$ for all $s \in S$ and $n \in N$. We say that M_S is a cyclic S-act if $M_S = uS$ for some $u \in M_S$. An element $z \in M_S$ is called a fixed element of M_S if $zs = z$ for all $s \in S$. The set of all fixed elements of M_S will be denoted by $F(M)$. If M_S has a unique fixed element z, then z is called zero element of M_S. We will denote the zero element of M_S by O. Every S-act M_S can be extended to an S-act with fixed element z by taking the disjoint union: $M_S \cup \{za\}s$. A nonempty subset $K \subseteq S$ is called left ideal of a monoid S if $SK \subseteq K$; a right ideal of S if $KS \subseteq K$; an ideal of S if $KS \subseteq K$ and $SK \subseteq K$. Recall that, for two S-acts A_S, B_S a mapping $\theta : A_S \longrightarrow B_S$ is called a homomorphism of S-acts or just an S-homomorphism if $\theta(as) = \theta(a)s$ for all $a \in A_S, s \in S$. The set of all S-homomorphisms from A_S into B_S will be denoted by $Hom(A_S, B_S)$ or sometimes by $Hom_S(A, B)$. Note that if $\theta : A_S \longrightarrow B_S$ is an S-homomorphism then $Im\theta = \theta(A_S)$ is a subact of B_S, and the S-homomorphism $f : M_S \longrightarrow M_S$ is called an endomorphism of M_S. The set $Hom_S(M, M)$ which forms a monoid under composition of mappings is denoted by $End_S(M)$ and is called the endomorphism monoid of M_S. An equivalence relation ρ on an S-act M_S is called an S-act congruence or a congruence on M_S, if $(m, n) \in \rho$ implies $(ms, ns) \in \rho$ for $m, n \in M_S, s \in S$" [1]. "For a congruence ρ on an S-act M, the quotient set $M/\rho = \{[m] : m \in M\}$ becomes an S-act by defining $[m]s = [ms]$. Given an S-act A and a subact B of A, we define the

Rees congruence ρ_B on A by $a\rho_B b$ if and only if $a = b$ or $a, b \in B forall a, b \in A$. We denote the quotient act A/ρ_B by A/B and call it the Rees quotient of A by B. We usually identify the ρ_B -class $\{a\} \in A/B$ with a for each $a \in A/B$, and denote the ρ-class $B \in A/B$ by 0"[2]. Recall that "let S be a semigroup An S-subsystem N of a right S-system (S-act) M_S is called R-pure (pure) in S if $N \cap Ma = Na$ for all $a \in S$ or N is R-pure (pure) if and only if $N \cap MI = NI$ for every ideal I of S"[3]. Recall "that a subact B of a right S-act A is called essential (or large) if for every right S-act C, every homomorphism $f : A \longrightarrow C$ with $f|_B$ a monomorphism, is itself a monomorphism. Also, a subact B of a right S-act A is called intersection large if $B \cap C \neq 0$, for each subact C of A. If S contains a zero, then B is intersection large in A if $B \cap C \neq 0_A$, for each non-zero subact C of A. It is easily checked that B is intersection large in A if and only if for each $a \in A$ (resp.$a \neq 0_A$) there exists $s \in S$ (resp. $s \neq 0_S$) such that $as \in B$ (resp. $0_A \neq as \in B$). Also, it is easy to see that every large subact B of a right S-act A with a zero 0 is intersection large"[4].

2. Approximately pure subacts and properties of S-acts which has approximately-pure intersection property

In this section the concept of approximately- pure subacts have been introduced.

Definition 2.1. Let A be S-act . A subact N of A is said to be approximately -pure if for each ideal I of S,$N \cap IA = IN \cup J(S)A \cap (N \cap IA)) = (IN \cup J(S)A) \cap (N \cap IA)$. where $J(S)$ is the Jacobson radical of S. It is clear that each pure subact is approximately pure subact.

Remark 2.2. 1. Let A be an S-act and N be approximately - pure subact of A. If H is approximately -pure subact of N, then H is approximately - pure submodule of A, where $J(S)$ is the Jacobson radical of S.

2. Let A be an S-act and N be approximately -pure subact of A. if K is a subact of A containing N, then N is a approximately -pure subact of K

Proof. 1. Let I be an ideal of S, since N is approximately - pure in A and H is approximately -pure in N, then $N \cap IA = (IN \cup J(S)A) \cap (N \cap IA)$ and $H \cap IN = (IN \cup J(S)A) \cap (N \cap IA)$ but H subact of N, therefore $H \cap IA \subseteq N \cap IA = (IN \cup J(S)A) \cap (N \cap IA)$ and hence $H \cap IA \subseteq H \cap [(IN \cap J(S)A) \cup (N \cap IA)] = [H \cap (IN \cup J(S)A)] \cap (N \cap IA) = H \cap [(IN \cup J(S)A) \cap (N \cap IA)] = [(H \cap IN) \cap (H \cup J(S)A)] \cap (N \cap IA) = [(H \cap IN) \cap (N \cap IA)] \cup [(H \cap J(S)A) \cap (N \cap IA)] = [(H \cap (IN \cap N) \cap IA)] \cup [(H \cap J(S)A) \cap (N \cap IA)] = [H \cap (IN \cap IA)] \cup [J(S)A \cap (H \cap IA)] = (H \cap IN) \cup [J(S)A \cap (H \cap IA)] = (H \cap IN) \cup [(H \cap IA) \cap J(S)A] = [(H \cap IN) \cup (H \cap IA)] \cap [(H \cap IN) \cup J(S)A] \subseteq [(H \cap IN) \cup (H \cap IA)] = [(IH \cup J(S)A) \cap (H \cap IN)] \cup (H \cap IA) = [[(IH \cup F) \cap (H \cap IN)] \cup H] \cap [[(IH \cup J(S)A) \cap (H \cap IN)] \cup IH]$ $\subseteq [(IH \cup J(S)A) \cup IH] \cap [(H \cap IN) \cup IH] = (IH \cup J(S)A) \cap [(H \cap IN) \cup IH] = [(IH \cup J(S)A) \cap (H \cap IN)] \cup [(IH \cup J(S)A) \cap IH] = (IH \cup J(S)A) \cap [(H \cap IN) \cup IH]$ $\subseteq (IH \cap J(S)A) \cap [(H \cup IH) \cap (IN \cup IH)] = (IH \cup J(S)A) \cap (H \cap IN) \subseteq$

$[(IH \cup J(S)A) \cap (H \cap IA)]$. Since $(IH \cup J(S)A) \cap (H \cap IA) \subseteq H \cap IA$, then $H \cap IA = (IH \cup J(S)A) \cap (H \cap IA)$.

2. Let I be an ideal of S, since N is approximately pure in A, then $N \cap IA = (IN \cup J(S)A) \cap (N \cap IA)$. But $K \subseteq A$, therefore, $N \cap IK \subseteq N \cap IA = (IN \cup J(S)A) \cap (N \cap IA)$, and hence $N \cap IK \subseteq [(IN \cup J(S)A) \cap (N \cap IA)] \cap IK = (IN \cup J(S)A) \cap [(N \cap IA) \cap IK]$. Since $(IN \cup J(S)A) \cap (N \cap IK) \subseteq N \cap IK$, then $N \cap IK = (IN \cup J(S)A) \cap (N \cap IK)$. □

Definition 2.3. S-act A is said to have the approximately -pure intersection property if the intersection of any two approximately - pure subacts is again approximately -pure.

Proposition 2.4. *1. If S-act A has the approximately - PIP, then every approximately - pure subact of A has the approximately - PIP.*

2. Let N be approximately - pure subact of S-act A . A has approximately - PIP if and only if A/N has approximately -PIP.

Proof. 1. Clear

2. Let L/N, K/N be two approximately pure subacts of A/N and let $J(S)$ be an ideal in S. we have to show that $(L/N \cap K/N) \cap F(A/N) = [F(L/N \cap K/N) \cup J(S)A/N] \cap [(L/N \cap K/N) \cap F(A/N)]$, where F is an ideal in S. We claim that each of L and K is approximately pure in A. To show this, let I be an ideal in S and let $x \in L \cap IA$. Since L/N is approximately pure in A/N, then $L/N \cap I(A/N) = [I(L/N) \cup J(S)A/N] \cap [(L/N \cap I(A/N)]$, thus $L/N \cap (IA \cup N)/N = [(IL \cup N)/N \cup J(S)A/N] \cap [(L/N \cap (IA \cup N)/N]$, and this implies that $[L \cap (IA \cup N)]/N = [(IL \cup N)/N \cup J(S)A/N] \cap [(L/N \cap (IA \cup N)/N] = [[(IL \cup N) \cup (J(S)A \cup N)] \cap [(L \cap (IA \cup N)]]/N$, therefore $L \cap (IA \cup N) = [(IL \cup J(S)A) \cup N] \cap [(L \cap IA) \cup (L \cap N)] = [(IL \cup J(S)A) \cup N] \cap [(L \cap IA) \cup N] = [(IL \cup J(S)A) \cap [(L \cap IA)] \cup N$, and hence $(L \cap IA) \cup N = [(IL \cup J(S)A) \cap [(L \cap IA) \cup N]$. Since $x \in L \cap IA \subseteq (L \cap IA) \cup N$, then $x \in [(IL \cup J(S)A) \cap [(L \cap IA) \cup N]$, thus $x \in (IL \cup J(S)A) \cap (L \cap IA)$ or $x \in N$, if $x \in (IL \cup J(S)A) \cap (L \cap IA)$, then $L \cap IA = (IL \cup J(S)A) \cap (L \cap IA)$, therefore L is approximately pure, if $x \in N$ thus $x \in (L \cap IA) \cup N = [(IL \cup J(S)A) \cap [(L \cap IA) \cup N]$, hence L is approximately-pure in A. Since A has the approximately -PIP, then $L \cap K$ is approximately - pure in A. Thus $(L \cap K) \cap FA = [F(L \cap K) \cup J(S)A] \cap [(L \cap K) \cap FA]$ where $F$is an ideal in S. Now, let Rees congruence $\rho_N \in [L/N \cap K/N] \cap J(S)(A/N)$, then Rees congruence $\rho_N \in J(S)(A/N)$ (Rees congruence on $J(S)A$) and $\rho_N \in L/N$ (Rees congruence on L) and $\rho_N \in K/N$ (Rees congruence on K), thus Rees congruence ρ_N on L and K therefore $\rho_N \in L \cap K$but $(L \cap K) \cap FA = [F(L \cap K) \cup J(S)A] \cap [(L \cap K) \cap FA]$ therefore $(L/N \cap K/N) \cap F(A/N) = [F(L/N \cap K/N) \cup J(S)A/N] \cap [(L/N \cap K/N) \cap F(A/N)]$. ($\Longleftarrow$) Conversely let$M$ and N be F- pure subacts of A, let L be a subact of M and L be a subact of N then M/L and N/L is approximately-pure subacts of A/L. Since A/L has approximately -PIP, then $M/L \cap N/L = (M \cap N)/L$ is approximately - pure subact of A/L. Therefore $M \cap N$ is approximately - pure subact of A. □

Theorem 2.5. *Let A be S-act, then A has the approximately -PIP if and only if* $[(IM\cap IN)\cup J(S)A]\cap[(M\cap N)\cap IA]=[I(M\cap N)\cup J(S)A]\cap[(M\cap N)\cap IA]$ *for every ideal I of S and for every approximately - pure subacts M and N of A.*

Proof. Suppose A has the approximately -PIP then for each approximately -pure subacts M and N, $M\cap N$ is approximately -pure. Let I be an ideal in S, then $(M\cap N)\cap IA=[I(M\cap N)\cup J(S)A]\cap[(M\cap N)\cap IA]$. It is clear that $[I(M\cap N)\cup J(S)A]\cap[(MN)\cap IA]\subseteq([(IM\cap IN)\cup J(S)A]\cap[(M\cap N)\cap IA]$. But $[(IM\cap IN)\cup J(S)A]\cap[(M\cap N)\cap IA]\subseteq M\cap(N\cap IA)=(M\cap N)\cap IA=[I(M\cap N)\cup J(S)A]\cap[(M\cap N)\cap IA]$. Thus $[(IM\cap IN)\cup J(S)A]\cap[(M\cap N)\cap IA]=[I(M\cap N)\cup J(S)A]\cap[(M\cap N)\cap IA]$ Conversely, let L and B be F-pure subact of A and J an ideal in S. then $(L\cap B)\cap JA=L\cap(B\cap JA)=L\cap[(JB\cup F)\cap(B\cap JA)]$. Similarly $(L\cap B)\cap JA=B\cap(L\cap JA)=B\cap(JA\cup F)\cap(B\cap JA)$. But L, B are F- pure in A. Thus $(L\cap B)\cap JA\subseteq[(JL\cap JB)\cup J(S)A]\cap[(L\cap B)\cap JA]=[J(L\cap B)\cup J(S)A]\cap[(L\cap B)\cap JA]$. □

3. Approximately pure intersection large subacts

In this section the concept of approximately- pure intersection large subacts have been introduced.

Definition 3.1. A subact K of an S-act A is called approximately pure intersection large in A, if for every approximately pure subact L of A with $K\cap L\subseteq J(S)A$ implies $L\subseteq J(S)A$. A is called approximately pure intersection large of K.

Theorem 3.2. *Let K, N be subacts of A such that K is subact of N then:*

1. *If K is approximately pure intersection large in A, then K is approximately pure intersection large in N and N is approximately pure intersection large in A.*

2. *If N is approximately pure intersection large in A and $J(S)A\cap N=J(S)N$ and K is approximately pure intersection large in A then K is approximately pure intersection large in A.*

3. *If A has approximately pure intersection large finite intersection property and if N is approximately pure intersection large in A and $J(S)A\cap N=J(S)N$ then K is approximately pure intersection large in A if and only if K is approximately pure intersection large in N and N is approximately pure intersection large in M.*

Proof. 1. We have to show that N is approximately pure intersection large in A. Let L be approximately pure subact of A with $N\cap L\subseteq J(S)A$, since K is subact of N, then $K\cap L\subseteq N\cap L$ and $N\cap L\subseteq J(S)A$ thus $K\cap L\subseteq J(S)A$.

Since K is approximately pure intersection large in A, then $L \subseteq J(S)A$. Hence N is approximately pure intersection large in A.

2. Let L be an approximately pure of N with $K \cap L$ is subact of $J(S)N$, thus $K \cap L \subseteq J(S)A$, since N is subact of A and K is approximately pure intersection large in A therefore $L \subseteq J(S)A$, therefore $L = L \cap N \subseteq J(S)A \cap N$, thus $L \subseteq J(S)N$, hence K is approximately pure intersection large in N. Now we have to show that N is approximately pure intersection large in A. Let L be an approximately pure subact of A with $N \cap L \subseteq J(S)A$, thus $K \cap L \subseteq N \cap L \subseteq J(S)A$ therefore $L \subseteq J(S)A$. Hence N is approximately pure intersection large in A.

3. Suppose that K is approximately pure intersection large in N and N is approximately pure intersection large in A, we have to show that K is approximately pure intersection large in A. Let L be approximately pure subact of A with $K \cap L \subseteq J(S)A$. By assumption $N \cap L$ is approximately pure in A, thus $N \cap L$ is approximately pure in N by Remark 2.2. Since N is approximately pure intersection large in A, thus $N \cap L \subseteq J(S)N \subseteq J(S)A$, hence $L \subseteq J(S)A$. Hence K is approximately pure intersection large in A. Conversely It is clear. □

Corollary 3.3. *Let A be an S-act that has the pure finite intersection property. If H is approximately pure intersection large of N and $J(S)N \cap H = J(S)H$. Then $K \cap H$ is approximately pure intersection large in A if and only if H is approximately pure intersection large in A and K is approximately pure intersection large in A, for any subact K of A.*

Proof. $\Longrightarrow$ the proof follows by Theorem 3.2.

$\Longleftarrow$ Let L be an approximately pure subact of A with $(K \cap H) \cap L \subseteq J(S)A$, by assumption $H \cap K$ is approximately pure in A. Since K is approximately pure intersection large in A, then $H \cap L \subseteq J(S)A$. So since H is approximately pure intersection large in A, hence $L \subseteq J(S)A$, therefore $K \cap H$ is approximately pure intersection large in A. □

Recall that an S-act M_s is called decomposable if there exist two subacts A , B of M_s such that $M_s = A \cup B and A \cap B = 0$[5]. We introduce the concept of approximately pure -direct summed subact.

Definition 3.4. A subact N of S-act A is approximately pure -direct summed if there exists approximately pure subact L of A such that $A = N \cup L$ and $N \cap L = 0$.

Remark 3.5.

1. Every approximately pure-direct summed of an S-act A is approximately pure
2. Let A be an S-act. If P is approximately pure subact of A and Q is any subact of A then $P \cap Q$ is approximately pure in Q.

Definition 3.6. A subact N of an S-act A is called approximately pure closed subact of A if A does not contain a proper approximately pure intersection large extension of N.

Proposition 3.7. *Let $J(S)A \cap K = J(S)K$ for every approximately pure subact K of A, then every approximately pure -direct summed is approximately pure -closed.*

Proof. Let $A = N \oplus L$, we want to show that A is approximately pure -closed. Suppose that N is approximately pure intersection large in K we have to show that $N = K$. L is approximately pure in A and K is any subact of A. then $L \cap K$ is approximately pure in K by Remark3.5 since $N \cap L \subseteq J(S)A$. Thus $N \cap (L \cap K) \subseteq J(S)K \subseteq J(S)A$ since N is approximately pure intersection large in A therefore $L \cap K \subseteq J(S)A$. Then $K = N$, hence N is approximately pure -closed subact of A. □

Theorem 3.8. *Let M, L and N be subacts of S-act A then there exists an approximately -pure closed subact H in A which is approximately pure and $J(S)A \cap H = J(S)H$ such that N is approximately pure closed in H.*

Proof. Let $V = \{K : K$ is an approximately-pure subact of H and $J(S)A \cap K = J(S)K$ such that N is approximately pure intersection large in $K\}$. It is clear that V is S-act .By Zorn's lemma, V has a maximal element say H. To show that H is approximately pure -closed in A. Let D is a subact of A such that H is approximately pure intersection large in D. Since N is approximately pure intersection large in H and $H$is approximately pure intersection large in D. Then by Theorem3.2, N is approximately pure intersection large in D, thus $H = D$. □

Definition 3.9. Let N and K be subacts of S-act A with K is pure in A, K is called pure relative complements of N in A if K is maximal with the property $K \cap N = 0$.

Definition 3.10. Let N and K be subacts of S-act A with K is approximately pure in A, K is called relative approximately pure complement of N in A if K is maximal with $K \cap N \subseteq J(S)A$.

Proposition 3.11. *Every subact of A has a relative approximately pure complement in A.*

Proof. Let N be a given subact of A and consider the set $B = \{K \cap A, K$ is approximately pure in A with $N \cap KJ(S)A\}$. It is clear that and every chain of B has an upper bound. By Zorn's Lemma B has maximal element, which means that N has relative approximately pure complement in A. □

The following propositions give the relation between approximately pure closed subact and relative approximately pure complement subact.

Proposition 3.12. *Let N be a subact of an S-act A and $J(S)A \cap F = J(R)F$ for every approximately pure subact F of A. If N is relative approximately pure complement for some subact K of A, then N is approximately pure closed subact in A.*

Proof. Let L be an approximately pure intersection large subact of A with N is approximately pure intersection large in L. We have $N \cap K \subseteq J(S)A$, $(N \cap K) \cap L \subseteq J(S)A \cap L$. since L is approximately pure in A, then $K \cap L$ is approximately pure in L -by Remark 3.5, thus $N \cap (K \cap L) \subseteq J(S) \subseteq J(S)A$, hence $L = N$, hence N is approximately pure closed in M. □

4. Pure subacts relative to subact

Definition 4.1. Let A be S-act and F be a subact of A. A subact N of A is said to be F-pure if for each ideal I of S, $N \cap IA = IN \cup (F \cap (N \cap IA)) = (IN \cup F) \cap (N \cap IA)$.

Remark 4.2.
1. Let A be an S-act and N be F- pure subact of A. If H is F-pure subact of N, then H is F- pure submodule of A, where F is subact of A.
2. Let A be an S-act and N be F-pure subact of A. if K is a subact of A containing N, then N is a F-pure subact of K, where F is subact of A.
3. Let A be an S-act and N be F -pure subact of M. If H is a subact of N and H is subact of F, then N/H is F/H -pure subact of A/H , where F is subact of A.
4. Let A be an S-act. Let N and H be subacts of A, If H is F-pure subact of A andN/H is F/H -pure subact of A/H, then N is F-pure subact of A, where F is subact of A.

Proof.
1. Let I be an ideal of S, since N is F- pure in A and H is F-pure in N, then $N \cap IA = (IN \cap F) \cap (N \cap IA)$ and $H \cap IN = (IN \cup F) \cap (N \cap IA)$ but H subact of N, therefore $H \cap IA \subseteq N \cap IA = (IN \cup F) \cap (N \cap IA)$ and hence $H \cap IA \subseteq H \cap [(IN \cup F) \cap (N \cap IA)] = [H \cap (IN \cup F)] \cap (N \cup IA) = H \cap [(IN \cup F) \cap (N \cap IA)] = [(H \cap IN) \cup (H \cap F)] \cap (N \cap IA) = [(H \cap IN) \cap (N \cap IA)] \cup [(H \cap F) \cap (N \cap IA)] = [(H \cap (IN \cap N) \cap IA)] \cup [(H \cap F)(N \cap IA)] = [H \cap (IN \cap IA)] \cup [F \cap (H \cap IA)] = (H \cap IN) \cup [F \cap (H \cap IA)] = (H \cap IN) \cup [(H \cap IA) \cap F] = [(H \cap IN) \cup (H \cap IA)] \cap [(H \cap IN) \cup F] \subseteq [(H \cap IN) \cup (H \cap IA)] = [(IH \cup F) \cap (H \cap IN)] \cup (H \cap IA) = [[(IH \cup F) \cap (H \cap IN)] \cup H] \cap [[(IH \cup F) \cap (H \cap IN)] \cup IH] \subseteq [(IH \cup F) \cup IH] \cap [(H \cap IN) \cup IH] = (IH \cup F) \cap [(H \cap IN) \cup IH] = [(IH \cup F) \cap (H \cap IN)] \cup [(IH \cup F) \cap IH] = (IH \cup F) \cap [(H \cap IN) \cup IH] \subseteq (IH \cup F) \cap [(H \cup IH) \cap (IN \cup IH)] = (IH \cup F) \cap (H \cap IN) \subseteq [(IH \cup F) \cap (H \cap IA)]$. Since $(IH \cup F) \cap (H \cap IA) \subseteq H \cap IA$, then $H \cap IA = (IH \cup F) \cap (H \cap IA)$.

2. Let I be an ideal of S, since N is F-pure in A, then $N \cap IA = (IN \cup F) \cap (N \cap IA)$. But K subact of A, therefore, $N \cap IK \subseteq N \cap IA = (IN \cup F) \cap (N \cap IA)$, and hence $N \cap IK \subseteq [(IN \cup F) \cap (N \cap IA)] \cap IK = (IN \cup F) \cap [(N \cap IA) \cap IK]$. Since $(IN \cup F) \cap (N \cap IK) \subseteq N \cap IK$, then$N \cap IK = (IN \cup F) \cap [(N \cap IK)$.

3. Let I be an ideal of S, since N is F-pure subact of A, then $N \cap IA = (IN \cup F) \cap (N \cap IA)$. So $N/H \cap I(A/H) = N/H \cap (IA \cup H)/H = [N \cap (IA \cup H)]/H = [(N \cap IA) \cup (N \cap H)]/H = [(IN \cup F) \cap (N \cap IA)] \cup H/H = ([(IN \cup F) \cup H] \cap [(N \cap IA) \cup H])/H = [(IN \cup F) \cup H]/H \cap [(N \cap IA) \cup H])/H = [(IN \cup F)(H \cup H)]/H \cap [(N \cap IA) \cup H])/H = [(IN \cup H) \cup (F \cup H)]/H \cap [(N \cap IA) \cup H])/H = I(N/H) \cup F/H \cap [(N \cup H) \cap (IA \cup H)]/H = [I(N/H) \cup F/H] \cap (N \cup H)/H \cap (IA \cup H)/H = [I(N/H) \cup F/H] \cap [N/H \cap I(A/H)]$.

4. Clear .

□

Definition 4.3. S-act A is said to have the pure relative to subact intersection property (for short $F - PIP$) if the intersection of any two F- pure subacts is again F- pure.

Proposition 4.4. *1. If S-act A has the $F - PIP$, then every F- pure subact of A has the $F - PIP$.*

2. Let N be F- pure submodule of S-act A and N subact of F. A has $F - PIP$ if and only if A/N has $F/N - PIP$.

Proof. 1. Clear

2. Let L/N, K/N be two F/N- pure subacts of A/N and let P be an ideal in S. we have to show that $(L/N \cap K/N) \cap P(A/N) = [P(L/N \cap K/N) \cup F/N] \cap [(L/N \cap K/N) \cap P(A/N)]$. We claim that each of L and K is F-pure in A. To show this, let I be an ideal in S and let $x \in L \cap IA$. Since L/N is F/N -pure in A/N, then $L/N \cap I(A/N) = [I(L/N) \cup F/N] \cap [(L/N \cap I(A/N)]$, thus $L/N \cap (IA \cup N)/N = [(IL \cup N)/N \cup F/N] \cap [(L/N \cap (IA \cup N)/N]$, and this implies that $[L \cap (IA \cup N)]/N = [(IL \cup N)/N \cup F/N] \cap [(L/N \cap (IA \cup N)/N] = [[(IL \cup N) \cup (F \cup N)] \cap [(L \cap (IA \cup N)]]/N$, therefore $L \cap (IA \cup N) = [(IL \cup F) \cup N] \cap [(L \cap IA) \cup (L \cap N)] = [(IL \cup F) \cup N] \cap [(L \cap IA) \cup N] = [(IL \cup F) \cap [(L \cap IA)] \cup N$, and hence $(L \cap IA) \cup N = [(IL \cup F) \cap [(L \cap IA)] \cup N$. Since $x \in L \cap IA \subseteq (L \cap IA) \cup N$, then $x \in [(IL \cup F) \cap [(L \cap IA)] \cup N$, thus $x \in (IL \cup F) \cap [(L \cap IA)$ or $x \in N$, if$x \in (IL \cup F) \cap (L \cap IA)$, then $L \cap IA = (IL \cup F) \cap (L \cap IA)$, therefore L is F-pure, if $x \in N$ thus $x \in (L \cap IA) \cup N = [(IL \cup F) \cap (L \cap IA)] \cup N$, hence L is F-pure in A. Since A has the $F - PIP$, then $L \cap K$ is F- pure in A. Thus $(L \cap K) \cap PA = [P(L \cap K) \cup F] \cap [(L \cap K) \cap IA]$. Now, let Rees congruence $\rho_N \in [L/N \cap K/N] \cap P(A/N)$, then Rees congruence $\rho_N \in P(A/N)$ (Rees congruence on PA) and $\rho_N \in L/N$ (Rees congruence on L) and $\rho_N \in$

K/N (Rees congruence on K), thus Rees congruence ρ_N on L and K therefore $\rho_N \in LK$, but $(L \cap K) \cap PA = [P(L \cap K) \cup F] \cap [(L \cap K) \cap IA]$, therefore $(L/N \cap K/N) \cap P(A/N) = [P(L/N \cap K/N) \cup F/N] \cap [(L/N \cap K/N)$. Conversely let M and N be F- pure subacts of A, let L be a subact of M and L be a subact of N then M/L and N/L is F/L-pure subacts of A/L. Since A/L has $F/L-PIP$, then $M/L \cap N/L = (M \cap N)/L$ is F/L- pure subact of A/L. Therefore $M \cap N$ is F- pure subact of A. □

Theorem 4.5. *Let A be an S-act, then A has the $F-PIP$ if and only if $[(IM \cap IN) \cup F] \cap [(M \cap N) \cap IA] = [I(M \cap N) \cup F] \cap [(M \cap N) \cap IA]$ for every ideal I of S and for every F- pure subact M and N of A.*

Proof. Suppose A has the F-PIP then for each F-pure subact M and N, $M \cap N$ is F-pure. Let I be an ideal in S, then $(M \cap N) \cap IA = [I(M \cap N) \cup F] \cap [(M \cap N) \cap IA]$. It is clear that $[I(M \cap N) \cup F] \cap [(M \cap N) \cap IA] \subseteq ([(IM \cap IN) \cup F] \cap [(M \cap N) \cap IA]$. But $[(IM \cap IN) \cup F] \cap [(M \cap N) \cap IA] \subseteq M \cap (N \cap IA) = (M \cap N) \cap IA = [I(M \cap N) \cup F] \cap [(M \cap N) \cap IA]$. Thus $[(IM \cap IN) \cup F] \cap [(M \cap N) \cap IA] = [I(M \cap N) \cup F] \cap [(M \cap N) \cap IA]$ Conversely, let L and B be F-pure subact of A and J an ideal in S. Then $(L \cap B) \cap JA = L \cap (B \cap JA) = L \cap [(JB \cup F) \cap (B \cap JA)]$. Similarly $(L \cap B) \cap JA = B \cap (L \cap JA) = B \cap (JA \cup F) \cap (B \cap JA)$. But L, B are F- pure in A. Thus $(L \cap B) \cap JA \subseteq [(JL \cap JB) \cup F] \cap [(L \cap B) \cap JA] = [J(L \cap B) \cup F] \cap [(L \cap B) \cap JA]$. □

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