A multi criteria decision making method for cubic hesitant fuzzy sets based on Einstein operational laws

Faisal Mehmood*
School of Mathematics and Statistics
Beijing Institute of Technology
Beijing, 100081
P.R. China
faisal.mehmood007@gmail.com

Khizar Hayat
Department of Mathematics and Information Sciences
Guangzhou University
Guangzhou
China
khizar233@gmail.com

Tahir Mahmood
Department of Mathematics and Statistics
International Islamic University
Islamabad 44000
Pakistan
tahirbakhat@yahoo.com

Bing-Yuan Cao
School of Mathematics and Information Sciences
Guangzhou University
510000 Guangzhou
China
and
Department of Mathematics and Big Data
Foshan University
528000, China
Guangzhou Vocational College of Science and Technology
Guangzhou, Guangdong, 510550
China

Abstract. In this paper, Einstein operations for cubic hesitant fuzzy sets have been introduced and also proved its various results. Aggregation operators play an important role to aggregate the fuzzy information, in view of this fact cubic hesitant fuzzy Einstein weighted averaging operators, cubic hesitant fuzzy Einstein weighted geometric

∗. Corresponding author

operators have been introduced. Finally, by using these aggregation operators a multi criteria decision making problem of real life has been solved.

Keywords: Cubic hesitant fuzzy sets (CHFSs), aggregation operators (AOs), cubic hesitant fuzzy Einstein weighted averaging (CHFEWA) operators, cubic hesitant fuzzy Einstein weighted geometric (CHFEWG) operator, multi criteria decision making (MCDM).

1. Introduction

Fuzzy set theory which was developed by L. A. Zadeh [33] in 1965 is a useful tool to describe the situation when the information is indefinite or indeterminate. Since its inception, fuzzy set theory has been developed and used in many directions of engineering and mathematics. Fuzzy set is a necessary mathematical tool to handle a compilation of facts whose division line is not clear. These facts are given by the membership grade which belongs to $[0, 1]$. If the membership grade is 0 it means that the object does not belong to defined criteria and if this value is 1 it shows that it is fully agreed by the given condition. When defining the membership criteria someone has its own approach to determine the fact. For example when defining the term intelligent someone feels that a person is intelligent if he got 90 percent marks in his academic career while the other has the concept that he should have a better IQ level. In such types of generally fuzzy sets does not help to point being different between immaterial elements and opposite parts. In view of these facts several types of generalization of fuzzy sets have been made which includes interval valued fuzzy sets (IVFSs) [34], hesitant fuzzy sets (HFSs) [15, 16], interval valued hesitant fuzzy sets (IVHFSs) [3], intuitionistic fuzzy sets [1], interval valued intuitionistic fuzzy sets [2], cubic sets (CSs) [6], cubic hesitant fuzzy sets (CHFSs) [11, 12], etc. V. Torra [15, 16] introduced HFS that gives a set of possible values in which each value belongs to $[0, 1]$. Actually the concept of HFS was defined due to the phenomena when a group of people have the hesitancy to reach the ultimate decision. N. Chen et al [3] defined IVHFSs that generalize the concept of HFSs and allow membership value to be the closed subintervals of $[0, 1]$. T. Mahmood et al [11, 12] introduced CHFSs which is a very vital generalization and the only tool to solve such type of uncertain information when there is hesitancy in both IVFSs and fuzzy sets.

There are lots of problems in the world that needs choice of appropriate alternative amongst possible options. Such types of problems are called decision making problems and have applications in almost every field of life like economics, medical sciences, engineering and information technology etc. When decision makers suggest a collection of alternatives against a set of criteria then these sorts of arguments are called multi criteria decision making (MCDM) arguments [4, 5, 7, 13, 14, 17, 20, 22, 27, 32]. The MCDM method is acceptable when an instinctive method is not appropriate, for example considering the decision makers finger the visualization is too large and ramified to handle instinctiveness, because it has a number of inconsistent goals, or has several collaborators with contrary views. When reaching the ultimate result in MCDM process, information given by the decision makers need to be aggregated. For this purpose many aggregation operators (AOs) have been introduced. M. Xia and Z. S. Xu [23] introduced AOs for HFSs. N. Chen et al [3] gave AOs for IVHFSs and used them in group decision making. It is noted that these AOs [24, 25, 26] are established of algebraic operational laws. Many AOs [8, 9, 10, 18, 19, 28, 29, 31] for fuzzy information have been introduced by using Einstein operations. D. Yu [28] defined AOs in order to aggregate hesitant fuzzy facts through Einstein operational laws. G. Wei and X. Zhao [21] explored AOs for IVHFSs by using Einstein operational laws. W. Wang and X. Liu [19] used Einstein operations to aggregate intuitionistic fuzzy information. H. Zhao et al [30] provided aggregation for dual hesitant fuzzy information by using Einstein operations.

The structure of this paper is as follows: Section 2 consists of primary definitions which are utilized to form the paper. Section 3 has Einstein operations for CHFSs and the proof of its various results. Section 4 gives CHFEWA and CHFEWG operators. Section 5 describes the algortim for solving MCDM problem with an example of real life. Section 6 concludes the paper.

2. Notations and preliminaries

In this section some fundamental notions and results have been studied which are used to from the paper.

Definition 2.1 ([33])**.** Assume T be a non-empty set. A fuzzy set is defined by the function $f : T \to P$ where $P = [0, 1]$. Denote P^T as the collection of all fuzzy sets (FSs) in T.

Definition 2.2 ([34])**.** Assume T be a non-empty set. An interval valued fuzzy set (IVFS) is defined by the mapping s from T to the set of closed intervals in $[0, 1]$.

Definition 2.3 ([6])**.** Let T be a non-empty set. The cubic set on T is defined by

$$C = \{< t, s(t), f(t) > /t \in T\}.$$

Where $s(t)$ represents interval valued fuzzy set (IVFS) in T and $f(t)$ is a fuzzy set in T.

Definition 2.4 ([15,16])**.** Let T be a non-empty set. A hesitant fuzzy set (HFS) is a function that when applied on T returns a finite subset of $[0, 1]$ which is denoted and defined by

$$R = \{< t, g(t) > /t \in T\}.$$

Where $g(t)$ is a set of different values in $[0,1]$, which denotes the possible membership degrees of the element $t \in T$ to the set R.

Definition 2.5 ([3])**.** Assume T be a non-empty set and $C[0,1]$ denotes the set of all closed subintervals of $[0,1]$. An interval valued hesitant fuzzy set (IVHFS) on T is denoted and defined by

$$T = \{< t_r, u(t_r) > /t_r \in T, r = 1, 2, ..., n\}.$$

Where $u(t_r) : T \to C[0,1]$ denotes all possible interval valued membership degrees of the element $t_r \in T$ to the set T.

Definition 2.6 ([11,12])**.** Let T be a non-empty set. A cubic hesitant fuzzy set (for short, CHFS) is defined by

$$Z = \{< t, u(t), g(t) > /t \in T\}.$$

Where $u(t)$ is an interval valued hesitant fuzzy element (briefly IVHFE) and $g(t)$ is hesitant fuzzy element (briefly, HFE). A CHFS $Z = \{< t, u(t), g(t) > /t \in T\}$ is simply denoted by $Z =< u, g >$.

Definition 2.7 ([11,12])**.** A CHFS $< u, g >$ is called an internal cubic hesitant fuzzy set (for short, ICHFS) if $\forall i \in \wedge$ (where $\wedge$ is an index) and $\forall t \in T$ we have $\aleph_i^-(t) \leq \hbar_i \leq \aleph_i^+(t)$, where $\aleph_i = [\aleph_i^-, \aleph_i^+] \in s(t)$ and $\hbar_i \in g(t)$.

Definition 2.8 ([11,12])**.** A CHFS $< u, g >$ for a non-empty set T is said to be an external cubic hesitant fuzzy set (briefly, ECHFS) $\forall i \in \wedge$ (where $\wedge$ is an index) and $\forall t \in T$ we have $\hbar_i \notin (\aleph_i^-(t), \aleph_i^+(t))$, where $\aleph_i = [\aleph_i^-, \aleph_i^+] \in s(t)$ and $\hbar_i \in g(t)$.

Definition 2.9 ([11,12])**.** Let $Z = \{< t, u(t), g(t) > /t \in T\}$ be a CHFS. The cubic hesitant fuzzy element (briefly, CHFE) on a non-empty set T is defined by

$$ch = \{< \aleph_i = [\aleph_i^-, \aleph_i^+] \in u(t), \hbar_i \in g(t)/\{[\aleph_i^-, \aleph_i^+]\}, \{\hbar_i\} >\}.$$

Where $u(t)$ represents IVHFE and $g(t)$ represents HFE.

Definition 2.10 ([11,12])**.** Let ch=$\{< \aleph_i$=$[\aleph_i^-, \aleph_i^+] \in u(t), \hbar_i \in g(t)/\{[\aleph_i^-, \aleph_i^+]\}$, $\{\hbar_i\} >\}$ be the CHFE on a non-empty set T, the score of ch is defined by

$$S(ch) = \frac{1}{\circledast(ch)} \left(\aleph_i^- + \aleph_i^+ - \frac{\circledast(ch)}{2} + \hbar_i \right).$$

Where $\aleph_i = [\aleph_i^-, \aleph_i^+] \in s(t)$ (an IVHFE), $\hbar_i \in g(t)$ (HFE) $\forall t \in T, \circledast(ch)$ is the number of elements in ch.

Definition 2.11 ([28])**.** Let g, g_1, g_2 be the three HFEs, then the Einstein operational laws are defined as follows,

1. $g_1 \oplus g_2 = \frac{\hbar_1+\hbar_2}{1+\hbar_1\hbar_2}$
2. $g_1 \otimes g_2 = \frac{\hbar_1\hbar_2}{1+(1-\hbar_1)(1-\hbar_2)}$
3. $\sigma g = \frac{(1+\hbar)^\sigma-(1-\hbar)^\sigma}{(1+\hbar)^\sigma+(1-\hbar)^\sigma}$
4. $g^\sigma = \frac{2\hbar^\sigma}{(2-\hbar)^\sigma+\hbar^\sigma}$

3. Einstein operations for CHFSs

In this section, based on Einstein operational laws Einstein operations for CHFSs are proposed.

Definition 3.1. Let $ch = \{< \{[\aleph_i^-, \aleph_i^+]\}, \{\hbar_i\} >\}, ch_1 = \{< \{[\aleph_{i_1}^-, \aleph_{i_1}^+]\}, \{\hbar_{i_1}\} >\}, ch_2 = \{< \{[\aleph_{i_2}^-, \aleph_{i_2}^+]\}, \{\hbar_{i_2}\} >\}$ be CHFEs. Then we define the following Einstein operational laws as follows:

1. $ch_1 \oplus_e ch_2 = \{< \{[\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-}, \frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+}]\}, \{\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)}\} >\}$;
2. $ch_1 \otimes_e ch_2 = \{< \{[\frac{\aleph_{i_1}^-\aleph_{i_2}^-}{1+\left(1-\aleph_{i_1}^-\right)\left(1-\aleph_{i_2}^-\right)}, \frac{\aleph_{i_1}^+\aleph_{i_2}^+}{1+\left(1-\aleph_{i_1}^+\right)\left(1-\aleph_{i_2}^+\right)}]\}, \{\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}}\} >\}$;
3. $\sigma ch = \{< \{[\frac{\left(1+\aleph_i^-\right)^\sigma-\left(1-\aleph_i^-\right)^\sigma}{\left(1+\aleph_i^-\right)^\sigma+\left(1-\aleph_i^-\right)^\sigma}, \frac{\left(1+\aleph_i^+\right)^\sigma-\left(1-\aleph_i^+\right)^\sigma}{\left(1+\aleph_i^+\right)^\sigma+\left(1-\aleph_i^+\right)^\sigma}]\}, \{\frac{2(\hbar_i)^\sigma}{(2-\hbar_i)^\sigma+(\hbar_i)^\sigma}\} >\}$;
4. $ch^\sigma = \{< \{[\frac{2(\aleph_i^-)^\sigma}{(2-\aleph_i^-)^\sigma+(\aleph_i^-)^\sigma}, \frac{2(\aleph_i^+)^\sigma}{(2-\aleph_i^+)^\sigma+(\aleph_i^+)^\sigma}]\}, \{\frac{(1+\hbar_i)^\sigma-(1-\hbar_i)^\sigma}{(1+\hbar_i)^\sigma+(1-\hbar_i)^\sigma}\} >\}$.

Theorem 3.2. *Assume ch, ch_1, ch_2 be the three CHFEs and $\sigma \geq 0$. Then $ch_1 \oplus_e ch_2, ch_1 \otimes_e ch_2, \sigma ch, ch^\sigma$ are also CHFEs.*

Proof. We have $ch = \{< \{[\aleph_i^-, \aleph_i^+]\}, \{\hbar_i\} >\}, ch_1 = \{< \{[\aleph_{i_1}^-, \aleph_{i_1}^+]\}, \{\hbar_{i_1}\} >\}, ch_2 = \{< \{[\aleph_{i_2}^-, \aleph_{i_2}^+]\}, \{\hbar_{i_2}\} >\}$ are CHFEs.

$$
\begin{aligned}
&0 \leq \aleph_i^-, \aleph_i^+, \aleph_{i_1}^-, \aleph_{i_1}^+, \aleph_{i_2}^-, \aleph_{i_2}^+, \hbar_i, \hbar_{i_1}, \hbar_{i_2} \leq 1,\\
&0 \leq \left(1-\aleph_{i_1}^-\right)\left(1-\aleph_{i_2}^-\right),\\
&0 \leq 1-\aleph_{i_1}^- - \aleph_{i_2}^- + \aleph_{i_1}^-\aleph_{i_2}^-,\\
&\aleph_{i_1}^- + \aleph_{i_2}^- \leq 1 + \aleph_{i_1}^-\aleph_{i_2}^-,\\
&\frac{\aleph_{i_1}^- + \aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-} \leq 1.
\end{aligned}
$$

Similarly $\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+} \leq 1$, also $\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-} \geq 0$ and $\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+} \geq 0$,

$$
\begin{aligned}
&0 \leq \hbar_{i_1} \leq 1, 0 \leq \hbar_{i_2} \leq 1,\\
&0 \leq \hbar_{i_1} + \hbar_{i_2} \leq 2,\\
&0 \leq 2 - \hbar_{i_1} - \hbar_{i_2},\\
&\hbar_{i_1}\hbar_{i_2} \leq 2 - \hbar_{i_1} - \hbar_{i_2} + \hbar_{i_1}\hbar_{i_2},
\end{aligned}
$$

$$\begin{aligned}
&\hbar_{i_1}\hbar_{i_2} \leq 1 + (1-\hbar_{i_1})(1-\hbar_{i_2}),\\
&\frac{\hbar_{i_1}\hbar_{i_2}}{1+(1-\hbar_{i_1})(1-\hbar_{i_2})} \leq 1,\\
&\frac{\hbar_{i_1}\hbar_{i_2}}{1+(1-\hbar_{i_1})(1-\hbar_{i_2})} \geq 0.
\end{aligned}$$

Thus $0 \leq \frac{\aleph_{i_1}^{+}+\aleph_{i_2}^{+}}{1+\aleph_{i_1}^{+}\aleph_{i_2}^{+}} \leq 1, 0 \leq \frac{\aleph_{i_1}^{-}+\aleph_{i_2}^{-}}{1+\aleph_{i_1}^{-}\aleph_{i_2}^{-}} \leq 1, 0 \leq \frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)} \leq 1.$

So $ch_1 \oplus_e ch_2$ is a CHFE.

$$\begin{aligned}
&0 \leq \aleph_{i_1}^{-} \leq 1, 0 \leq \aleph_{i_1}^{+} \leq 1, 0 \leq \aleph_{i_2}^{-} \leq 1, 0 \leq \aleph_{i_2}^{+} \leq 1,\\
&0 \leq \aleph_{i_1}^{-} + \aleph_{i_2}^{-} \leq 2,\\
&0 \leq 2 - \aleph_{i_1}^{-} - \aleph_{i_2}^{-},\\
&\aleph_{i_1}^{-}\aleph_{i_2}^{-} \leq 2 - \aleph_{i_1}^{-} - \aleph_{i_2}^{-} + \aleph_{i_1}^{-}\aleph_{i_2}^{-},\\
&\aleph_{i_1}^{-}\aleph_{i_2}^{-} \leq 1 + \left(1-\aleph_{i_1}^{-}\right)\left(1-\aleph_{i_2}^{-}\right),\\
&\frac{\aleph_{i_1}^{-}\aleph_{i_2}^{-}}{1+\left(1-\aleph_{i_1}^{-}\right)\left(1-\aleph_{i_2}^{-}\right)} \leq 1,\\
&\frac{\aleph_{i_1}^{-}\aleph_{i_2}^{-}}{1+\left(1-\aleph_{i_1}^{-}\right)\left(1-\aleph_{i_2}^{-}\right)} \geq 0.
\end{aligned}$$

Similarly

$$\begin{aligned}
&0 \leq \frac{\aleph_{i_1}^{+}\aleph_{i_2}^{+}}{1+\left(1-\aleph_{i_1}^{+}\right)\left(1-\aleph_{i_2}^{+}\right)} \leq 1,\\
&0 \leq (1-\hbar_{i_1})(1-\hbar_{i_2}),\\
&0 \leq 1 - \hbar_{i_1} - \hbar_{i_2} + \hbar_{i_1}\hbar_{i_2},\\
&\hbar_{i_1} + \hbar_{i_2} \leq 1 + \hbar_{i_1}\hbar_{i_2},\\
&\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}} \leq 1,\\
&\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}} \geq 0,
\end{aligned}$$

Thus $0 \leq \frac{\aleph_{i_1}^{-}\aleph_{i_2}^{-}}{1+(1-\aleph_{i_1}^{-})(1-\aleph_{i_2}^{-})} \leq 1, 0 \leq \frac{\aleph_{i_1}^{+}\aleph_{i_2}^{+}}{1+\left(1-\aleph_{i_1}^{+}\right)\left(1-\aleph_{i_2}^{+}\right)} \leq 1$, and $0 \leq \frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}} \leq 1$.

So $ch_1 \otimes_e ch_2$ is a CHFE.

$$\begin{aligned}
&\left(1+\aleph_{i}^{-}\right)^{\sigma} - \left(1-\aleph_{i}^{-}\right)^{\sigma} \leq \left(1+\aleph_{i}^{-}\right)^{\sigma} + \left(1-\aleph_{i}^{-}\right)^{\sigma},\\
&\frac{\left(1+\aleph_{i}^{-}\right)^{\sigma} - \left(1-\aleph_{i}^{-}\right)^{\sigma}}{\left(1+\aleph_{i}^{-}\right)^{\sigma} + \left(1-\aleph_{i}^{-}\right)^{\sigma}} \leq 1.
\end{aligned}$$

Similarly

$$\frac{\left(1+\aleph_{i}^{+}\right)^{\sigma} - \left(1-\aleph_{i}^{+}\right)^{\sigma}}{\left(1+\aleph_{i}^{+}\right)^{\sigma} + \left(1-\aleph_{i}^{+}\right)^{\sigma}} \leq 1.$$

Also $\frac{\left(1+\aleph_i^-\right)^\sigma-\left(1-\aleph_i^-\right)^\sigma}{\left(1+\aleph_i^-\right)^\sigma+\left(1-\aleph_i^-\right)^\sigma}\geq 0, \frac{\left(1+\aleph_i^+\right)^\sigma-\left(1-\aleph_i^+\right)^\sigma}{\left(1+\aleph_i^+\right)^\sigma+\left(1-\aleph_i^+\right)^\sigma}\geq 0,$

$$\begin{aligned}
&0\leq\hbar_i\leq 1,\\
&2\left(\hbar_i\right)^\sigma\leq\left(2-\hbar_i\right)^\sigma+\left(\hbar_i\right)^\sigma,\\
&\frac{2\left(\hbar_i\right)^\sigma}{\left(2-\hbar_i\right)^\sigma+\left(\hbar_i\right)^\sigma}\leq 1,\\
&\frac{2\left(\hbar_i\right)^\sigma}{\left(2-\hbar_i\right)^\sigma+\left(\hbar_i\right)^\sigma}\geq 0.
\end{aligned}$$

Thus $0\leq\frac{(1+\aleph_i^-)^\sigma-(1-\aleph_i^-)^\sigma}{(1+\aleph_i^-)^\sigma+(1-\aleph_i^-)^\sigma}\leq 1, 0\leq\frac{(1+\aleph_i^+)^\sigma-(1-\aleph_i^+)^\sigma}{(1+\aleph_i^+)^\sigma+(1-\aleph_i^+)^\sigma}\leq 1$ and $0\leq\frac{2(\hbar_i)^\sigma}{(2-\hbar_i)^\sigma+(\hbar_i)^\sigma}\leq 1$.

Hence σch is a CHFE. In the same way we can show that ch^σ is a CHFE.

Proposition 3.3. *Suppose ch, ch_1, ch_2 be the three CHFEs and $\sigma,\sigma_1,\sigma_2>0$ then,*

1. $ch_1\oplus_e ch_2=ch_2\oplus_e ch_1$
2. $ch_1\otimes_e ch_2=ch_2\otimes_e ch_1$
3. $(\sigma_1\oplus_e\sigma_2)ch=\sigma_1 ch\oplus_e\sigma_2 ch$
4. $\sigma(ch_1\oplus_e ch_2)=\sigma ch_1\oplus_e\sigma ch_2$
5. $(ch_1\otimes_e ch_2)^\sigma=ch_1^\sigma\otimes_e ch_2^\sigma$

Proof.

1. $ch_1\oplus_e ch_2=ch_2\oplus_e ch_1$

$ch_1\oplus_e ch_2=\{<\{[\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-},\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+}]\},\{\frac{\hbar_{i_1}\hbar_{i_2}}{1+\left(1-\hbar_{i_1}\right)\left(1-\hbar_{i_2}\right)}\}>\},$

$=\{<\{[\frac{\aleph_{i_2}^-+\aleph_{i_1}^-}{1+\aleph_{i_2}^-\aleph_{i_1}^-},\frac{\aleph_{i_2}^++\aleph_{i_1}^+}{1+\aleph_{i_2}^+\aleph_{i_1}^+}]\},\{\frac{\hbar_{i_2}\hbar_{i_1}}{1+\left(1-\hbar_{i_2}\right)\left(1-\hbar_{i_1}\right)}\}>\}=ch_2\oplus_e ch_1.$

2. $ch_1\otimes_e ch_2=ch_2\otimes_e ch_1$

$ch_1\otimes_e ch_2=\{<\{[\frac{\aleph_{i_1}^-\aleph_{i_2}^-}{1+\left(1-\aleph_{i_1}^-\right)\left(1-\aleph_{i_2}^-\right)},\frac{\aleph_{i_1}^+\aleph_{i_2}^+}{1+\left(1-\aleph_{i_1}^+\right)\left(1-\aleph_{i_2}^+\right)}]\},\{\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}}\}>\},$

$=\{<\{[\frac{\aleph_{i_2}^-\aleph_{i_1}^-}{1+\left(1-\aleph_{i_2}^-\right)\left(1-\aleph_{i_1}^-\right)},\frac{\aleph_{i_2}^+\aleph_{i_1}^+}{1+\left(1-\aleph_{i_2}^+\right)\left(1-\aleph_{i_1}^+\right)}]\},\{\frac{\hbar_{i_2}+\hbar_{i_1}}{1+\hbar_{i_2}\hbar_{i_1}}\}>\}=ch_2\otimes_e ch_1.$

3. $(\sigma_1\oplus_e\sigma_2)ch=\sigma_1 ch\oplus_e\sigma_2 ch$

$(\sigma_1\oplus_e\sigma_2)ch=\{<\{[\frac{(1+\aleph_i^-)^{\sigma_1+\sigma_2}-(1-\aleph_i^-)^{\sigma_1+\sigma_2}}{(1+\aleph_i^-)^{\sigma_1+\sigma_2}+(1-\aleph_i^-)^{\sigma_1+\sigma_2}},\frac{(1+\aleph_i^+)^{\sigma_1+\sigma_2}-(1-\aleph_i^+)^{\sigma_1+\sigma_2}}{(1+\aleph_i^+)^{\sigma_1+\sigma_2}+(1-\aleph_i^+)^{\sigma_1+\sigma_2}}]\},$
$\{\frac{2(\hbar_i)^{\sigma_1+\sigma_2}}{(2-\hbar_i)^{\sigma_1+\sigma_2}+(\hbar_i)^{\sigma_1+\sigma_2}}\}>\},$

$\sigma_1 ch\oplus_e\sigma_2 ch=\{<\{[\frac{(1+\aleph_i^-)^{\sigma_1}-(1-\aleph_i^-)^{\sigma_1}}{(1+\aleph_i^-)^{\sigma_1}+(1-\aleph_i^-)^{\sigma_1}},\frac{(1+\aleph_i^+)^{\sigma_1}-(1-\aleph_i^+)^{\sigma_1}}{(1+\aleph_i^+)^{\sigma_1}+(1-\aleph_i^+)^{\sigma_1}}]\},\{\frac{2(\hbar_i)^{\sigma_1}}{(2-\hbar_i)^{\sigma_1}+(\hbar_i)^{\sigma_1}}\}>\}\oplus_e\{<\{[\frac{(1+\aleph_i^-)^{\sigma_2}-(1-\aleph_i^-)^{\sigma_2}}{(1+\aleph_i^-)^{\sigma_2}+(1-\aleph_i^-)^{\sigma_2}},\frac{(1+\aleph_i^+)^{\sigma_2}-(1-\aleph_i^+)^{\sigma_2}}{(1+\aleph_i^+)^{\sigma_2}+(1-\aleph_i^+)^{\sigma_2}}]\},\{\frac{2(\hbar_i)^{\sigma_2}}{(2-\hbar_i)^{\sigma_2}+(\hbar_i)^{\sigma_2}}\}>\},$

$=\{<\{[\frac{\frac{(1+\aleph_i^-)^{\sigma_1}-(1-\aleph_i^-)^{\sigma_1}}{(1+\aleph_i^-)^{\sigma_1}+(1-\aleph_i^-)^{\sigma_1}}+\frac{(1+\aleph_i^-)^{\sigma_2}-(1-\aleph_i^-)^{\sigma_2}}{(1+\aleph_i^-)^{\sigma_2}+(1-\aleph_i^-)^{\sigma_2}}}{1+\frac{(1+\aleph_i^-)^{\sigma_1}-(1-\aleph_i^-)^{\sigma_1}}{(1+\aleph_i^-)^{\sigma_1}+(1-\aleph_i^-)^{\sigma_1}}\times\frac{(1+\aleph_i^-)^{\sigma_2}-(1-\aleph_i^-)^{\sigma_2}}{(1+\aleph_i^-)^{\sigma_2}+(1-\aleph_i^-)^{\sigma_2}}},$

$$\frac{\frac{(1+\aleph_i^+)^{\sigma_1}-(1-\aleph_i^+)^{\sigma_1}}{(1+\aleph_i^+)^{\sigma_1}+(1-\aleph_i^+)^{\sigma_1}}+\frac{(1+\aleph_i^+)^{\sigma_2}-(1-\aleph_i^+)^{\sigma_2}}{(1+\aleph_i^+)^{\sigma_2}+(1-\aleph_i^+)^{\sigma_2}}}{1+\frac{(1+\aleph_i^-)^{\sigma_1}-(1-\aleph_i^-)^{\sigma_1}}{(1+\aleph_i^+)^{\sigma_1}+(1-\aleph_i^+)^{\sigma_1}}\times\frac{(1+\aleph_i^-)^{\sigma_2}-(1-\aleph_i^-)^{\sigma_2}}{(1+\aleph_i^+)^{\sigma_2}+(1-\aleph_i^+)^{\sigma_2}}}]\},$$

$$\{\frac{\frac{2(\hbar_i)^{\sigma_1}}{(2-\hbar_i)^{\sigma_1}+(\hbar_i)^{\sigma_1}}\times\frac{2(\hbar_i)^{\sigma_2}}{(2-\hbar_i)^{\sigma_2}+(\hbar_i)^{\sigma_2}}}{1+(1-\frac{2(\hbar_i)^{\sigma_1}}{(2-\hbar_i)^{\sigma_1}+(\hbar_i)^{\sigma_1}})(1-\frac{2(\hbar_i)^{\sigma_2}}{(2-\hbar_i)^{\sigma_2}+(\hbar_i)^{\sigma_2}})}\}>\},$$

$$=\{<\{[\frac{((C^-)^{\sigma_1}-(D^-)^{\sigma_1})((C^-)^{\sigma_2}+(D^-)^{\sigma_2})+((C^-)^{\sigma_1}+(D^-)^{\sigma_1})((C^-)^{\sigma_2}-(D^-)^{\sigma_2})}{((C^-)^{\sigma_1}+(D^-)^{\sigma_1})((C^-)^{\sigma_2}+(D^-)^{\sigma_2})+((C^-)^{\sigma_1}-(D^-)^{\sigma_1})((C^-)^{\sigma_2}-(D^-)^{\sigma_2})},$$

$$\frac{((C^+)^{\sigma_1}-(D^+)^{\sigma_1})((C^+)^{\sigma_2}+(D^+)^{\sigma_2})+((C^+)^{\sigma_1}+(D^+)^{\sigma_1})((C^+)^{\sigma_2}-(D^+)^{\sigma_2})}{((C^+)^{\sigma_1}+(D^+)^{\sigma_1})((C^+)^{\sigma_2}+(D^+)^{\sigma_2})+((C^+)^{\sigma_1}-(D^+)^{\sigma_1})((C^+)^{\sigma_2}-(D^+)^{\sigma_2})}]\},$$

$$\{\frac{2(E)^{\sigma_1}\times 2(E)^{\sigma_2}}{((2-E)^{\sigma_1}+(E)^{\sigma_1})((2-E)^{\sigma_2}+(E)^{\sigma_2})+((2-E)^{\sigma_1}-(E)^{\sigma_1})((2-E)^{\sigma_2}-(E)^{\sigma_2})}\}>\}.$$

Where $C^-=1+\aleph_i^-, C^+=1+\aleph_i^+, D^-=1-\aleph_i^-, D^+=1-\aleph_i^+$ and $E=\hbar_i$,

$$=\{<\{[\frac{((C^-)^{\sigma_1+\sigma_2}-(D^-)^{\sigma_1+\sigma_2})}{((C^-)^{\sigma_1+\sigma_2}+(D^-)^{\sigma_1+\sigma_2})},\frac{((C^+)^{\sigma_1+\sigma_2}-(D^+)^{\sigma_1+\sigma_2})}{((C^+)^{\sigma_1+\sigma_2}+(D^+)^{\sigma_1+\sigma_2})}]\},$$

$$\{\frac{2(E)^{\sigma_1+\sigma_2}}{((2-E)^{\sigma_1+\sigma_2}+(E)^{\sigma_1+\sigma_2})}\}>\},$$

$$=\{<\{[\frac{(1+\aleph_i^-)^{\sigma_1+\sigma_2}-(1-\aleph_i^-)^{\sigma_1+\sigma_2}}{(1+\aleph_i^-)^{\sigma_1+\sigma_2}+(1-\aleph_i^-)^{\sigma_1+\sigma_2}},\frac{(1+\aleph_i^+)^{\sigma_1+\sigma_2}-(1-\aleph_i^+)^{\sigma_1+\sigma_2}}{(1+\aleph_i^+)^{\sigma_1+\sigma_2}+(1-\aleph_i^+)^{\sigma_1+\sigma_2}}]\},$$

$$\{\frac{2(\hbar_i)^{\sigma_1+\sigma_2}}{((2-\hbar_i)^{\sigma_1+\sigma_2}+(\hbar_i)^{\sigma_1+\sigma_2})}\}>\},$$

Hence $(\sigma_1\oplus_e\sigma_2)ch=\sigma_1 ch\oplus_e\sigma_2 ch$.

4. $\sigma(ch_1\oplus_e ch_2)=\sigma ch_1\oplus_e\sigma ch_2$

$$(ch_1\oplus_e ch_2)=\{<\{[\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-},\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+}]\},\{\frac{\hbar_{i_1}\hbar_{i_2}}{1+\big(1-\hbar_{i_1}\big)\big(1-\hbar_{i_2}\big)}\}>\},$$

$$\sigma(ch_1\oplus_e ch_2)=\{<\{[\frac{(1+\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-})^\sigma-(1-\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-})^\sigma}{(1+\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-})^\sigma+(1-\frac{\aleph_{i_1}^-+\aleph_{i_2}^-}{1+\aleph_{i_1}^-\aleph_{i_2}^-})^\sigma},\frac{(1+\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+})^\sigma-(1-\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+})^\sigma}{(1+\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+})^\sigma+(1-\frac{\aleph_{i_1}^++\aleph_{i_2}^+}{1+\aleph_{i_1}^+\aleph_{i_2}^+})^\sigma}]\},$$

$$\{\frac{2(\frac{\hbar_{i_1}\hbar_{i_2}}{1+\big(1-\hbar_{i_1}\big)\big(1-\hbar_{i_2}\big)})^\sigma}{(2-\frac{\hbar_{i_1}\hbar_{i_2}}{1+\big(1-\hbar_{i_1}\big)\big(1-\hbar_{i_2}\big)})^\sigma+(\frac{\hbar_{i_1}\hbar_{i_2}}{1+\big(1-\hbar_{i_1}\big)\big(1-\hbar_{i_2}\big)})^\sigma}\}>\},$$

$$=\{<\{[\frac{(1+\aleph_{i_1}^-)^\sigma(1+\aleph_{i_2}^-)^\sigma-(1-\aleph_{i_1}^-)^\sigma(1-\aleph_{i_2}^-)^\sigma}{(1+\aleph_{i_1}^-)^\sigma(1+\aleph_{i_2}^-)^\sigma+(1-\aleph_{i_1}^-)^\sigma(1-\aleph_{i_2}^-)^\sigma},\frac{(1+\aleph_{i_1}^+)^\sigma(1+\aleph_{i_2}^+)^\sigma-(1-\aleph_{i_1}^+)^\sigma(1-\aleph_{i_2}^+)^\sigma}{(1+\aleph_{i_1}^+)^\sigma(1+\aleph_{i_2}^+)^\sigma+(1-\aleph_{i_1}^+)^\sigma(1-\aleph_{i_2}^+)^\sigma}]\},$$

$$\{\frac{2\hbar_{i_1}^\sigma\hbar_{i_2}^\sigma}{(2-\hbar_{i_1})^\sigma(2-\hbar_{i_2})^\sigma+\hbar_{i_1}^\sigma\hbar_{i_2}^\sigma}\}>\},$$

$$\sigma ch_1=\{<\{[\frac{(1+\aleph_{i_1}^-)^\sigma-(1-\aleph_{i_1}^-)^\sigma}{(1+\aleph_{i_1}^-)^\sigma+(1-\aleph_{i_1}^-)^\sigma},\frac{(1+\aleph_{i_1}^+)^\sigma-(1-\aleph_{i_1}^+)^\sigma}{(1+\aleph_{i_1}^+)^\sigma+(1-\aleph_{i_1}^+)^\sigma}]\},\{\frac{2(\hbar_{i_1})^\sigma}{(2-\hbar_{i_1})^\sigma+(\hbar_{i_1})^\sigma}\}>\},$$

$$\sigma ch_2=\{<\{[\frac{(1+\aleph_{i_2}^-)^\sigma-(1-\aleph_{i_2}^-)^\sigma}{(1+\aleph_{i_2}^-)^\sigma+(1-\aleph_{i_2}^-)^\sigma},\frac{(1+\aleph_{i_2}^+)^\sigma-(1-\aleph_{i_2}^+)^\sigma}{(1+\aleph_{i_2}^+)^\sigma+(1-\aleph_{i_2}^+)^\sigma}]\},\{\frac{2(\hbar_{i_2})^\sigma}{(2-\hbar_{i_2})^\sigma+(\hbar_{i_2})^\sigma}\}>\},$$

$$\sigma ch_1=\{<\{[\frac{K_1^--L_1^-}{K_1^-+L_1^-},\frac{K_1^+-L_1^+}{K_1^++L_1^+}]\},\{\frac{2M_1}{N_1+M_1}\}>\},$$

$$\sigma ch_2=\{<\{[\frac{K_2^--L_2^-}{K_2^-+L_2^-},\frac{K_2^+-L_2^+}{K_2^++L_2^+}]\},\{\frac{2M_2}{N_2+M_2}\}>\}.$$

Where $K_1^-=(1+\aleph_{i_1}^-)^\sigma, K_1^+=(1+\aleph_{i_1}^+)^\sigma, L_1^-=(1-\aleph_{i_1}^-)^\sigma, L_1^+=(1-\aleph_{i_1}^+)^\sigma, M_1=(\hbar_{i_1})^\sigma, N_1=(2-\hbar_{i_1})^\sigma, K_2^-=(1+\aleph_{i_2}^-)^\sigma, K_2^+=(1+\aleph_{i_2}^+)^\sigma, L_2^-=(1-\aleph_{i_2}^-)^\sigma, L_2^+=(1-\aleph_{i_2}^+)^\sigma, M_2=(\hbar_{i_2})^\sigma, N_2=(2-\hbar_{i_2})^\sigma$,

$$\sigma ch_1 \oplus_e \sigma ch_2 = \{<\{[\frac{\frac{K_1^- - L_1^-}{K_1^- + L_1^-} + \frac{K_2^- - L_2^-}{K_2^- + L_2^-}}{1+(\frac{K_1^- - L_1^-}{K_1^- + L_1^-})(\frac{K_2^- - L_2^-}{K_2^- + L_2^-})}, \frac{\frac{K_1^+ - L_1^+}{K_1^+ + L_1^+} + \frac{K_2^+ - L_2^+}{K_2^+ + L_2^+}}{1+(\frac{K_1^+ - L_1^+}{K_1^+ + L_1^+})(\frac{K_2^+ - L_2^+}{K_2^+ + L_2^+})}]\},$$

$\{\frac{\frac{2M_1}{N_1+M_1} \times \frac{2M_2}{N_2+M_2}}{1+(1-\frac{2M_1}{N_1+M_1})(1-\frac{2M_2}{N_2+M_2})}\} >\},$

$= \{<\{[\frac{K_1^- K_2^- - L_1^- L_2^-}{K_1^- K_2^- + L_1^- L_2^-}, \frac{K_1^+ K_2^+ - L_1^+ L_2^+}{K_1^+ K_2^+ + L_1^+ L_2^+}]\}, \{\frac{2M_1 M_2}{N_1 N_2 + M_1 M_2}\} >\},$

$= \{<\{[\frac{(1+\aleph_{i_1}^-)^\sigma(1+\aleph_{i_2}^-)^\sigma - (1-\aleph_{i_1}^-)^\sigma(1-\aleph_{i_2}^-)^\sigma}{(1+\aleph_{i_1}^-)^\sigma(1+\aleph_{i_2}^-)^\sigma + (1-\aleph_{i_1}^-)^\sigma(1-\aleph_{i_2}^-)^\sigma}, \frac{(1+\aleph_{i_1}^+)^\sigma(1+\aleph_{i_2}^+)^\sigma - (1-\aleph_{i_1}^+)^\sigma(1-\aleph_{i_2}^+)^\sigma}{(1+\aleph_{i_1}^+)^\sigma(1+\aleph_{i_2}^+)^\sigma + (1-\aleph_{i_1}^+)^\sigma(1-\aleph_{i_2}^+)^\sigma}]\},$

$\{\frac{2\hbar_{i_1}^\sigma \hbar_{i_2}^\sigma}{(2-\hbar_{i_1})^\sigma(2-\hbar_{i_2})^\sigma + \hbar_{i_1}^\sigma \hbar_{i_2}^\sigma}\} >\},$

Thus $\sigma(ch_1 \oplus_e ch_2) = \sigma ch_1 \oplus_e \sigma ch_2$.

5.$(ch_1 \otimes_e ch_2)^\sigma = ch_1^\sigma \otimes_e ch_2^\sigma$

$ch_1^\sigma = \{<\{[\frac{2(\aleph_{i_1}^-)^\sigma}{(2-\aleph_{i_1}^-)^\sigma + (\aleph_{i_1}^-)^\sigma}, \frac{2(\aleph_{i_1}^+)^\sigma}{(2-\aleph_{i_1}^+)^\sigma + (\aleph_{i_1}^+)^\sigma}]\}, \{\frac{(1+\hbar_{i_1})^\sigma - (1-\hbar_{i_1})^\sigma}{(1+\hbar_{i_1})^\sigma + (1-\hbar_{i_1})^\sigma}\} >\},$

$ch_2^\sigma = \{<\{[\frac{2(\aleph_{i_2}^-)^\sigma}{(2-\aleph_{i_2}^-)^\sigma + (\aleph_{i_2}^-)^\sigma}, \frac{2(\aleph_{i_2}^+)^\sigma}{(2-\aleph_{i_2}^+)^\sigma + (\aleph_{i_2}^+)^\sigma}]\}, \{\frac{(1+\hbar_{i_2})^\sigma - (1-\hbar_{i_2})^\sigma}{(1+\hbar_{i_2})^\sigma + (1-\hbar_{i_2})^\sigma}\} >\},$

$ch_1^\sigma = \{<\{[\frac{2P_1^-}{Q_1^- + P_1^-}, \frac{2P_1^+}{Q_1^+ + P_1^+}]\}, \{\frac{R_1 - S_1}{R_1 + S_1}\} >\},$

$ch_2^\sigma = \{<\{[\frac{2P_2^-}{Q_2^- + P_2^-}, \frac{2P_2^+}{Q_2^+ + P_2^+}]\}, \{\frac{R_2 - S_2}{R_2 + S_2}\} >\}.$

Where $P_1^- = (\aleph_{i_1}^-)^\sigma, P_1^+ = (\aleph_{i_1}^+)^\sigma, Q_1^- = (2-\aleph_{i_1}^-)^\sigma, Q_1^+ = (2-\aleph_{i_1}^+)^\sigma, R_1 = (1+\hbar_{i_1})^\sigma, S_1 = (1-\hbar_{i_1})^\sigma, P_2^- = (\aleph_{i_2}^-)^\sigma, P_2^+ = (\aleph_{i_2}^+)^\sigma, Q_2^- = (2-\aleph_{i_2}^-)^\sigma, Q_2^+ = (2-\aleph_{i_2}^+)^\sigma, R_2 = (1+\hbar_{i_2})^\sigma, S_2 = (1-\hbar_{i_2})^\sigma,$

$$ch_1^\sigma \otimes_e ch_2^\sigma = \{<\{[\frac{\frac{2P_1^-}{Q_1^- + P_1^-} \times \frac{2P_2^-}{Q_2^- + P_2^-}}{1+(1-\frac{2P_1^-}{Q_1^- + P_1^-})(1-\frac{2P_2^-}{Q_2^- + P_2^-})}, \frac{\frac{2P_1^+}{Q_1^+ + P_1^+} \times \frac{2P_2^+}{Q_2^+ + P_2^+}}{1+(1-\frac{2P_1^+}{Q_1^+ + P_1^+})(1-\frac{2P_2^+}{Q_2^+ + P_2^+})}]\},$$

$\{\frac{\frac{R_1 - S_1}{R_1 + S_1} + \frac{R_2 - S_2}{R_2 + S_2}}{1+(\frac{R_1 - S_1}{R_1 + S_1})(\frac{R_2 - S_2}{R_2 + S_2})}\} >\},$

$= \{<\{[\frac{4P_1^- P_2^-}{(Q_1^- + P_1^-)(Q_2^- + P_2^-) + (Q_1^- - P_1^-)(Q_2^- - P_2^-)}, \frac{4P_1^+ P_2^+}{(Q_1^+ + P_1^+)(Q_2^+ + P_2^+) + (Q_1^+ - P_1^+)(Q_2^+ - P_2^+)}]\},$

$\{\frac{R_1 R_2 - S_1 S_2}{R_1 R_2 + S_1 S_2}\} >\},$

$= \{<\{[\frac{2P_1^- P_2^-}{Q_1^- Q_2^- + P_1^- P_2^-}, \frac{2P_1^+ P_2^+}{Q_1^+ Q_2^+ + P_1^+ P_2^+}]\}, \{\frac{R_1 R_2 - S_1 S_2}{R_1 R_2 + S_1 S_2}\} >\},$

$= \{<\{[\frac{2(\aleph_{i_1}^-)^\sigma(\aleph_{i_2}^-)^\sigma}{(2-\aleph_{i_1}^-)^\sigma(2-\aleph_{i_2}^-)^\sigma + (\aleph_{i_1}^-)^\sigma(\aleph_{i_2}^-)^\sigma}, \frac{2(\aleph_{i_1}^+)^\sigma(\aleph_{i_2}^+)^\sigma}{(2-\aleph_{i_1}^+)^\sigma(2-\aleph_{i_2}^+)^\sigma + (\aleph_{i_1}^+)^\sigma(\aleph_{i_2}^+)^\sigma}]\},$

$\{\frac{(1+\hbar_{i_1})^\sigma(1+\hbar_{i_2})^\sigma - (1-\hbar_{i_1})^\sigma(1-\hbar_{i_2})^\sigma}{(1+\hbar_{i_1})^\sigma(1+\hbar_{i_2})^\sigma + (1-\hbar_{i_1})^\sigma(1-\hbar_{i_2})^\sigma}\} >\}.$

Now $ch_1 \otimes_e ch_2 = \{<\{[\frac{\aleph_{i_1}^- \aleph_{i_2}^-}{1+(1-\aleph_{i_1}^-)(1-\aleph_{i_2}^-)}, \frac{\aleph_{i_1}^+ \aleph_{i_2}^+}{1+(1-\aleph_{i_1}^+)(1-\aleph_{i_2}^+)}]\}, \{\frac{\hbar_{i_1} + \hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}}\} >\},$

$$(ch_1 \otimes_e ch_2)^\sigma = \{<\{[\frac{2(\frac{\aleph_{i_1}^- \aleph_{i_2}^-}{1+(1-\aleph_{i_1}^-)(1-\aleph_{i_2}^-)})^\sigma}{(2-(\frac{\aleph_{i_1}^- \aleph_{i_2}^-}{1+(1-\aleph_{i_1}^-)(1-\aleph_{i_2}^-)}))^\sigma + (\frac{\aleph_{i_1}^- \aleph_{i_2}^-}{1+(1-\aleph_{i_1}^-)(1-\aleph_{i_2}^-)})^\sigma},$$

$\frac{2(\frac{\aleph^+_{i_1}\aleph^+_{i_2}}{1+(1-\aleph^+_{i_1})(1-\aleph^+_{i_2})})^\sigma}{(2-(\frac{\aleph^+_{i_1}\aleph^+_{i_2}}{1+(1-\aleph^+_{i_1})(1-\aleph^+_{i_2})}))^\sigma+(\frac{\aleph^+_{i_1}\aleph^+_{i_2}}{1+(1-\aleph^+_{i_1})(1-\aleph^+_{i_2})})^\sigma}]\}, \{\frac{(1+\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}})^\sigma-(1-\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}})^\sigma}{(1+\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}})^\sigma+(1-\frac{\hbar_{i_1}+\hbar_{i_2}}{1+\hbar_{i_1}\hbar_{i_2}})^\sigma}\} >\},$

$= \{< \{[\frac{2(\aleph^-_{i_1})^\sigma(\aleph^-_{i_2})^\sigma}{(2-\aleph^-_{i_1})^\sigma(2-\aleph^-_{i_2})^\sigma+(\aleph^-_{i_1})^\sigma(\aleph^-_{i_2})^\sigma}, \frac{2(\aleph^+_{i_1})^\sigma(\aleph^+_{i_2})^\sigma}{(2-\aleph^+_{i_1})^\sigma(2-\aleph^+_{i_2})^\sigma+(\aleph^+_{i_1})^\sigma(\aleph^+_{i_2})^\sigma}]\},$
$\{\frac{(1+\hbar_{i_1})^\sigma(1+\hbar_{i_2})^\sigma-(1-\hbar_{i_1})^\sigma(1-\hbar_{i_2})^\sigma}{(1+\hbar_{i_1})^\sigma(1+\hbar_{i_2})^\sigma+(1-\hbar_{i_1})^\sigma(1-\hbar_{i_2})^\sigma}\} >\},$

So $(ch_1 \otimes_e ch_2)^\sigma = ch_1^\sigma \otimes_e ch_2^\sigma$.

4. Cubic hesitant fuzzy Einstein weighted averaging and geometric operators

In this section Einstein weighted averaging and weighted geometric operators for CHFSs are defined which helps to aggregate the given cubic hesitant fuzzy information.

Definition 4.1. Let $ch_s(s = 1, 2, ..., n)$ be the number of CHFEs. The cubic hesitant fuzzy Einstein weighted averaging (briefly, CHFEWA) operator is defined by the mapping $G^n \to G$ such that

$$CHFEWA(ch_1, ch_2, ..., ch_n) = (\oplus_e)^n_{s=1}(w_s ch_s).$$

Here $w = (w_1, w_2, ..., w_n)^T$ be a weighting vector of CHFEs where $w_s \in [0, 1]$ and $\sum_{s=1}^n w_s = 1$. As a special case when $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ then CHFEWA operator becomes CHFEA operator as

$$CHFEA(ch_1, ch_2, ..., ch_n) = (\oplus_e)^n_{s=1}(\frac{1}{n} ch_s).$$

Theorem 4.2. *Suppose $ch_s(s = 1, 2, ..., n)$ be the number of CHFEs. Their aggregated result obtained by using CHFEWA operator is a CHFE,*

$CHFEWA(ch_1, ch_2, ..., ch_n) = (\oplus_e)^n_{s=1}(w_s ch_s)$
$= \{< \{[\frac{\Pi^n_{s=1}(1+\aleph^-_{i_s})^{w_s}-\Pi^n_{s=1}(1-\aleph^-_{i_s})^{w_s}}{\Pi^n_{s=1}(1+\aleph^-_{i_s})^{w_s}+\Pi^n_{s=1}(1-\aleph^-_{i_s})^{w_s}}, \frac{\Pi^n_{s=1}(1+\aleph^+_{i_s})^{w_s}-\Pi^n_{s=1}(1-\aleph^+_{i_s})^{w_s}}{\Pi^n_{s=1}(1+\aleph^+_{i_s})^{w_s}+\Pi^n_{s=1}(1-\aleph^+_{i_s})^{w_s}}]\},$
$\{\frac{2\Pi^n_{s=1}(\hbar_{i_s})^{w_s}}{\Pi^n_{s=1}(2-\hbar_{i_s})^{w_s}+\Pi^n_{s=1}(\hbar_{i_s})^{w_s}}\} >\}.$

Proof. We show the result by using mathematical induction.

When $n = 2$ and by applying operational law 3 of Definition 3.1 it results,

$w_1ch_1 = \{< \{[\frac{(1+\aleph^-_{i_1})^{w_1}-(1-\aleph^-_{i_1})^{w_1}}{(1+\aleph^-_{i_1})^{w_1}+(1-\aleph^-_{i_1})^{w_1}}, \frac{(1+\aleph^+_{i_1})^{w_1}-(1-\aleph^+_{i_1})^{w_1}}{(1+\aleph^+_{i_1})^{w_1}+(1-\aleph^+_{i_1})^{w_1}}]\},$
$\{\frac{2(\hbar_{i_1})^{w_1}}{(2-\hbar_{i_1})^{w_1}+(\hbar_{i_1})^{w_1}}\} >\},$

$w_2ch_2 = \{< \{[\frac{(1+\aleph^-_{i_2})^{w_2}-(1-\aleph^-_{i_2})^{w_2}}{(1+\aleph^-_{i_2})^{w_2}+(1-\aleph^-_{i_2})^{w_2}}, \frac{(1+\aleph^+_{i_2})^{w_2}-(1-\aleph^+_{i_2})^{w_2}}{(1+\aleph^+_{i_2})^{w_2}+(1-\aleph^+_{i_2})^{w_2}}]\},$
$\{\frac{2(\hbar_{i_2})^{w_2}}{(2-\hbar_{i_2})^{w_2}+(\hbar_{i_2})^{w_2}}\} >\}.$

By applying operational law 1 of Definition 3.1 we have,

$w_1ch_1 \oplus_e w_2ch_2 = \{< \{[\frac{\frac{(1+\aleph^-_{i_1})^{w1}-(1-\aleph^-_{i_1})^{w1}}{(1+\aleph^-_{i_1})^{w1}+(1-\aleph^-_{i_1})^{w1}}+\frac{(1+\aleph^-_{i_2})^{w2}-(1-\aleph^-_{i_2})^{w2}}{(1+\aleph^-_{i_2})^{w2}+(1-\aleph^-_{i_2})^{w2}}}{1+(\frac{(1+\aleph^-_{i_1})^{w1}-(1-\aleph^-_{i_1})^{w1}}{(1+\aleph^-_{i_1})^{w1}+(1-\aleph^-_{i_1})^{w1}})(\frac{(1+\aleph^-_{i_2})^{w2}-(1-\aleph^-_{i_2})^{w2}}{(1+\aleph^-_{i_2})^{w2}+(1-\aleph^-_{i_2})^{w2}})},$
$\frac{\frac{(1+\aleph^+_{i_1})^{w1}-(1-\aleph^+_{i_1})^{w1}}{(1+\aleph^+_{i_1})^{w1}+(1-\aleph^+_{i_1})^{w1}}+\frac{(1+\aleph^+_{i_2})^{w2}-(1-\aleph^+_{i_2})^{w2}}{(1+\aleph^+_{i_2})^{w2}+(1-\aleph^+_{i_2})^{w2}}}{1+(\frac{(1+\aleph^+_{i_1})^{w1}-(1-\aleph^+_{i_1})^{w1}}{(1+\aleph^+_{i_1})^{w1}+(1-\aleph^+_{i_1})^{w1}})(\frac{(1+\aleph^+_{i_2})^{w2}-(1-\aleph^+_{i_2})^{w2}}{(1+\aleph^+_{i_2})^{w2}+(1-\aleph^+_{i_2})^{w2}})}]\},$
$\{\frac{(\frac{2(\hbar_{i_1})^{w1}}{(2-\hbar_{i_1})^{w1}+(\hbar_{i_1})^{w1}})(\frac{2(\hbar_{i_2})^{w2}}{(2-\hbar_{i_2})^{w2}+(\hbar_{i_2})^{w2}})}{1+(1-\frac{2(\hbar_{i_1})^{w1}}{(2-\hbar_{i_1})^{w1}+(\hbar_{i_1})^{w1}})(1-\frac{2(\hbar_{i_2})^{w2}}{(2-\hbar_{i_2})^{w2}+(\hbar_{i_2})^{w2}})}\} >\},$
$= \{< \{[\frac{\Pi^2_{s=1}(1+\aleph^-_{i_s})^{w_s}-\Pi^2_{s=1}(1-\aleph^-_{i_s})^{w_s}}{\Pi^2_{s=1}(1+\aleph^-_{i_s})^{w_s}+\Pi^2_{s=1}(1-\aleph^-_{i_s})^{w_s}}, \frac{\Pi^2_{s=1}(1+\aleph^+_{i_s})^{w_s}-\Pi^2_{s=1}(1-\aleph^+_{i_s})^{w_s}}{\Pi^2_{s=1}(1+\aleph^+_{i_s})^{w_s}+\Pi^2_{s=1}(1-\aleph^+_{i_s})^{w_s}}]\},$
$\{\frac{2\Pi^2_{s=1}(\hbar_{i_s})^{w_s}}{\Pi^2_{s=1}(2-\hbar_{i_s})^{w_s}+\Pi^2_{s=1}(\hbar_{i_s})^{w_s}}\} >\}.$

Suppose the result is true for $n = t$,
$CHFEWA(ch_1, ch_2, ..., ch_t) = (\oplus_e)^t_{s=1}(w_s ch_s)$
$== \{< \{[\frac{\Pi^t_{s=1}(1+\aleph^-_{i_s})^{w_s}-\Pi^t_{s=1}(1-\aleph^-_{i_s})^{w_s}}{\Pi^t_{s=1}(1+\aleph^-_{i_s})^{w_s}+\Pi^t_{s=1}(1-\aleph^-_{i_s})^{w_s}}, \frac{\Pi^t_{s=1}(1+\aleph^+_{i_s})^{w_s}-\Pi^t_{s=1}(1-\aleph^+_{i_s})^{w_s}}{\Pi^t_{s=1}(1+\aleph^+_{i_s})^{w_s}+\Pi^t_{s=1}(1-\aleph^+_{i_s})^{w_s}}]\},$
$\{\frac{2\Pi^t_{s=1}(\hbar_{i_s})^{w_s}}{\Pi^t_{s=1}(2-\hbar_{i_s})^{w_s}+\Pi^t_{s=1}(\hbar_{i_s})^{w_s}}\} >\}.$

Now we have to show that the result holds for $n = t + 1$,
$CHFEWA(ch_1, ch_2, ..., ch_t, ch_{t+1}) = (\oplus_e)^{t+1}_{s=1}(w_s ch_s) = (\oplus_e)^t_{s=1}(w_s ch_s) \oplus_e (w_{t+1}ch_{t+1})$
$= \{< \{[\frac{\Pi^t_{s=1}(1+\aleph^-_{i_s})^{w_s}-\Pi^t_{s=1}(1-\aleph^-_{i_s})^{w_s}}{\Pi^t_{s=1}(1+\aleph^-_{i_s})^{w_s}+\Pi^t_{s=1}(1-\aleph^-_{i_s})^{w_s}}, \frac{\Pi^t_{s=1}(1+\aleph^+_{i_s})^{w_s}-\Pi^t_{s=1}(1-\aleph^+_{i_s})^{w_s}}{\Pi^t_{s=1}(1+\aleph^+_{i_s})^{w_s}+\Pi^t_{s=1}(1-\aleph^+_{i_s})^{w_s}}]\},$
$\{\frac{2\Pi^t_{s=1}(\hbar_{i_s})^{w_s}}{\Pi^t_{s=1}(2-\hbar_{i_s})^{w_s}+\Pi^t_{s=1}(\hbar_{i_s})^{w_s}}\} >\} \oplus_e \{< \{[\frac{(1+\aleph^-_{i_{t+1}})^{w_{t+1}}-(1-\aleph^-_{i_{t+1}})^{w_{t+1}}}{(1+\aleph^-_{i_{t+1}})^{w_{t+1}}+(1-\aleph^-_{i_{t+1}})^{w_{t+1}}},$
$\frac{(1+\aleph^+_{i_{t+1}})^{w_{t+1}}-(1-\aleph^+_{i_{t+1}})^{w_{t+1}}}{(1+\aleph^+_{i_{t+1}})^{w_{t+1}}+(1-\aleph^+_{i_{t+1}})^{w_{t+1}}}]\}, \{\frac{2(\hbar_{i_{t+1}})^{w_{t+1}}}{(2-\hbar_{i_{t+1}})^{w_{t+1}}+(\hbar_{i_{t+1}})^{w_{t+1}}}\} >\}.$

Now again applying operational law 1 of Definition 3.1 we obtain,
$CHFEWA(ch_1, ch_2, ..., ch_t, ch_{t+1}) = (\oplus_e)^{t+1}_{s=1}(w_s ch_s)$
$= \{< \{[\frac{\Pi^{t+1}_{s=1}(1+\aleph^-_{i_s})^{w_s}-\Pi^{t+1}_{s=1}(1-\aleph^-_{i_s})^{w_s}}{\Pi^{t+1}_{s=1}(1+\aleph^-_{i_s})^{w_s}+\Pi^{t+1}_{s=1}(1-\aleph^-_{i_s})^{w_s}}, \frac{\Pi^{t+1}_{s=1}(1+\aleph^+_{i_s})^{w_s}-\Pi^{t+1}_{s=1}(1-\aleph^+_{i_s})^{w_s}}{\Pi^{t+1}_{s=1}(1+\aleph^+_{i_s})^{w_s}+\Pi^{t+1}_{s=1}(1-\aleph^+_{i_s})^{w_s}}]\},$
$\{\frac{2\Pi^{t+1}_{s=1}(\hbar_{i_s})^{w_s}}{\Pi^{t+1}_{s=1}(2-\hbar_{i_s})^{w_s}+\Pi^{t+1}_{s=1}(\hbar_{i_s})^{w_s}}\} >\}.$

Hence the result holds for all n.

Definition 4.3. Let $ch_s(s = 1, 2, ..., n)$ be the number of CHFEs. The cubic hesitant fuzzy Einstein weighted geometric (briefly, CHFEWG) operator is defined by the mapping $G^n \to G$ such that

$$CHFEWG(ch_1, ch_2, ..., ch_n) = (\otimes_e)^n_{s=1}(ch_s^{w_s}).$$

Here $w = (w_1, w_2, ..., w_n)^T$ is a weighting vector of CHFEs where $w_s \in [0, 1]$ and $\sum_{s=1}^n w_s = 1$. As a special case when $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$ then CHFEWG operator becomes CHFEG operator as,

$$CHFEG(ch_1, ch_2, ..., ch_n) = (\otimes_e)^n_{s=1}(ch_s)^{\frac{1}{n}}.$$

Theorem 4.4. *Suppose $ch_s(s=1,2,...,n)$ be the number of CHFEs. Their aggregated result obtained by using CHFEWG operator is a CHFE,*

$$CHFEWG(ch_1, ch_2, ..., ch_n) = (\otimes_e)_{s=1}^{n}(ch_s^{w_s})$$
$$= \{< \{[\frac{2\Pi_{s=1}^{n}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{n}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{n}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{n}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{n}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{n}(\aleph_{i_s}^{+})^{w_s}}]\},$$
$$\{\frac{\Pi_{s=1}^{n}(1+\hbar_{i_s})^{w_s}-\Pi_{s=1}^{n}(1-\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{n}(1+\hbar_{i_s})^{w_s}+\Pi_{s=1}^{n}(1-\hbar_{i_s})^{w_s}}\} >\}.$$

Proof. We show the result by using mathematical induction.

When $n=2$ and by applying operational law 4 of Definition 3.1 we have,

$$ch_1^{w_1} = \{< \{[\frac{2(\aleph_{i_1}^{-})^{w_1}}{(2-\aleph_{i_1}^{-})^{w_1}+(\aleph_{i_1}^{-})^{w_1}}, \frac{2(\aleph_{i_1}^{+})^{w_1}}{(2-\aleph_{i_1}^{+})^{w_1}+(\aleph_{i_1}^{+})^{w_1}}]\}, \{\frac{(1+\hbar_{i_1})^{w_1}-(1-\hbar_{i_1})^{w_1}}{(1+\hbar_{i_1})^{w_1}+(1-\hbar_{i_1})^{w_1}}\} >\},$$
$$ch_2^{w_2} = \{< \{[\frac{2(\aleph_{i_2}^{-})^{w_2}}{(2-\aleph_{i_2}^{-})^{w_2}+(\aleph_{i_2}^{-})^{w_2}}, \frac{2(\aleph_{i_2}^{+})^{w_2}}{(2-\aleph_{i_2}^{+})^{w_2}+(\aleph_{i_2}^{+})^{w_2}}]\}, \{\frac{(1+\hbar_{i_2})^{w_2}-(1-\hbar_{i_2})^{w_2}}{(1+\hbar_{i_2})^{w_2}+(1-\hbar_{i_2})^{w_2}}\} >\}.$$

By applying operational law 2 of Definition 3.1 we have,

$$ch_1^{w_1} \otimes_e ch_2^{w_2} = \{< \{[\frac{(\frac{2(\aleph_{i_1}^{-})^{w_1}}{(2-\aleph_{i_1}^{-})^{w_1}+(\aleph_{i_1}^{-})^{w_1}})(\frac{2(\aleph_{i_2}^{-})^{w_2}}{(2-\aleph_{i_2}^{-})^{w_2}+(\aleph_{i_2}^{-})^{w_2}})}{1+(1-\frac{2(\aleph_{i_1}^{-})^{w_1}}{(2-\aleph_{i_1}^{-})^{w_1}+(\aleph_{i_1}^{-})^{w_1}})(1-\frac{2(\aleph_{i_2}^{-})^{w_2}}{(2-\aleph_{i_2}^{-})^{w_2}+(\aleph_{i_2}^{-})^{w_2}})},$$
$$\frac{(\frac{2(\aleph_{i_1}^{+})^{w_1}}{(2-\aleph_{i_1}^{+})^{w_1}+(\aleph_{i_1}^{+})^{w_1}})(\frac{2(\aleph_{i_2}^{+})^{w_2}}{(2-\aleph_{i_2}^{+})^{w_2}+(\aleph_{i_2}^{+})^{w_2}})}{1+(1-\frac{2(\aleph_{i_1}^{+})^{w_1}}{(2-\aleph_{i_1}^{+})^{w_1}+(\aleph_{i_1}^{+})^{w_1}})(1-\frac{2(\aleph_{i_2}^{+})^{w_2}}{(2-\aleph_{i_2}^{+})^{w_2}+(\aleph_{i_2}^{+})^{w_2}})}]\},$$
$$\{\frac{\frac{(1+\hbar_{i_1})^{w_1}-(1-\hbar_{i_1})^{w_1}}{(1+\hbar_{i_1})^{w_1}+(1-\hbar_{i_1})^{w_1}}+\frac{(1+\hbar_{i_2})^{w_2}-(1-\hbar_{i_2})^{w_2}}{(1+\hbar_{i_2})^{w_2}+(1-\hbar_{i_2})^{w_2}}}{1+(\frac{(1+\hbar_{i_1})^{w_1}-(1-\hbar_{i_1})^{w_1}}{(1+\hbar_{i_1})^{w_1}+(1-\hbar_{i_1})^{w_1}})(\frac{(1+\hbar_{i_2})^{w_2}-(1-\hbar_{i_2})^{w_2}}{(1+\hbar_{i_2})^{w_2}+(1-\hbar_{i_2})^{w_2}})}\} >\},$$
$$ch_1^{w_1} \otimes_e ch_2^{w_2} = \{< \{[\frac{2\Pi_{s=1}^{2}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{2}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{2}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{2}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{2}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{2}(\aleph_{i_s}^{+})^{w_s}}]\},$$
$$\{\frac{\Pi_{s=1}^{2}(1+\hbar_{i_s})^{w_s}-\Pi_{s=1}^{2}(1-\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{2}(1+\hbar_{i_s})^{w_s}+\Pi_{s=1}^{2}(1-\hbar_{i_s})^{w_s}}\} >\}.$$

Suppose the result is true for $n=t$,

$$CHFEWG(ch_1, ch_2, ..., ch_t) = (\otimes_e)_{s=1}^{t}(ch_s^{w_s})$$
$$= \{< \{[\frac{2\Pi_{s=1}^{t}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{t}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{t}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{t}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{t}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{t}(\aleph_{i_s}^{+})^{w_s}}]\}.$$

Now we have to show that the result holds for $n=t+1$,

$$CHFEWG(ch_1, ch_2, ..., ch_t, ch_{t+1}) = (\otimes_e)_{s=1}^{t+1}(ch_s^{w_s}) = (\otimes_e)_{s=1}^{t}(ch_s^{w_s})\otimes_e ch_{t+1}^{w_{t+1}}$$
$$= \{< \{[\frac{2\Pi_{s=1}^{t}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{t}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{t}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{t}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{t}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{t}(\aleph_{i_s}^{+})^{w_s}}]\},$$
$$\{\frac{\Pi_{s=1}^{t}(1+\hbar_{i_s})^{w_s}-\Pi_{s=1}^{t}(1-\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{t}(1+\hbar_{i_s})^{w_s}+\Pi_{s=1}^{t}(1-\hbar_{i_s})^{w_s}}\} >\} \otimes_e \{< \{[\frac{2(\aleph_{i_{t+1}}^{-})^{w_{t+1}}}{(2-\aleph_{i_{t+1}}^{-})^{w_{t+1}}+(\aleph_{i_{t+1}}^{-})^{w_{t+1}}},$$
$$\frac{2(\aleph_{i_{t+1}}^{+})^{w_{t+1}}}{(2-\aleph_{i_{t+1}}^{+})^{w_{t+1}}+(\aleph_{i_{t+1}}^{+})^{w_{t+1}}}]\}, \{\frac{(1+\hbar_{i_{t+1}})^{w_{t+1}}-(1-\hbar_{i_{t+1}})^{w_{t+1}}}{(1+\hbar_{i_{t+1}})^{w_{t+1}}+(1-\hbar_{i_{t+1}})^{w_{t+1}}}\} >\}.$$

Again by applying operational law 2 of Definition 3.1 we get,

$$CHFEWG(ch_1, ch_2, ..., ch_t, ch_{t+1}) = (\otimes_e)_{s=1}^{t+1}(ch_s^{w_s})$$
$$= \{< \{[\frac{2\Pi_{s=1}^{t+1}(\aleph_{i_s}^{-})^{w_s}}{\Pi_{s=1}^{t+1}(2-\aleph_{i_s}^{-})^{w_s}+\Pi_{s=1}^{t+1}(\aleph_{i_s}^{-})^{w_s}}, \frac{2\Pi_{s=1}^{t+1}(\aleph_{i_s}^{+})^{w_s}}{\Pi_{s=1}^{t+1}(2-\aleph_{i_s}^{+})^{w_s}+\Pi_{s=1}^{t+1}(\aleph_{i_s}^{+})^{w_s}}]\},$$
$$\{\frac{\Pi_{s=1}^{t+1}(1+\hbar_{i_s})^{w_s}-\Pi_{s=1}^{t+1}(1-\hbar_{i_s})^{w_s}}{\Pi_{s=1}^{t+1}(1+\hbar_{i_s})^{w_s}+\Pi_{s=1}^{t+1}(1-\hbar_{i_s})^{w_s}}\} >\}.$$

Hence the result holds for all n.

5. Multi criteria decision making (MCDM) using einstein weighted averaging and geometric aggregation operators for CHFSs

In this section, CHFEWA and CHFEWG operators are applied to MCDM with cubic hesitant fuzzy information. Let $\check{D} = \{\check{D}_1, \check{D}_2, \check{D}_3, ...\check{D}_q\}$ be a finite set of q alternatives. Let $\breve{U} = \{\breve{U}_1, \breve{U}_2, \breve{U}_3, ..., \breve{U}_p\}$ be a finite set of p criteria. Let $w = (w_1, w_2, ..., w_p)^T$ be the weight vector of criteria's $\breve{U} = \{\breve{U}_1, \breve{U}_2, \breve{U}_3, ..., \breve{U}_p\}$ with $w_l, 1 \leq l \leq p$ and $\sum_{l=1}^{p} w_l = 1$. Suppose $\breve{E} = (ch_{js})_{m\times n}$ is cubic hesitant fuzzy matrix, where $ch_{js} = \{< \{[\aleph_i^-, \aleph_i^+]\}, \{\hbar_i\} >\}$ is the CHFE given for alternative $\check{D}_j (j = 1, 2, ..., q)$ with respect to criteria $\breve{U}_s (s = 1, 2, ...p)$. The CHFEWA and CHFEWG operators are utilized to develop a MCDM method with cubic hesitant fuzzy information by the following steps.

Algorithm 1;

Algorithm 1

Input: A set of criteria associated with alternatives.
Output: Best alternative.

1: Obtian a set of criteria associated with a set of alternatives.
2: Obtain CHFEWA and CHFEWG result through the weight vector associated with the mapping $G^n \to G$ and by applying Definition 4.1 and Definition 4.3 respectively.
3: Calculate the scores of CHFEWA and CHFEWG results by using Definition 2.10.
4: Rank given alternatives $\check{D}_m, 1 \leq m \leq q$ by applying accuracy procedure through score values.
5: Obtain a best choice on a maximum score.

Next, an example is presented to select best alternative from a set of alternatives utilizing Algorithm-1.

5.1 Illustrative example

Let an organization interested to build a sugar industry in Guangzhou, China. The three districts namely, Conghua ($\check{D}_1$), Nansha ($\check{D}_2$) and Huangpu ($\check{D}_3$) are alternative districts to build a sugar industry. In each district a place is selected to build a sugar industry. Since different factors need to manage before selecting a district for industry. Then the organization mainly consider three main criterias "Waste management ($\breve{U}_1$)", "Water resources planning ($\breve{U}_2$)" and "Population of area ($\breve{U}_3$)". A team of experts are selected to collect data from three districts and experts give their data on each criteria as a CHF number, that is, $< t, u(t), g(t) >$. In Table 1 CHFS values are given.

A set of weights on CHFEs is given as $w = (w_1, w_2, w_3)^T = (0.2, 0.3, 0.5)^T$. Now to calculate CHFEWG and CHFEWA, first ch_{mn}, $n, m = 1, 2, 3$ cumulated as follows:

Table 1: Matrix containing ICHFS values

	$\check{U}_1$	$\check{U}_2$	$\check{U}_3$
$\check{D}_1$	$\left\{ \begin{array}{c} <[0.1,0.2],[0.3,0.5],[0.7,0.9], \\ \{0.15,0.4,0.8\}> \end{array} \right\}$	$\{<[0.4,0.6],[0.7,0.9],\{0.5,0.8\}>\}$	$\{<[0.6,0.8],\{0.7\}>\}$,
$\check{D}_2$	$\{<[0.4,0.7],\{0.6\}>\}$	$\left\{ \begin{array}{c} <[0.2,0.4],[0.5,0.7],[0.8,0.9], \\ \{0.3,0.6,0.85\}> \end{array} \right\}$	$\{<[0.3,0.5],[0.4,0.7],\{0.4,0.6\}>\}$,
$\check{D}_3$	$\{<[0.2,0.5],[0.6,0.9],\{0.4,0.7\}>\}$	$\{<[0.5,0.9],\{0.7\}>\}$	$\left\{ \begin{array}{c} <[0.3,0.4],[0.5,0.6],[0.7,0.9], \\ \{0.35,0.55,0.8\}> \end{array} \right\}$,

$ch_{11} = \{<\{[0.1, 0.2], [0.3, 0.5], [0.7, 0.9]\}, \{0.15, 0.4, 0.8\}>\}$,
$ch_{12} = \{<\{[0.4, 0.6], [0.7, 0.9]\}, \{0.5, 0.8\}>\}$,
$ch_{13} = \{<\{[0.6, 0.8]\}, \{0.7\}>\}$.
Next, CHFEWG are calculated as follows;
$CHFEWG(ch_{11}, ch_{12}, ch_{13}) = \phi_1 = ((\otimes_e)_{s=1}^3)_{j=1}(ch_{js}^{w_s})$
$= \{<\{[\frac{2\Pi_{s=1}^3(\aleph_{i_{1s}}^-)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{1s}}^-)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{1s}}^-)^{w_s}}, \frac{2\Pi_{s=1}^3(\aleph_{i_{1s}}^+)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{1s}}^+)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{1s}}^+)^{w_s}}]\}$,
$\{\frac{\Pi_{s=1}^3(1+\hbar_{i_{1s}})^{w_s}-\Pi_{s=1}^3(1-\hbar_{i_{1s}})^{w_s}}{\Pi_{s=1}^3(1+\hbar_{i_{1s}})^{w_s}+\Pi_{s=1}^3(1-\hbar_{i_{1s}})^{w_s}}\}>\}$,
$CHFEWG(ch_{11}, ch_{12}, ch_{13}) = \phi_1 = \{<\{[0.3867, 0.5796], [0.4678, 0.6740]$,
$[0.5524, 0.7565], [0.4636, 0.6626], [0.5552, 0.7632], [0.6490, 0.8497]\}$,
$\{0.5571, 0.5936, 0.6741, 0.6604, 0.6900, 0.7543\}>\}$,
$ch_{21} = \{<\{[0.4, 0.7]\}, \{0.6\}>\}$,
$ch_{22} = \{<\{[0.2, 0.4], [0.5, 0.7], [0.8, 0.9]\}, \{0.3, 0.6, 0.85\}>\}$,
$ch_{23} = \{<\{[0.3, 0.5], [0.4, 0.7]\}, \{0.4, 0.6\}>\}$,
$CHFEWG(ch_{21}, ch_{22}, ch_{23}) = \phi_2 = ((\otimes_e)_{s=1}^3)_{j=2}(ch_{js}^{w_s})$
$= \{<\{[\frac{2\Pi_{s=1}^3(\aleph_{i_{2s}}^-)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{2s}}^-)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{2s}}^-)^{w_s}}, \frac{2\Pi_{s=1}^3(\aleph_{i_{2s}}^+)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{2s}}^+)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{2s}}^+)^{w_s}}]\}$,
$\{\frac{\Pi_{s=1}^3(1+\hbar_{i_{2s}})^{w_s}-\Pi_{s=1}^3(1-\hbar_{i_{2s}})^{w_s}}{\Pi_{s=1}^3(1+\hbar_{i_{2s}})^{w_s}+\Pi_{s=1}^3(1-\hbar_{i_{2s}})^{w_s}}\}>\}$,
$CHFEWG(ch_{21}, ch_{22}, ch_{23}) = \phi_2 = \{<\{[0.2828, 0.5036], [0.3278, 0.5992]$,
$[0.3726, 0.5952], [0.4283, 0.7000], [0.4398, 0.6489], [0.5024, 0.7581]\}$,
$\{0.4164, 0.5213, 0.5068, 0.6000, 0.6214, 0.6973\}>\}$,
$ch_{31} = \{<\{[0.2, 0.5], [0.6, 0.9]\}, \{0.4, 0.7\}>\}$,
$ch_{32} = \{<\{[0.5, 0.9]\}, \{0.7\}>\}$,
$ch_{33} = \{<\{[0.3, 0.4], [0.5, 0.6], [0.7, 0.9]\}, \{0.35, 0.55, 0.8\}>\}$,
$CHFEWG(ch_{31}, ch_{32}, ch_{33}) = \phi_3 = ((\otimes_e)_{s=1}^3)_{j=3}(ch_{js}^{w_s})$
$= \{<\{[\frac{2\Pi_{s=1}^3(\aleph_{i_{3s}}^-)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{3s}}^-)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{3s}}^-)^{w_s}}, \frac{2\Pi_{s=1}^3(\aleph_{i_{3s}}^+)^{w_s}}{\Pi_{s=1}^3(2-\aleph_{i_{3s}}^+)^{w_s}+\Pi_{s=1}^3(\aleph_{i_{3s}}^+)^{w_s}}]\}$,
$CHFEWG(ch_{31}, ch_{32}, ch_{33}) = \phi_3 = \{<\{[0.3259, 0.5485], [0.4222, 0.6621]$,
$[0.5076, 0.8121], [0.4064, 0.6229], [0.5191, 0.7439], [0.6164, 0.9000]\}$,
$\{0.4836, 0.5744, 0.7135, 0.5486, 0.6308, 0.7543\}>\}$.
The operators CHFEWA are calculated as in following:
$CHFEWA(ch_{11}, ch_{12}, ch_{13}) = \delta_1 = \left((\oplus_e)_{s=1}^3\right)_{j=1}(w_s ch_{js})$
$= \{<\{[\frac{\Pi_{s=1}^3(1+\aleph_{i_{1s}}^-)^{w_s}-\Pi_{s=1}^3(1-\aleph_{i_{1s}}^-)^{w_s}}{\Pi_{s=1}^3(1+\aleph_{i_{1s}}^-)^{w_s}+\Pi_{s=1}^3(1-\aleph_{i_{1s}}^-)^{w_s}}, \frac{\Pi_{s=1}^3(1+\aleph_{i_{1s}}^+)^{w_s}-\Pi_{s=1}^3(1-\aleph_{i_{1s}}^+)^{w_s}}{\Pi_{s=1}^3(1+\aleph_{i_{1s}}^+)^{w_s}+\Pi_{s=1}^3(1-\aleph_{i_{1s}}^+)^{w_s}}]\}$,

$\{\frac{2\Pi_{s=1}^{3}(\hbar_{i_{1s}})^{w_s}}{\Pi_{s=1}^{3}(2-\hbar_{i_{1s}})^{w_s}+\Pi_{s=1}^{3}(\hbar_{i_{1s}})^{w_s}}\} >\}$,

$CHFEWA(ch_{11}, ch_{12}, ch_{13}) = \delta_1 = \{< \{[0.4572, 0.6628], [0.5559, 0.7745], [0.4896, 0.6999], [0.5841, 0.8008], [0.5697, 0.7824], [0.6528, 0.8579]\}, \{0.4841, 0.5644, 0.5714, 0.6599, 0.6547, 0.7494\} >\}$,

$CHFEWA(ch_{21}, ch_{22}, ch_{23}) = \delta_2 = \left((\oplus_e)_{s=1}^{3}\right)_{j=2}(w_s ch_{js})$

$= \{< \{[\frac{\Pi_{s=1}^{3}(1+\aleph_{i_{2s}}^{-})^{w_s}-\Pi_{s=1}^{3}(1-\aleph_{i_{2s}}^{-})^{w_s}}{\Pi_{s=1}^{3}(1+\aleph_{i_{2s}}^{-})^{w_s}+\Pi_{s=1}^{3}(1-\aleph_{i_{2s}}^{-})^{w_s}}, \frac{\Pi_{s=1}^{3}(1+\aleph_{i_{2s}}^{+})^{w_s}-\Pi_{s=1}^{3}(1-\aleph_{i_{2s}}^{+})^{w_s}}{\Pi_{s=1}^{3}(1+\aleph_{i_{2s}}^{+})^{w_s}+\Pi_{s=1}^{3}(1-\aleph_{i_{2s}}^{+})^{w_s}}]\}$,

$\{\frac{2\Pi_{s=1}^{3}(\hbar_{i_{2s}})^{w_s}}{\Pi_{s=1}^{3}(2-\hbar_{i_{2s}})^{w_s}+\Pi_{s=1}^{3}(\hbar_{i_{2s}})^{w_s}}\} >\}$,

$CHFEWA(ch_{21}, ch_{22}, ch_{23}) = \delta_2 = \{< \{[0.2916, 0.5192], [0.3429, 0.6256], [0.3836, 0.6096], [0.4312, 0.7000], [0.5147, 0.7113], [0.5554, 0.7813]\}, \{0.4010, 0.4944, 0.4932, 0.6000, 0.5564, 0.6708\} >\}$,

$CHFEWA(ch_{31}, ch_{32}, ch_{33}) = \delta_3 = \left((\oplus_e)_{s=1}^{3}\right)_{j=3}(w_s ch_{js})$

$= \{< \{[\frac{\Pi_{s=1}^{3}(1+\aleph_{i_{3s}}^{-})^{w_s}-\Pi_{s=1}^{3}(1-\aleph_{i_{3s}}^{-})^{w_s}}{\Pi_{s=1}^{3}(1+\aleph_{i_{3s}}^{-})^{w_s}+\Pi_{s=1}^{3}(1-\aleph_{i_{3s}}^{-})^{w_s}}, \frac{\Pi_{s=1}^{3}(1+\aleph_{i_{3s}}^{+})^{w_s}-\Pi_{s=1}^{3}(1-\aleph_{i_{3s}}^{+})^{w_s}}{\Pi_{s=1}^{3}(1+\aleph_{i_{3s}}^{+})^{w_s}+\Pi_{s=1}^{3}(1-\aleph_{i_{3s}}^{+})^{w_s}}]\}$,

$\{\frac{2\Pi_{s=1}^{3}(\hbar_{i_{3s}})^{w_s}}{\Pi_{s=1}^{3}(2-\hbar_{i_{3s}})^{w_s}+\Pi_{s=1}^{3}(\hbar_{i_{3s}})^{w_s}}\} >\}$,

$CHFEWA(ch_{31}, ch_{32}, ch_{33}) = \delta_3 = \{< \{[0.3453, 0.6431], [0.4462, 0.7154], [0.5642, 0.8585], [0.4286, 0.7388], [0.5212, 0.7942], [0.6274, 0.9000]\}, \{0.4495, 0.5587, 0.6789, 0.5052, 0.6226, 0.7494\} >\}$.

Step 2

$S(\phi_1) = 0.4408, S(\phi_2) = 0.2935, S(\phi_3) = 0.3994$,

$S(\delta_1) = 0.4643, S(\delta_2) = 0.3069, S(\delta_3) = 0.4289$.

Step 3

Since $S(\phi_1) > S(\phi_3) > S(\phi_2)$ so by accuracy procedure $\phi_1 > \phi_3 > \phi_2$.

Also $S(\delta_1) > S(\delta_3) > S(\delta_2)$ hence by accuracy procedure we have $\delta_1 > \delta_3 > \delta_2$.

So we conclude that by applying both averaging and geometric operators we have the same result.

Hence $\check{D}_1$ is the best choice.

6. Conclusion

In our work we defined Einstein operations for CHFSs and also proved their important results. We defined CHFEWA and CHFEWG operators which are necessary and important to handle and aggregate the data given by decision makers. Cubic hesitant fuzzy sets provide such type of information which is so complicated because of its hesitancy factor. T. Mahmood et al [12] gave some generalized aggregation operators for CHFSs which worked very effectively to solve these problems. But there was a need to provide an easy and effective approach which provide more accuracy then the previous methods. So Einstein operations are the one which gave the solution to our problem.

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Accepted: 8.11.2018