The Crank-Nicolson-Galerkin FEM for a nonlocal parabolic system

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Abstract. A theoretical analysis of a linearized Crank-Nicolson Galerkin finite element method for the nonlocal nonlinear coupled system of the reaction-diffusion problem is presented here. For sufficiently smooth solutions, the maximal error in the L^2-norm possesses the optimal rate of convergence $O(\delta^2 + h^{r+1})$ (where h is the mesh size and δ is the time step size with $r \geq 1$) without any time step restriction. Some important results on the energy decay and vanishing of the solutions in finite time are also presented. To confirm our theoretical analysis, some numerical experiments are performed using Matlab.
Keywords: Optimal error, Crank-Nicolson schemes, finite element method, nonlocal diffusion.

1. Introduction

The following system of reaction-diffusion equations with a specific type nonlocal non-linearity is the focus of this research article.

$$(1.1)\quad \begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \alpha_1 |u|^{p-2}u = f_1(x,t) & \text{in} \quad \Omega \times (0,T) \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \alpha_2 |v|^{p-2}v = f_2(x,t) & \text{in} \quad \Omega \times (0,T) \\ u(x,t) = v(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

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Where $\Omega \subset \mathbb{R}^d$ is a domain with a smooth boundary $\partial\Omega$.
The reaction-diffusion equation-model is one of the most successful models for describing physical, chemical, biological and ecological systems. Amongst them, the most successful and crucial one is the following model of parabolic partial differential equations, called the reaction-diffusion problem

$$u_t - A\Delta u - f(u) = 0, \tag{1.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear function. For $n \geq 2$, A is an $n\times n$ real matrix of diffusion (see [1] and references therein) and for $n = 1$, A is a real function called diffusion coefficient (see [2, 3]). However, cross-diffusion phenomena are not uncommon (see [4] and references therein) (1.2) can be treated as a system of equations in which A is not even diagonalizable. In 2008, Raposo et al. [5] studied the reaction-diffusion coupled system in a parallel way via a parameter $\alpha = const > 0$, of the form

$$\begin{cases} u_t - a(l(u))\Delta u + f(u - v) = \alpha(u - v) & \text{in} \quad \Omega \times (0, T) \\ v_t - a(l(v))\Delta v - f(u - v) = \alpha(v - u) & \text{in} \quad \Omega \times (0, T), \end{cases} \tag{1.3}$$

and they proved the existence, uniqueness and exponential decay of solutions using the energy method. They also presented some numerical simulations using an implicit finite difference scheme in one dimension and a finite volume method without an analysis in two dimension.

Recently, Duque et al.[6] considered problem (1.1). Here u and v described the densities of two populations that interact through the functions a_1 and a_2. The death in species u is assumed to be proportional to $|u|^{p-2}u$ by the factor $\alpha_1 \geq 0$ and in species v to be proportional to $|v|^{p-2}v$ by the factor $\alpha_2 \geq 0$. The external sources are denoted by f_1 and f_2. Existence and uniqueness of weak and strong solutions were proved, globally in time, and conditions on the data so that these solutions have the waiting time and stable localization properties were given. Important results on polynomial and exponential decay and vanishing of the solutions in finite time were also presented.

There is a vast literature on finite element methods for nonlinear elliptic and parabolic problems, for instance [7, 8, 9] and references therein. But there are few studies on numerical approximations of nonlocal problems. Mbehou et al. [10] proposed a linearized Cranck-Nicolson finite element methods for a nonlocal nonlinear reaction-diffusion equation. Duque et al. [11] proposed the $P1$ or linear approximation finite element method in space variables and the Euler method for the time variable for the reaction-diffusion system (1.1) and derived convergence of order $O(\delta + h^2)$ in the L^2 norm. Almeida et al. [7] presented the convergence analysis of the fully discretized in the finite element method in space variables and the Crank-Nicolson method in time variables for a nonlocal parabolic equation with moving boundaries.

The main objective of this research article is to prove the convergence of a total discrete solution using the Crank-Nicolson-Galerkin finite element method.

To the best of our knowledge, these results are new for the nonlocal reaction-diffusion coupled system (1.1). This paper is organized as follows. Section 2 is concerned with the weak variational formulation of the problem and the hypotheses on the data. In Section 3, we present the semidiscrete solution and its convergence while in Section 4, we derive the optimal L^2 error estimate of the fully discrete Galerkin method unconditionnally. Finally numerical results are presented in Section 5 to demonstrate our theoretical analysis.

2. Preliminaries and weak formulation

$H^k(\Omega)$ is the usual Sobolev space of order $k \in \mathbb{N}$ with norm $\|\cdot\|_k$. $H_0^k(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^k(\Omega)$. $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. The Lebesgue space is denoted as $L^r(\Omega)$, $1 < r \le \infty$, with norm $\|\cdot\|_{L^r}$ (the $L^2(\Omega)$-norm is denoted by $\|\cdot\|$). Moreover, we employ the standard notation of Bochner spaces such as $L^q(0,T,X)$, with norm $\|\cdot\|_{L^q(X)}$, where X is a Hilbert space.

Following Duque et al. [6],the existence and uniqueness of strong solutions of the reaction–diffusion system (1.1) in which α_1, α_2 are nonnegative constants and $p > 1$ were proven under the following:

Hyp 1: $u_0,\, v_0 \in L^2(\Omega)$.

Hyp 2: $a_i : \mathbb{R}^2 \to \mathbb{R}$ is bounded with $0 < m \le a_i(p,q) \le M$, $p, q \in \mathbb{R}$, $i = 1, 2$.

Hyp 3: $a_i : \mathbb{R}^2 \to \mathbb{R}$ and $|a_i(p_1,q_1) - a_i(p_2,q_2)| \le M_i|p_1 - p_2| + K_i|q_1 - q_2|$, $p_i, q_i \in \mathbb{R}$, $i = 1, 2$, a_i Lipschitz–continuous.

Hyp 4: $l_i : L^2(\Omega) \to \mathbb{R}$ is a continuous linear form, that is, there exists a positive function $g_i \in L^2(\Omega)$ such that $l_i(w) = l_{g_i}(w) = \int_\Omega g_i(x)w(x)dx, \quad \text{for all} \quad w \in L^2(\Omega)$, $i = 1, 2$.

Hyp 5: $f_1,\, f_2 \in L^2(0,T,L^2(\Omega))$.

The proof of the following results (Theorem 2.1, Theorem 2.2 and Theorem 2.3) can be found in [6].

Theorem 2.1. *Let $p > 1$ and $0 < T < \infty$. If* (**Hyp 1**) – (**Hyp 5**) *hold, then system* (1.1) *admits a unique solution, that is, there exists a unique couple* (u, v) *such that*

$$u,\, v \in L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H_0^1(\Omega)) \cap L^p(0,T,L^p(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)), \tag{2.1}$$

$$u_t,\, v_t \in L^2(0,T,H^{-1}(\Omega)), \tag{2.2}$$

$$\frac{d}{dt}\int_\Omega u\phi\,dx + a_1(l_1(u),l_2(v))\int_\Omega \nabla u\cdot\nabla\phi\,dx + \alpha_1\int_\Omega |u|^{p-2}u\phi\,dx = \int_\Omega f_1\phi\,dx, \tag{2.3}$$

$$\frac{d}{dt}\int_\Omega v\psi\,dx + a_2(l_1(u), l_2(v))\int_\Omega \nabla v\cdot\nabla\psi\,dx + \alpha_2\int_\Omega |v|^{p-2}v\psi\,dx$$
$$= \int_\Omega f_2\psi\,dx, \tag{2.4}$$

for all $\phi, \psi \in H_0^1(\Omega)$, *where* (2.3) *and* (2.4) *must be understood as an equality in* $\mathcal{D}'(0,T)$,

$$u(x,t) = v(x,t) = 0, \quad on \quad \partial\Omega\times]0,T], \tag{2.5}$$
$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad in \quad \Omega. \tag{2.6}$$

If more regularity is imposed on u_0 and v_0, then

$$u_t,\ v_t, \Delta u,\ \Delta v \in L^2(0,T,L^2(\Omega)). \tag{2.7}$$

Denote the global energy function by

$$E(t) = \frac{1}{2}(\|u(t)\|^2 + \|v(t)\|^2).$$

Theorem 2.2. *If* $f_1 \equiv f_2 \equiv 0$ *and* (u,v) *the solution of* (1.1) *then*

$$E(t) \le E(0)\exp(-\beta t),$$

where β *is a positive constant which depends on* m *and* Ω.

Moreover if $p > 2$, *then*

$$E(t)^{\mu-1} \le \frac{E(0)^{\mu-1}}{1 + tC(\mu-1)E(0)^{\mu-1}},$$

where $\mu = p/2$ *and* C *a positive constant which depends only on* N, p *and* Ω.

Theorem 2.3. *Suppose that* $1 < p < 2$. *If* $f_1 \equiv f_2 \equiv 0$ *and* (u,v) *the solution of* (1.1) *then* (u,v) *vanishes in a finite time, that is, there exists* t^* *which depends only on* $\|u_0\|$, $\|v_0\|$, p *and* Ω *such that*

$$u(x,t) \equiv 0 \quad and \quad v(x,t) \equiv 0 \quad in \quad \Omega \quad for \quad t > t^*.$$

3. Semi discrete approximation

Following classical finite element method theory [12], we define $\mathcal{T}_h$ as a regular triangulation of Ω into elements K. Let $h = \max_{K\in\mathcal{T}_h}\{h_K\}$ denote the mesh size, where $h_K = diam(K) = \max\{\|\boldsymbol{x}-\boldsymbol{y}\|,\ \boldsymbol{x},\ \boldsymbol{y} \in K\}$. Let V_h be the finite dimensional subspace of $H_0^1(\Omega)$, which consists of continuous piecewise polynomials of degree $r \ge 1$ on $\mathcal{T}_h$.

Let Π_h be an interpolation operator and let $R_h : H_0^1(\Omega) \to V_h$ be a Ritz projection operator defined by

$$\int_\Omega \nabla(u - R_h u)\cdot\nabla w dx = 0 \quad \forall w \in H_0^1(\Omega). \tag{3.1}$$

Lemma 3.1 (cf. Ref. [12]). *If* $u \in H^{r+1}(\Omega) \cap H_0^1(\Omega)$, *then*

$$\|u - \Pi_h u\| + h\|\nabla(u - \Pi_h u)\| \leq Ch^{r+1}\|u\|_{H^{r+1}} \tag{3.2}$$

$$\|u - R_h u\| + h\|\nabla(u - R_h u)\| \leq Ch^{r+1}\|u\|_{H^{r+1}}, \tag{3.3}$$

where C *is a positive constant which does not depend on* h *and* r.

Lemma 3.2 (cf. Ref. [13]). *For all* $p \in (1,\infty)$ *and* $\tau \geq 0$, *there exist a generic constant* $C = C(p,d)$ *such that for all* ξ, $\eta \in \mathbb{R}^d$ *with* $d \geq 1$

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq C|\xi - \eta|^{1-\tau}(|\xi| + |\eta|)^{p-2+\tau} \tag{3.4}$$

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi - \eta) \geq C|\xi - \eta|^{2+\tau}(|\xi| + |\eta|)^{p-2-\tau}. \tag{3.5}$$

The semidiscrete problem, based on Theorem 2.1 consists of finding u_h, $v_h \in V_h$, for $t \geq 0$, such that

$$\begin{aligned}
&\int_\Omega (u_h)_t\phi_h\,dx + a_1(l_1(u_h), l_2(v_h))\int_\Omega \nabla u_h\cdot\nabla\phi_h\,dx \\
&+ \alpha_1\int_\Omega |u_h|^{p-2}u_h\phi_h\,dx = \int_\Omega f_1\phi_h\,dx, \\
&\int_\Omega (v_h)_t\psi_h\,dx + a_2(l_1(u_h), l_2(v_h))\int_\Omega \nabla v_h\cdot\nabla\psi_h\,dx \\
&+ \alpha_2\int_\Omega |v_h|^{p-2}v_h\psi_h\,dx = \int_\Omega f_2\psi_h\,dx, \\
&u_h(x,0) = \Pi_h u_0, \quad v_h(x,0) = \Pi_h v_0,
\end{aligned} \tag{3.6}$$

for all $\phi_h, \psi_h \in V_h$.

We have the following result.

Theorem 3.1. *The semidiscrete problem* (3.6) *admits a unique solution* (u_h, v_h) *and if* (u,v) *is the solution of* (1.1), *then*

$$\|u - u_h\| + \|v - v_h\| \leq Ch^{r+1}, \quad \textit{for all} \quad t \in (0,T), \tag{3.7}$$

where C *does not depend on* h.

Proof. Note that (3.6) is equivalent to a system of ordinary differential equations of first order, so the existence of the solution follows from Caratheodory's theorem. The proof of uniqueness is similar to that of Theorem 6 in Ref. [6]. To prove the convergence, let us write $u_h - u = (u_h - R_h u) + (R_h u - u) = \Theta_h^u + \eta^u$ and $v_h - v = (v_h - R_h v) + (R_h v - v) = \Theta_h^v + \eta^v$ From Lemma 3.1, we have

$$\|\eta^u\| + \|\eta^v\| \leq Ch^{r+1}\|u\|_{H^{r+1}}\ .$$

Now to estimate Θ_h^u and Θ_h^v, we have

$$\begin{aligned}
&\int_\Omega (\Theta_h^u)_t \phi_h \, dx + a_1(l_1(u_h), l_2(v_h)) \int_\Omega \nabla \Theta_h^u \cdot \nabla \phi_h \, dx \\
&\qquad = \int_\Omega (u_h)_t \phi_h dx + a_1(l_1(u_h), l_2(v_h)) \int_\Omega \nabla u_h \cdot \nabla \phi_h \, dx \\
&\qquad - \int_\Omega (R_h u)_t \phi_h \, dx - a_1(l_1(u_h), l_2(v_h)) \int_\Omega \nabla (R_h u) \cdot \nabla \phi_h \, dx \\
&\qquad = \int_\Omega (u - R_h u)_t \phi_h \, dx + \alpha_1 \int_\Omega (|u|^{p-2} u - |u_h|^{p-2} u_h) \phi_h \, dx \\
&\qquad + (a_1(l_1(u), l_2(v)) - a_1(l_1(u_h), l_2(v_h))) \int_\Omega \nabla u \cdot \nabla \phi_h \, dx.
\end{aligned}$$

If we take $\phi_h = \Theta_h^u$, on using the hypotheses (**Hyp 2**), (**Hyp 3**), (**Hyp 4**) and Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Theta_h^u\|^2 + m\|\nabla \Theta_h^u\|^2 &\le \|\eta_t^u\|\|\Theta_h^u\| + C_1(u, u_h)\|u - u_h\|\|\Theta_h^u\| \\
&+ C_2(M_1, K_1, u)(\|u - u_h\| + \|v - v_h\|)\|\nabla \Theta_h^u\| \,,
\end{aligned} \tag{3.8}$$

where $C_1(u, u) = C(\|u\|_{L^\infty(L^2(\Omega))}, \|u_h\|_{L^\infty(L^2(\Omega))})$ and $C_2(M_1, K_1, u) = C(M_1, K_1, \|\nabla u\|_{L^\infty(L^2(\Omega))})$ are constants. Then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Theta_h^u\|^2 + m\|\nabla \Theta_h^u\|^2 \le C\left\{\|\eta_t^u\|^2 + \|\eta^u\|^2 + \|\eta^v\|^2 + \|\Theta_h^u\|^2 + \|\Theta_h^v\|^2\right\} \\
&+ m\|\nabla \Theta_h^u\|^2 \,.
\end{aligned} \tag{3.9}$$

If we use the same argument for Θ_h^v, we will have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Theta_h^v\|^2 + m\|\nabla \Theta_h^v\|^2 \\
&\le C\left\{\|\eta_t^v\|^2 + \|\eta^u\|^2 + \|\eta^v\|^2 + \|\Theta_h^u\|^2 + \|\Theta_h^v\|^2\right\} + m\|\nabla \Theta_h^v\|^2 \,.
\end{aligned} \tag{3.10}$$

Adding (3.9) and (3.10), we have

$$\frac{1}{2}\frac{d}{dt}\|\Theta_h^u\|^2 + \frac{1}{2}\frac{d}{dt}\|\Theta_h^v\|^2 \le C\left\{\|\eta_t^v\|^2 + \|\eta^u\|^2 + \|\eta^v\|^2 + \|\Theta_h^u\|^2 + \|\Theta_h^v\|^2\right\}$$

and integrating with respect to time on the interval $[0, t]$ we obtain

$$\begin{aligned}
\|\Theta_h^u\|^2 + \|\Theta_h^v\|^2 &\le \|\Theta_h^u(0)\|^2 + \|\Theta_h^v(0)\|^2 + C\int_0^T (\|\eta_t^u(s)\|^2 \\
&+ \|\eta_t^v(s)\|^2 + \|\eta^u(s)\|^2 + \|\eta^v(s)\|^2) \, ds \,,
\end{aligned}$$

so the inequality (3.7) follows from Lemma 3.1. □

4. Fully discrete problem

Let $\{t_n| \, t_n = n\delta;\, 0 \le n \le N\}$ be a uniform partition of $[0,T]$ with the time step $\delta = T/N$, and $t_{n-\frac{1}{2}} = \frac{1}{2}(t_n + t_{n-1})$. We denote w^m by $w^m = w(x,t_m)$. For a sequence of functions $\{w^m\}_{n=0}^N$, we define

$$\bar{\partial} w^n = \frac{w^n - w^{n-1}}{\delta}, \quad \bar{w}^n = \frac{1}{2}(w^n + w^{n-1}),\ n = 1, 2..., N,$$
$$\hat{w}^n = \frac{1}{2}(3w^{n-1} - w^{n-2}), \quad n = 2, ..., N.$$

With above notations, a linearized Crank-Nicolson FEM is as follows: Find $u_h^n,\ v_h^n \in V_h$ such that

$$\begin{aligned}
&\int_\Omega \bar{\partial} u_h^n \phi_h \, dx + a_1(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n)) \int_\Omega \nabla \bar{u}_h^n \cdot \nabla \phi_h \, dx \\
&+ \alpha_1 \int_\Omega |\hat{u}_h^n|^{p-2} \bar{u}_h^n \phi_h \, dx = \int_\Omega f_1^{n-1/2} \phi_h \, dx, \\
&\int_\Omega \bar{\partial} v_h^n \psi_h \, dx + a_2(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n)) \int_\Omega \nabla \bar{v}_h^n \cdot \nabla \psi_h \, dx \\
&+ \alpha_2 \int_\Omega |\hat{v}_h^n|^{p-2} \bar{v}_h^n \psi_h \, dx = \int_\Omega f_2^{n-1/2} \psi_h \, dx, \\
&u_h^0 = \Pi_h u_0, \quad v_h^0 = \Pi_h v_0, \quad n = 1, ..., N, \quad \text{for all} \quad \phi_h, \psi_h \in V_h,
\end{aligned} \tag{4.1}$$

where $(\hat{u}_h^1,\ \hat{v}_h^1)$ is the solution of the following system of equations.

$$\begin{aligned}
&\int_\Omega \frac{\hat{u}_h^1 - u_h^0}{\delta/2} \phi_h \, dx + a_1(l_1(u_h^0), l_2(v_h^0)) \int_\Omega \nabla \hat{u}_h^1 \cdot \nabla \phi_h \, dx \\
&+ \alpha_1 \int_\Omega |u_h^0|^{p-2} \hat{u}_h^1 \phi_h \, dx = \int_\Omega f_1^{1/2} \phi_h \, dx, \\
&\int_\Omega \frac{\hat{v}_h^1 - v_h^0}{\delta/2} \psi_h \, dx + a_2(l_1(u_h^0), l_2(v_h^0)) \int_\Omega \nabla \hat{v}_h^1 \cdot \nabla \psi_h \, dx \\
&+ \alpha_2 \int_\Omega |v_h^0|^{p-2} \hat{v}_h^1 \psi_h \, dx = \int_\Omega f_2^{1/2} \psi_h \, dx.
\end{aligned} \tag{4.2}$$

Theorem 4.1. *Let $p \ge 2$. If (u,v) is the solution of system* (1.1) *and (u_h^n, v_h^n) is the solution of* (4.1) – (**??**) *then*

$$\|u^n - u_h^n\| + \|v^n - v_h^n\| \le C(h^{r+1} + \delta^2), \tag{4.3}$$

where C is a generic constant which does not depend on h and δ.

Proof. We write

$$u_h^n - u^n = (u_h^n - R_h u^n) + (R_h u^n - u^n) = \Theta_n^u + \Psi_n^u$$

and

$$v_h^n - v^n = (v_h^n - R_h v^n) + (R_h v^n - v^n) = \Theta_n^v + \Psi_n^v.$$

The estimations of Ψ_n^u and Ψ_n^v are given by Lemma 3.1. To estimate Θ_n^u and Θ_n^v, we have

$$\begin{aligned}
&\int_\Omega \bar\partial \Theta_n^u \phi_h dx + a_1(l_1(\hat u_h^n), l_2(\hat v_h^n)) \int_\Omega \nabla \bar\Theta_n^u \cdot \nabla \phi_h dx + \alpha_1 \int_\Omega |\hat u_h^n|^{p-2} \bar\Theta_n^u \phi_h dx \\
&= \int_\Omega \bar\partial u_h^n \phi_h dx + a_1(l_1(\hat u_h^n), l_2(\hat v_h^n)) \int_\Omega \nabla \bar u_h^n \cdot \nabla \phi_h dx + \alpha_1 \int_\Omega |\hat u_h^n|^{p-2} \bar u_h^n \phi_h dx \\
&- \int_\Omega \bar\partial R_h u^n \phi_h dx - a_1(l_1(\hat u_h^n), l_2(\hat v_h^n)) \int_\Omega \nabla \bar R_h u^n \cdot \nabla \phi_h dx \\
&- \alpha_1 \int_\Omega |\hat u_h^n|^{p-2} \bar R_h u^n \phi_h dx = \int_\Omega f_1^{n-1/2} \phi_h dx - \int_\Omega (u_t)^{n-1/2} \phi_h dx \\
&- a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) \int_\Omega \nabla u^{n-1/2} \cdot \nabla \phi_h dx \\
&- \alpha_1 \int_\Omega |u^{n-1/2}|^{p-2} u^{n-1/2} \phi_h dx + \int_\Omega ((u_t)^{n-1/2} - \bar\partial R_h u^n) \phi_h dx \\
&+ a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) \int_\Omega \nabla (u^{n-1/2} - \bar R_h u^n) \cdot \nabla \phi_h dx \\
&+ (a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) - a_1(l_1(\hat u_h^n), l_2(\hat v_h^n))) \int_\Omega \nabla \bar R_h u^n \cdot \nabla \phi_h dx \\
&+ \alpha_1 \int_\Omega (|u^{n-1/2}|^{p-2} u^{n-1/2} - |\bar R_h u^n|^{p-2} \bar R_h u^n) \phi_h dx \\
&+ \alpha_1 \int_\Omega (|\bar R_h u^n|^{p-2} - |\hat u_h^n|^{p-2}) \bar R_h u^n \phi_h dx.
\end{aligned}$$

By applying the definition of Ritz projection operator (3.1), we obtain

$$\begin{aligned}
&\int_\Omega \bar\partial \Theta_n^u \phi_h dx + a_1(l_1(\hat u_h^n), l_2(\hat v_h^n)) \int_\Omega \nabla \bar\Theta_n^u \cdot \nabla \phi_h dx \\
&+ \alpha_1 \int_\Omega |\hat u_h^n|^{p-2} \bar\Theta_n^u \phi_h dx = \int_\Omega ((u_t)^{n-1/2} - \bar\partial R_h u^n) \phi_h dx \\
&+ a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) \int_\Omega \nabla (u^{n-1/2} - \bar u^n) \cdot \nabla \phi_h dx \\
&+ (a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) - a_1(l_1(\hat u_h^n), l_2(\hat v_h^n))) \int_\Omega \nabla \bar u^n \cdot \nabla \phi_h dx \\
&+ \alpha_1 \int_\Omega (|u^{n-1/2}|^{p-2} u^{n-1/2} - |\bar R_h u^n|^{p-2} \bar R_h u^n) \phi_h dx \\
&+ \alpha_1 \int_\Omega (|\bar R_h u^n|^{p-2} - |\hat u_h^n|^{p-2}) \bar R_h u^n \phi_h dx.
\end{aligned} \tag{4.4}$$

Choosing $\phi_h = \bar{\Theta}_n^u$ and applying Holder's inequality, we arrive at

$$\begin{aligned}
&\frac{1}{2}\bar{\partial}\|\Theta_n^u\|^2 + a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2}))\|\nabla\bar{\Theta}_n^u\|^2 + \alpha_1 \int_\Omega |\hat{u}_h^n|^{p-2}|\bar{\Theta}_n^u|^2 dx \\
&\leq \|(u_t)^{n-1/2} - \bar{\partial}R_h u^n\|\|\bar{\Theta}_n^u\| \\
&+ a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2}))\|\nabla(u^{n-1/2} - \bar{u}^n)\|\|\nabla\bar{\Theta}_n^u\| \\
&+ |(a_1(l_1(u^{n-1/2}), l_2(v^{n-1/2})) - a_1(l_1(\hat{u}_h^n), l_2(\hat{v}_h^n)))|\|\nabla\bar{u}^n\|\|\nabla\bar{\Theta}_n^u\| \\
&+ \alpha_1 \int_\Omega |(|u^{n-1/2}|^{p-2}u^{n-1/2} - |\bar{R}_h u^n|^{p-2}\bar{R}_h u^n)||\bar{\Theta}_n^u| dx \\
&+ \alpha_1 \int_\Omega |(|\bar{R}_h u^n|^{p-2} - |\hat{u}_h^n|^{p-2})||\bar{R}_h u^n||\bar{\Theta}_n^u| dx.
\end{aligned}$$

We apply Poincare's inequality, $(Hyp\,2) - (Hyp\,3)$ and Lemma 3.2, to obtain

$$\begin{aligned}
&\frac{1}{2}\bar{\partial}\|\Theta_n^u\|^2 + m\|\nabla\bar{\Theta}_n^u\|^2 \leq \Big\{C\|(u_t)^{n-1/2} - \bar{\partial}R_h u^n\| + M\|\nabla(u^{n-1/2} - \bar{u}^n)\| \\
&+ C_1(u)(M_1\|u^{n-1/2} - \hat{u}_h^n\| + K_1\|v^{n-1/2} - \hat{v}_h^n\|) \\
&+ C_2(u)\|u^{n-1/2} - \bar{R}_h u^n\| + C_2(u, u_h)\|\bar{R}_h u^n - \hat{u}_h^n\|\Big\}\|\nabla\bar{\Theta}_n^u\|,
\end{aligned}$$

where $C_1(u) = C(\|\nabla u\|_{L^\infty(0,T,L^2(\Omega))})$ and $C_2(w) = C(\|w\|_{L^\infty(0,T,L^\infty(\Omega))})$. Therefore,

$$\begin{aligned}
\frac{1}{2}\bar{\partial}\|\Theta_n^u\|^2 \leq C\Big\{&\|(u_t)^{n-1/2} - \bar{\partial}R_h u^n\| + \|\nabla(u^{n-1/2} - \bar{u}^n)\| + \|u^{n-1/2} - \hat{u}_h^n\| \\
&+ \|v^{n-1/2} - \hat{v}_h^n\| + \|u^{n-1/2} - \bar{R}_h u^n\| + \|\bar{R}_h u^n - \hat{u}_h^n\|\Big\}^2.
\end{aligned} \tag{4.5}$$

In the same way we can prove that

$$\begin{aligned}
\frac{1}{2}\bar{\partial}\|\Theta_n^v\|^2 \leq C\Big\{&\|(v_t)^{n-1/2} - \bar{\partial}R_h v^n\| + \|\nabla(v^{n-1/2} - \bar{v}^n)\| + \|u^{n-1/2} - \hat{u}_h^n\| \\
&+ \|v^{n-1/2} - \hat{v}_h^n\| + \|v^{n-1/2} - \bar{R}_h v^n\| + \|\bar{R}_h v^n - \hat{v}_h^n\|\Big\}^2.
\end{aligned} \tag{4.6}$$

By numerical differentiation, interpolation theories and Lemma 3.1, we can prove that for $w \in \{u, v\}$,

$$\begin{aligned}
\|(w_t)^{n-1/2} - \bar{\partial}R_h w^n\| &\leq \|(w_t)^{n-1/2} - \bar{\partial}w^n\| + \|\bar{\partial}\Psi_n^w\| \\
&\leq C\delta^2\|w_{ttt}\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|w_t\|_{L^\infty(0,T,H^{r+1}(\Omega))}
\end{aligned} \tag{4.7}$$

$$\|\nabla(w^{n-1/2} - \bar{w}^n)\| \leq C\delta^2\|\nabla w_{tt}\|_{L^\infty(0,T,L^2(\Omega))} \tag{4.8}$$

$$\begin{aligned}
\|w^{n-1/2} - \bar{R}_h w^n\| &\leq \|w^{n-1/2} - \bar{w}^n\| + \|\bar{w}^n - \bar{R}_h w^n\| \\
&\leq C\delta^2\|w_{tt}\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|w\|_{L^\infty(0,T,H^{r+1}(\Omega))}
\end{aligned} \tag{4.9}$$

$$\|w^{n-1/2}-\hat{w}_h^n\|\leq\|w^{n-1/2}-\hat{w}^n\|+\|\hat{\Psi}_n^w\|+\|\hat{\Theta}_n^w\|\leq C\delta^2\|w_{tt}\|_{L^\infty(0,T,L^2(\Omega))}$$
$$+Ch^{r+1}\|w\|_{L^\infty(0,T,H^{r+1}(\Omega))}+C(\|\Theta_{n-1}^w\|+\|\Theta_{n-2}^w\|)\quad \text{for } n\geq 2 \tag{4.10}$$

$$\|\bar{R}_h w^n-\hat{w}_h^n\|\leq\|\bar{R}_h w^n-w^{n-1/2}\|+\|w^{n-1/2}-\hat{w}_h^n\|. \tag{4.11}$$

Hence from above inequalities (4.7)-(4.11), we obtain

$$\|\Theta_n^u\|^2+\|\Theta_n^v\|^2\leq C\delta(\|\Theta_{n-1}^u\|^2+\|\Theta_{n-1}^v\|^2+\|\Theta_{n-2}^u\|^2+\|\Theta_{n-2}^v\|^2)$$
$$+C\delta(\delta^2+h^{r+1})^2 \tag{4.12}$$

By iterating and noting that for all $1\leq n\leq N$, $\sum_{k=1}^n\delta\leq T$, we obtain

$$\|\Theta_n^u\|^2+\|\Theta_n^v\|^2\leq C(\|\Theta_0^u\|^2+\|\Theta_0^v\|^2+\|\Theta_1^u\|^2+\|\Theta_1^v\|^2)+C(\delta^2+h^{r+1})^2. \tag{4.13}$$

Note that by the definition of w_h^0 with $w_h^0\in\{u_h^0,\, v_h^0\}$ and Lemma 3.1, we have

$$\|\Theta_0^w\|\leq Ch^{r+1}\|w_0\|_{L^\infty(0,T,H^{r+1}(\Omega))}.$$

To conclude the proof, we have to estimate $\|\Theta_1^w\|$ for $w\in\{u,\, v\}$.

From (4.5), we have

$$\frac{1}{2}\bar{\partial}\|\Theta_1^u\|^2\leq C\Big\{\|(u_t)^{1/2}-\bar{\partial}R_h u^1\|+\|\nabla(u^{1/2}-\bar{u}^1)\|+\|u^{1/2}-\hat{u}_h^1\|$$
$$+\|v^{1/2}-\hat{v}_h^1\|+\|u^{1/2}-\bar{R}_h u^1\|+\|\bar{R}_h u^1-\hat{u}_h^1\|\Big\}^2, \tag{4.14}$$

where $(\hat{u}_h^1,\hat{v}_h^1)$ is the solution of (4.2)–(**??**). From inequalities (4.7)-(4.11) and the triangle inequality, we can write

$$\frac{1}{2}\bar{\partial}\|\Theta_1^u\|^2\leq C\{\delta^2+h^{r+1}\}^2+C\Big\{\|R_h v^{1/2}-\hat{v}_h^1\|+\|R_h u^{1/2}-\hat{u}_h^1\|\Big\}^2. \tag{4.15}$$

Let us denote $\hat{\Theta}_0^u$ and $\hat{\Theta}_1^u$ by $\hat{\Theta}_0^u=\Theta_0^u$ and $\hat{\Theta}_1^u=\hat{u}_h^1-R_h\hat{u}^1$ with $\hat{u}^1=u^{1/2}=u(t_{1/2})$. Then from (4.2), using (3.1), we have

$$2\int_\Omega\bar{\partial}\hat{\Theta}_1^u\phi_h dx+a_1(l_1(u_h^0),l_2(v_h^0))\int_\Omega\nabla\hat{\Theta}_1^u\cdot\nabla\phi_h dx$$
$$+\alpha_1\int_\Omega|u_h^0|^{p-2}\hat{\Theta}_1^u\phi_h dx$$
$$=\int_\Omega((u_t)^{1/2}-2\bar{\partial}R_h u^{1/2})\phi_h dx+(a_1(l_1(u^{1/2}),l_2(v^{1/2}))$$
$$-a_1(l_1(u_h^0),l_2(v_h^0)))\int_\Omega\nabla u^{1/2}\cdot\nabla\phi_h dx \tag{4.16}$$
$$+\alpha_1\int_\Omega(|u^{1/2}|^{p-2}u^{1/2}-|R_h u^{1/2}|^{p-2}R_h u^{1/2})\phi_h dx$$
$$+\alpha_1\int_\Omega(|u_h^0|^{p-2}-|R_h u^{1/2}|^{p-2})R_h u^{1/2}\phi_h dx.$$

By choosing $\phi_h = \hat{\Theta}_1^u$ and applying the properties of $a_1(.,.)$, Poincare's inequality, Holder's inequality and Lemma 3.2, we arrive at

$$\begin{aligned}
&\frac{2}{\delta}\|\hat{\Theta}_1^u\|^2 - \frac{2}{\delta}\int_\Omega \Theta_0^u \hat{\Theta}_1^u dx + m\|\nabla\hat{\Theta}_1^u\|^2 \\
&\leq C\left\{\|(u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\| + \|u^{1/2} - u_h^0\| + \|v^{1/2} - v_h^0\|\right. \\
&\left. + \|u^{1/2} - R_h u^{1/2}\| + \|R_h u^{1/2} - u_h^0\|\right\}\|\nabla\hat{\Theta}_1^u\|,
\end{aligned}$$

that is

$$\begin{aligned}
\frac{1}{\delta}\|\hat{\Theta}_1^u\|^2 \leq \frac{1}{\delta}\|\Theta_0^u\|^2 + C&\left\{\|(u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\| + \|u^{1/2} - u_h^0\| + \|v^{1/2} - v_h^0\|\right. \\
&\left. + \|u^{1/2} - R_h u^{1/2}\| + \|R_h u^{1/2} - u_h^0\|\right\}^2. \quad (4.17)
\end{aligned}$$

Again applying numerical differentiation and interpolation theories and Lemma 3.1, we can prove that

$$\begin{aligned}
\|(u_t)^{1/2} - 2\bar{\partial}R_h u^{1/2}\| &\leq \|(u_t)^{1/2} - 2\bar{\partial}u^{1/2}\| + 2\|\bar{\partial}u^{1/2} - \bar{\partial}R_h u^{1/2}\| \\
&\leq C\delta\|u_t\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|u_t\|_{L^\infty(0,T,H^{r+1}(\Omega))} \\
\|w^{1/2} - w_h^0\| &\leq \|w^{1/2} - w_0\| + \|w_0 - w_h^0\| \\
&\leq C\delta\|w_t\|_{L^\infty(0,T,L^2(\Omega))} + Ch^{r+1}\|w_0\|_{L^\infty(0,T,H^{r+1}(\Omega))}.
\end{aligned}$$

Finally, we have

$$\|\hat{\Theta}_1^u\|^2 \leq C\{\delta^3 + h^{2(r+1)}\}. \quad (4.18)$$

By substituting (4.18) into (4.15), we conclude that

$$\begin{aligned}
\|\Theta_1^u\|^2 &\leq C\|\Theta_0^u\|^2 + C\delta\{\delta^3 + h^{2(r+1)}\} + C\delta\{\delta^2 + h^{r+1}\}^2 \\
&\leq C\{\delta^2 + h^{r+1}\}^2. \quad (4.19)
\end{aligned}$$

In the same way, wex can prove that

$$\|\Theta_1^v\|^2 \leq C\{\delta^2 + h^{r+1}\}^2. \quad (4.20)$$

5. Numerical simulations

5.1 Example 1: Energy decay and the vanishing in finite time

In order to verify the properties of the energy decay exponentially and the finite time extinction when $1 < p < 2$, we consider the system (1.1) in $\Omega = (0,1)^2$ with $\alpha_1 = \alpha_2 = 1$ and functions satisfying (**Hyp 1**) – (**Hyp 5**).

$$\begin{aligned}
&a_1(s,z) = 3 + \sin(s) + \cos(z), \\
&a_2(s,z) = 3 - \cos(s) + \sin(z), \quad l_1(w) = l_2(w) = \int_\Omega w dx \\
&u_0(x,y) = 2xy(1-x)(1-y), \quad v_0(x,y) = 0.5xy(1-x)(1-y), \\
&f_1(x,y,t) = 0.01ln(x+y+1)(0.1-t)^6, \quad f_2(x,y,t) = 0.01\sin(xy)(0.1-t)^4.
\end{aligned}$$

Figure 1 and Figure 2 show that, for $p = 1.2$, the solutions $u_h^n \equiv 0$ and $v_h^n \equiv 0$ for $t \geq 0.046$ while in Figure 3 and Figure 4 ($p = 3.5$), the solutions u_h^n and v_h^n are very small but not zeros for $t \geq 0.046$. For the asymptotic behavior, the energy function $E_n = 1/2(\|u_h^n\|^2 + \|v_h^n\|^2)$ decays and is extinguished at $t \approx 0.045$ for $p = 1.2$ (See Figure 5).

Figure 1: Evolution of the solution u_h^n for some values of t and $p = 1.2$

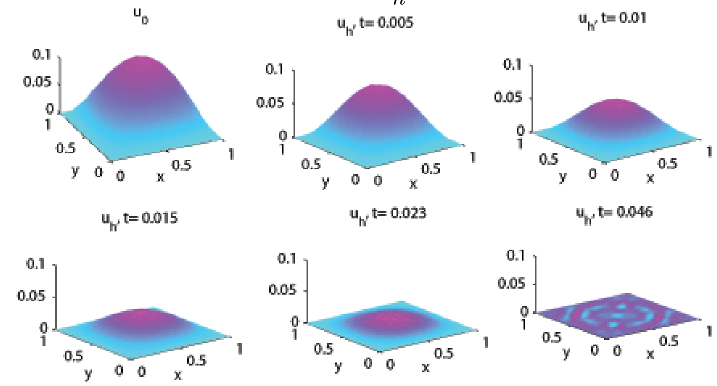


Figure 2: Evolution of the solution v_h^n for some values of t and $p = 1.2$

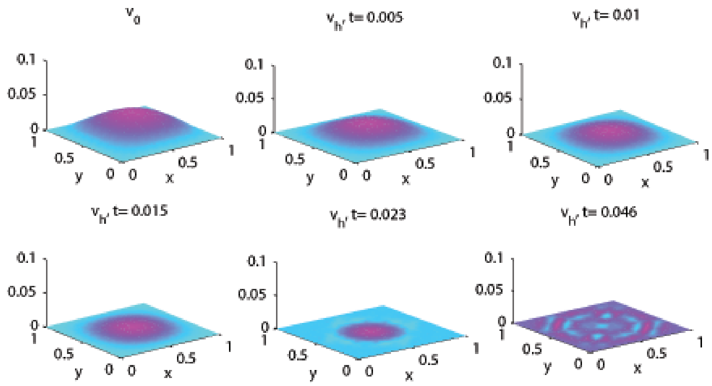


Figure 3: Evolution of the solution u_h^n for some values of t and $p = 3.5$

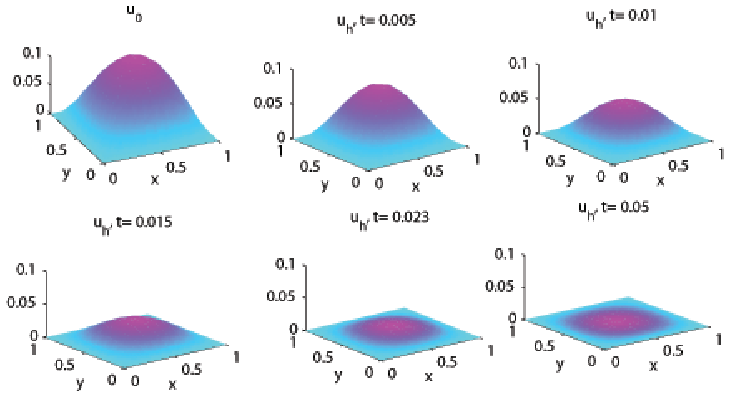
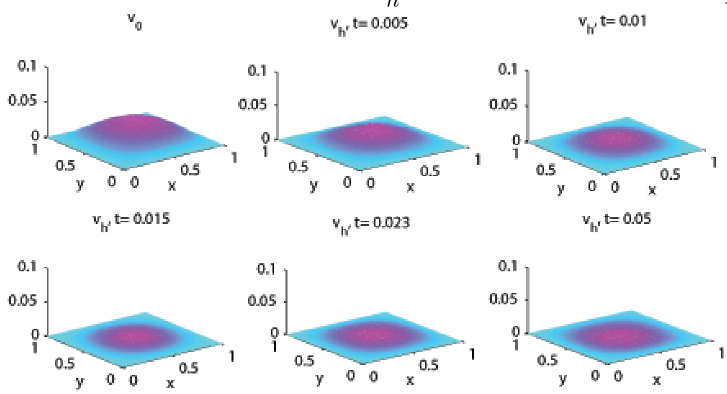


Figure 4: Evolution of the solution v_h^n for some values of t and $p = 3.5$



5.2 Example 2: Illustration of the diffusion property

We consider system (1.1) in $\Omega = (0,1)^2$ with $p = 3.5$, $\alpha_1 = \alpha_2 = 1$ and

$$a_1(s,z) = 3 + \sin(s) + \cos(z),$$

$$a_2(s,z) = 3 - \cos(s) + \sin(z), \quad l_1(w) = l_2(w) = \int_\Omega w dx,$$

$$u_0(x,y) = \begin{cases} 20xy(0.2 - (x-0.2)^2 - (y-0.2)^2), & \text{if} \quad 0 \le x,\, y \le 0.5 \\ 0 \quad \text{elsewhere} \end{cases}$$

$$v_0(x,y) = \begin{cases} 15(1-x)(1-y)(0.2 - (x-0.8)^2 - (y-0.8)^2), & \text{if} \quad 0.5 \le x,\, y \le 1 \\ 0 \quad \text{elsewhere} \end{cases}$$

$$f_1(x,y,t) = 0.01ln(x+y+1)(0.1-t)^6, \quad f_2(x,y,t) = 0.01\sin(xy)(0.1-t)^4.$$

Figure 5: Evolution of the energy function E_n for $p = 1.2$ and $p = 3.5$ respectively

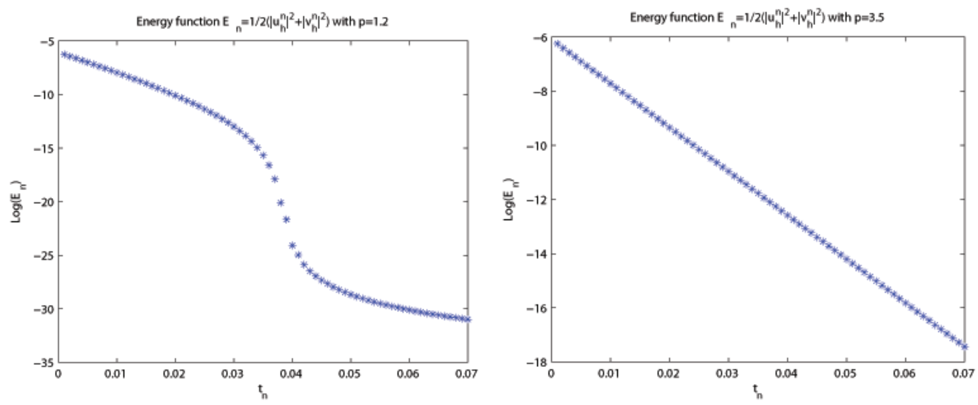


Figure 6 and Figure 7 show that, initially the solutions u_h^n and v_h^n are concentrated in two different small regions, and as time increases these solutions spread over the domain and decrease.

Figure 6: Evolution of the solution u_h^n for some values of t and $p = 3.5$

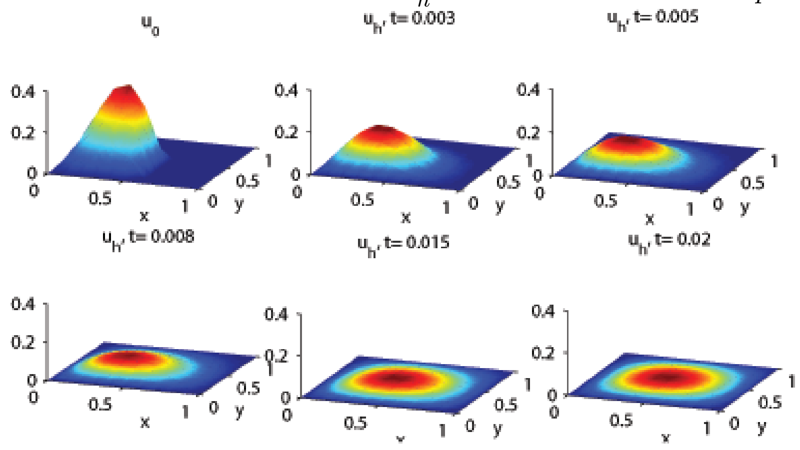
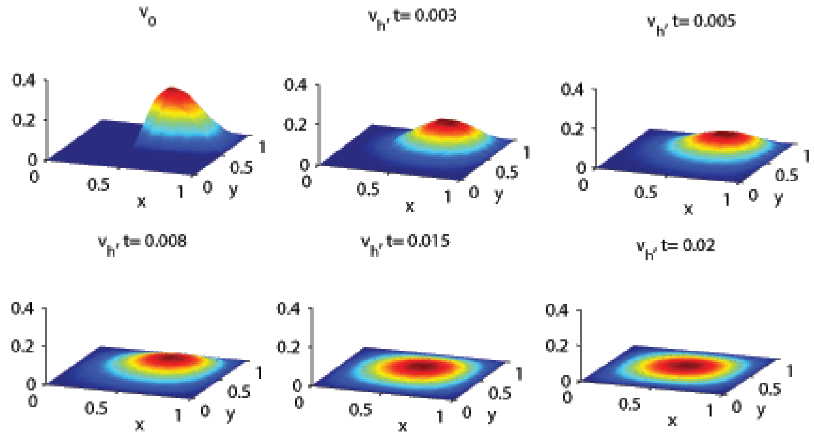


Figure 7: Evolution of the solution v_h^n for some values of t and $p = 3.5$



5.3 Example 3: Convergence rate

We consider the system (1.1) in $\Omega = (0,1)^2$ with $p = 3.5$, $\alpha_1 = \alpha_2 = 1$ and

$$a_1(s,z) = 3 + \sin(s) + \cos(z),$$
$$a_2(s,z) = 3 - \cos(s) + \sin(z), \quad l_1(w) = l_2(w) = \int_\Omega w dx$$
$$u(x,y,t) = \sin(\pi x)\sin(\pi y)\cos(\pi t), \quad v(x,y,t) = 0.5\sin(\pi x)\sin(\pi y)\cos(\pi t).$$

We solve the system (1.1) by the linearized Crank-Nicolson Galerkin finite element method (4.1)-(4.2) with a linear and quadratic finite element approximation. A uniform triangulation is used. For the convergence with respect to the mesh size h, $\delta = h^2$ and for the time step, $h = 0.01$. From Figure 8, we see that errors estimate in L^2-norm are in the form of $O(h^{r+1} + \delta^2)$, with $r = 1, 2$ which confirm our theoretical analysis.

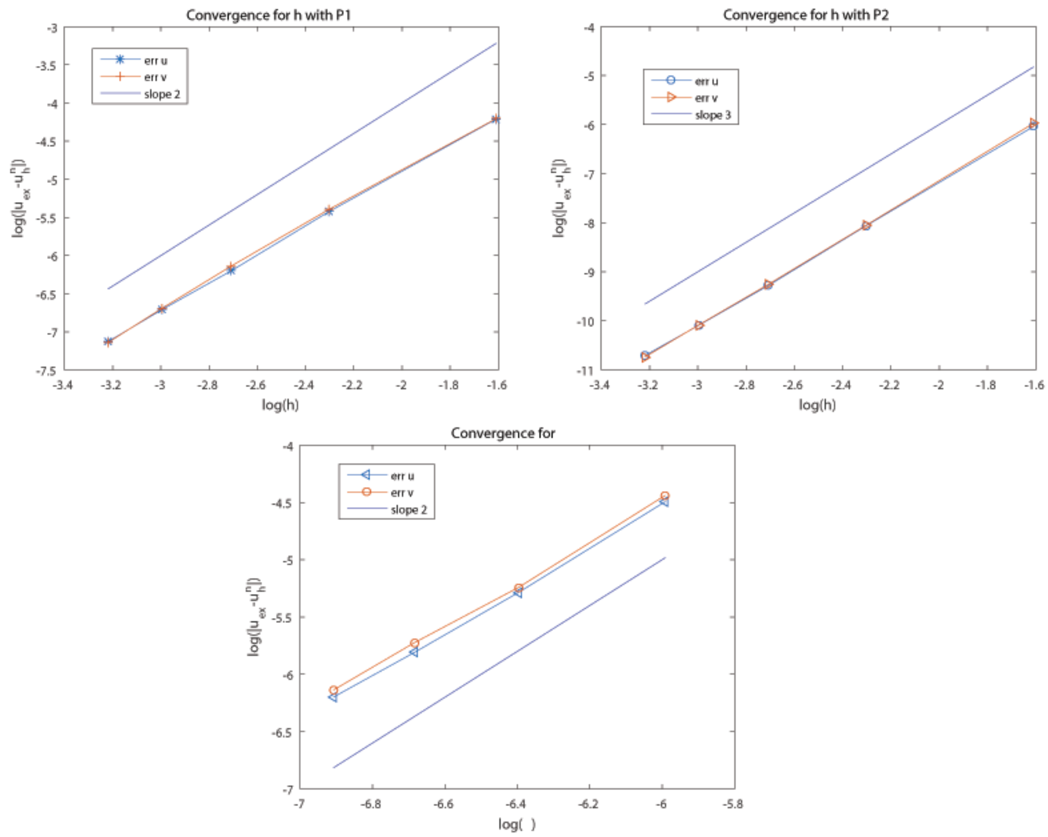
6. Conclusion

We have presented and analyzed a linearized Crank-Nicolson Galerkin finite element method for a nonlocal nonlinear parabolic system. The optimal L^2 error estimate has been obtained using sufficient conditions on the exact solution. Some numerical experiments on Matlab's environment have been carried out and our numerical results confirmed the theoretical analysis.

References

[1] J. K. Hale, *Large diffusivity and asymptotic behavior in parabolic systems,* Journal of mathematical analysis and applications, 118 (1986), 455-466.

Figure 8: Convergence rate



[2] M. Chipot and B. Lovat, *Some remarks on non local elliptic and parabolic problems,* Nonlinear Analysis: Theory, Methods & Applications, 30 (1997), 4619-4627.

[3] J. Djoko, J. Lubuma, and M. Mbehou, *On the numerical solution of the stationary power-law Stokes equations: A penalty finite element approach*, Journal of Scientific Computing, 69 (2016), 1058-1082.

[4] F. J. S. Corrêa, S. D. Menezes, and J. Ferreira, *On a class of problems involving a nonlocal operator*, Applied Mathematics and Computation, 147 (2004), 475-489.

[5] C. A. Raposo, M. Sepúlveda, O. V. Villagrán, D. C. Pereira, and M. L. Santos, *Solution and asymptotic behaviour for a nonlocal coupled system of reaction-diffusion*, Acta Applicandae Mathematicae, 102 (2008), 37-56.

[6] J. C. Duque, R. M. Almeida, S. N. Antontsev, and J. Ferreira, *A reaction-diffusion model for the nonlinear coupled system: existence, uniqueness,*

long time behavior and localization properties of solutions, IMA Journal of Applied Mathematics, 1-21, 2016.

[7] R. M. Almeida, J. Duque, J. Ferreira, and R. J. Robalo, *The Crank–Nicolson–Galerkin finite element method for a nonlocal parabolic equation with moving boundaries*, Numerical Methods for Partial Differential Equations, 31 (2015), 1515-1533.

[8] J. K. Djoko and M. Mbehou, *Finite element analysis of the stationary power-law Stokes equations driven by friction boundary conditions*, *Journal of Numerical Mathematics*, 23 (2015), 21-40.

[9] M. Mbehou, *The theta-Galerkin finite element method for coupled systems resulting from microsensor thermistor problems*, Mathematical Methods in the Applied Sciences, 41 (2018), 1480-1491.

[10] M. Mbehou, R. Maritz, and P. Tchepmo, *Numerical Analysis for a Nonlocal Parabolic Problem*, East Asian Journal on Applied Mathematics, 6 (2016), 434-447.

[11] J. C. Duque, R. M. Almeida, S. N. Antontsev, and J. Ferreira, *The Euler–Galerkin finite element method for a nonlocal coupled system of reaction–diffusion type*, Journal of Computational and Applied Mathematics, 296 (2016), 116-126.

[12] V. Thomée, *Galerkin finite element methods for parabolic problems*, 1054, Springer, 1984.

[13] J. W. Barrett and W. Liu, *Finite element approximation of the p-laplacian,* Mathematics of computation, 61 (1993), 523-537.

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