On hyper Q-fuzzy normal HX subgroup, conjugate and its normal level

Mourad Oqla Massa'deh*
Department of Applied Science
Ajloun College
Al-Balqa Applied University
Jordan
mourad.oqla@bau.edu.jo
and
Department of Mathematics
College of Science
Taibah University
Madinah
Saudi Arabia
mmassadeh@taibahu.edu.sa

Ahlam Fellatah
Department of Mathematics
Taibah University
College of Science
Madinah
Saudi Arabia
Afallatah@taibahu.edu.sa

Abstract. In this research, we introduce a hyper Q-fuzzy HX subgroup of a HX group ν notion also we discuss a few of its properties. Also we define a hyper Q-fuzzy normal HX subgroup, normal level subgroup HX of hyper Q-fuzzy normal HX subgroup of a HX group also self conjugate and we discussed some of their important properties.
Keywords: HX group, hyper fuzzy set, fuzzy $HX-$Subgroup, Q-fuzzy set, Q-fuzzy $HX-$subgroup, hyper Q-fuzzy $HX-$subgroup, normal level sub HX group.

1. Introduction

In 1965, Zadeh [20] introduced the notion of fuzzy sets and opened a new path of thinking to engineers, computer science, mathematicians, physicists and many others due to its diverse applications in various fields. In 1975, Zadeh [19] discussed the concepts of interval valued fuzzy sets. Where the values of membership functions are intervals of real numbers instead of the real points. The concept of fuzzy subgroups was introduced by Rosenfield [17] in 1971, which was the first fuzzification of any algebraic structure, after that Biswas [1] in 1994

∗. Corresponding author

define interval valued fuzzy subgroups. Piergiulio Corsini and Li Honyxing [2, 3, 5] studied HX group concept and there after Chengzhong., Mitlonghai., and Hongxing [6] discussed fuzzy HX group concept. Manikandan et al [14] introduce anti Q-fuzzy normal HX group and its lower level sub HX groups concept. A new algebraic structure of Q-fuzzy group are introduced and defined by Solairaju et al [18]. Hereafter, several results about the HX group with respect to Q-fuzzy and anti Q-fuzzy, bipolar fuzzy, intuitionistic fuzzy have been obtained see [7, 9, 10, 11, 12, 13, 15]. In 2012, Jayanta et al [4] introduced the concepts of hyper fuzzy sets, where the values of the membership functions are assumed to be a subsets of $[0, 1]$, which is a generalization of fuzzy sets and interval valued fuzzy sets. In this research we give a new algebraic structure of hyper Q-fuzzy HX subgroups also we study some of its properties. Also we define hyper Q-fuzzy normal HX subgroup and some of its properties are discussed.

2. Preliminaries

Definition 2.1 ([5]). An HX group on G is non empty set $\upsilon \subset 2^G - \{\phi\}$ such that υ is a group with respect to the algebraic operation defined by $WV = \{wv, w \in W \text{ and } v \in V\}$, which its unit element is denoted by U.

Definition 2.2 ([20]). If A is any non empty set. A fuzzy subset λ of A is a function $\lambda : A \to [0, 1]$.

Definition 2.3 ([16]). A fuzzy set μ is said to be fuzzy HX subgroup of υ if for $W, V \in \upsilon$, that is,

1. $\mu(WV) \geq \min\{\mu(W), \mu(V)\}$
2. $\mu(W^{-1}) = \mu(W)$.

Definition 2.4 ([16, 18]). Let Q and υ be any two sets. A mapping $\mu : \upsilon \times Q \to [0, 1]$ is called Q-fuzzy set in υ.

Definition 2.5 ([16, 18]). A Q-fuzzy set μ is called Q-fuzzy HX subgroup of υ if for all $W, V \in \upsilon$ and $q \in Q$:

1. $\mu(WV, q) \geq \min\{\mu(W, q), \mu(V, q)\}$
2. $\mu(W^{-1}, q) = \mu(W, q)$.

Definition 2.6 ([8]). A Q-fuzzy HX subgroup μ of υ is said to be normal if for all $W, V \in \upsilon$ and $q \in Q$. That is, $\mu(WV, q) = \mu(VW, q)$ or $\mu(WVW^{-1}, q) = \mu(V, q)$.

Definition 2.7 ([4]). Let υ, Q be any two sets. An interval valued Q-fuzzy set μ defined in υ is given by $\mu = \{((V, q), \bar{\lambda}_\mu \in (V, q)); V \in \upsilon \text{ and } q \in Q\}$ where $\bar{\lambda}_\mu : \upsilon \times Q \to F[0, 1]$ and $F[0, 1]$ denoted the family of all closed subinterval of $[0, 1]$ and $\bar{\lambda}_\mu(V, q) = [\lambda^L_\mu(V, q), \lambda^R_\mu(V, q)]$ such that $\lambda^L_\mu, \lambda^R_\mu$ are two Q-fuzzy subsets of υ such that $\lambda^L_\mu(V, q) \leq \lambda^R_\mu(V, q)$ for all $V \in \upsilon$ and $q \in Q$.

Definition 2.8 ([4])**.** Let $F_1, F_2 \in F[0,1]$. If $F_1 = [\alpha_1, \beta_1]$ and $F_2 = [\alpha_2, \beta_2]$, then $\max(F_1, F_2) = [\max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2)]$ and $\min(F_1, F_2) = [\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2)]$. Thus if $F_i = [\alpha_i, \beta_i] \in F[0,1]$ for $i = 1, 2, 3, \cdots$, then we define $r\sup_i F_i = [\sup_i \alpha_i, \sup_i \beta_i]$ and $r\inf_i F_i = [\inf_i \alpha_i, \inf_i \beta_i]$. Now we call $F_1 \geq F_2$ iff $\alpha_1 \geq \alpha_2$ and $\beta_1 \geq \beta_2$. Similarly we may have $F_1 \leq F_2$ and $F_1 = F_2$.

Definition 2.9 ([4])**.** An interval valued Q-fuzzy subset $\bar{\lambda}_\mu$ of a HX group υ is called an interval valued Q-fuzzy subgroup if :

1. $\bar{\lambda}_\mu(WV, q) \geq r\min\{\bar{\lambda}_\mu(W, q), \bar{\lambda}_\mu(V, q)\}$
2. $\bar{\lambda}_\mu(W^{-1}, q) = \bar{\lambda}_\mu(W, q)\}$,

   Where $W, V \in \upsilon$ and $q \in Q$.

3. Hyper Q-fuzzy HX subgroups

Definition 3.1. Let υ and Q be two sets. A mapping $\mu^* : \upsilon \times Q \to \wp([0,1])$ is said to be a hyper Q-fuzzy of υ, where $\wp([0,1])$ denotes the set of all non empty subset of $[0,1]$.

Definition 3.2. Let υ and Q be two sets and μ^*, λ^* be two hyper Q-fuzzy subset of υ. Then the intersection of μ^* and λ^* is denoted by $\mu^* \cap \lambda^*$ and defined by: $\mu^* \cap \lambda^*(V, q) = \{\min\{v_1, v_2\}; v_1 \in \mu^*(V, q), v_2 \in \lambda^*(V, q)\}$ for all $V \in \upsilon$ and $q \in Q$. The union of μ^* and λ^* is denoted by $\mu^* \cup \lambda^*$ and defined by: $(\mu^* \cup \lambda^*)(V, q) = \{\max\{v_1, v_2\}; v_1 \in \mu^*(V, q), v_2 \in \lambda^*(V, q)\}$ for all $V \in \upsilon$ and $q \in Q$.

Definition 3.3. Let υ be a HX group. A mapping $\mu^* : \upsilon \times Q \to \wp([0,1])$ is called a hyper Q-fuzzy HX subgroup of υ if $\forall W, V \in \upsilon$ and $q \in Q$ the following axioms hold:

1. $\inf \mu^*(WV, q) \geq \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}$
2. $\sup \mu^*(WV, q) \geq \min\{\sup \mu^*(W, q), \sup \mu^*(V, q)\}$
3. $\inf \mu^*(W^{-1}, q) \geq \inf \mu^*(W, q)$
4. $\sup \mu^*(W^{-1}, q) \geq \sup \mu^*(W, q)$.

Lemma 3.4. If μ^* is a hyper Q-fuzzy HX subgroup of υ having the identity U, then for all $W \in \upsilon$:

1. $\inf \mu^*(W^{-1}, q) = \inf \mu^*(W, q)$ and $\sup \mu^*(W^{-1}, q) = \sup \mu^*(W, q)$.
2. $\inf \mu^*(U, q) \geq \inf \mu^*(W, q)$ and $\sup \mu^*(U, q) \geq \sup \mu^*(W, q)$.

Proof. 1. As μ^* is a hyper Q-fuzzy HX subgroup of υ, then $\inf \mu^*(W^{-1}, q) \geq \inf \mu^*(W, q) \forall W \in \upsilon$ and $q \in Q$, also $\inf \mu^*(W, q) = \inf \mu^*((W^{-1})^{-1}, q) \geq \inf \mu^*(W^{-1}, q)$. Then $\inf \mu^*(W, q) \geq \inf \mu^*(W^{-1}, q)$, thus $\inf \mu^*(W, q) = \inf \mu^*(W^{-1}, q)$. Similarly, we can prove that $\sup \mu^*(W^{-1}, q) = \sup \mu^*(W, q)$.

2. $\inf \mu^*(U, q) = \inf \mu^*(WW^{-1}, q) \geq \min\{\inf \mu^*(W, q), \inf \mu^*(W^{-1}, q)\} = \inf \mu^*(W, q)$. Then $\inf \mu^*(U, q) \geq \inf \mu^*(W, q)$, also $\sup \mu^*(U, q) = \sup \mu^*(WW^{-1}, q) \geq \min\{\sup \mu^*(W, q), \sup \mu^*(W^{-1}, q)\} = \sup \mu^*(W, q)$, then $\sup \mu^*(U, q) \geq \sup \mu^*(W, q)$. □

Proposition 3.5. A hyper Q-fuzzy subset μ^* of a HX group υ is a hyper Q-fuzzy HX subgroup iff for all $W, V \in \upsilon, q \in Q$ followings are hold:

1. $\inf \mu^*(WV^{-1}, q) \geq \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}$.

2. $\sup \mu^*(WV^{-1}, q) \geq \min\{\sup \mu^*(W, q), \sup \mu^*(V, q)\}$.

Proof. ⇒ Let μ^* be a hyper Q-fuzzy HX subgroup of υ and $W, V \in \upsilon, q \in Q$. Then

$$\begin{aligned}\inf \mu^*(WV^{-1}, q) &\geq \min\{\inf \mu^*(W, q), \inf \mu^*(V^{-1}, q)\}\\ &= \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}.\end{aligned}$$

And

$$\begin{aligned}\sup \mu^*(WV^{-1}, q) &\geq \min\{\sup \mu^*(W, q), \sup \mu^*(V^{-1}, q)\}\\ &= \min\{\sup \mu^*(W, q), \sup \mu^*(V, q)\}.\end{aligned}$$

⇐ Let μ^* be a hyper Q-fuzzy subset of υ and given conditions hold. Then for all $W, V \in \upsilon, q \in Q$, we have $\inf \mu^*(U, q) = \inf \mu^*(VV^{-1}, q) \geq \min\{\inf \mu^*(V, q), \inf \mu^*(V^{-1}, q)\} = \inf \mu^*(V, q)$, $\sup \mu^*(U, q) = \sup \mu^*(VV^{-1}, q) \geq \min\{\sup \mu^*(V, q), \sup \mu^*(V^{-1}, q)\} = \sup \mu^*(V, q)$.

So, $\inf \mu^*(V^{-1}, q) = \inf \mu^*(UV^{-1}, q) \geq \min\{\inf \mu^*(U, q), \inf \mu^*(V, q)\} = \inf \mu^*(V, q)$. And $\sup \mu^*(V^{-1}, q) = \sup \mu^*(UV^{-1}, q) \geq \min\{\sup \mu^*(U, q), \sup \mu^*(V, q)\} = \sup \mu^*(V, q)$.

Again, $\inf \mu^*(WV, q) \geq \min\{\inf \mu^*(W, q), \inf \mu^*(V^{-1}, q)\} \geq \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}$. Therefore μ^* be a hyper Q-fuzzy HX subgroup of υ. □

Theorem 3.6. If λ^*, μ^* are two hyper Q-fuzzy $HX-$subgroups of υ, then $\lambda^* \cap \mu^*$ is hyper Q-fuzzy $HX-$subgroup of υ.

Proof. Since λ^*, μ^* are two hyper Q-fuzzy $HX-$subgroups of υ, and $W, V \in \upsilon, q \in Q$. Thus

$$\begin{aligned}&\inf(\lambda^* \cap \mu^*)(WV^{-1}, q) = \min\{\inf \lambda^*(WV^{-1}, q), \inf \mu^*(WV^{-1}, q)\}\\ &\geq \min\{\min\{\inf \lambda^*(W, q), \inf \lambda^*(V, q)\}, \min\{\inf \mu^*(W, q), \inf \mu^*(V, q)\}\}\\ &= \min\{\min\{\inf \lambda^*(W, q), \inf \mu^*(W, q)\}, \min\{\inf \lambda^*(V, q), \inf \mu^*(V, q)\}\}\\ &= \min\{\inf(\lambda^* \cap \mu^*)(W, q), \inf(\lambda^* \cap \mu^*)(V, q)\}.\end{aligned} \tag{1}$$

Also

$$
\begin{aligned}
&\sup(\lambda^* \cap \mu^*)(WV^{-1}, q) = \min\{\sup \lambda^*(WV^{-1}, q), \sup \mu^*(WV^{-1}, q)\} \\
&\geq \min\{\min\{\sup \lambda^*(W, q), \sup \lambda^*(V, q)\}, \min\{\sup \mu^*(W, q), \sup \mu^*(V, q)\}\} \\
&= \min\{\min\{\sup \lambda^*(W, q), \sup \mu^*(W, q)\}, \min\{\sup \lambda^*(V, q), \sup \mu^*(V, q)\}\} \qquad (2) \\
&= \min\{\sup(\lambda^* \cap \mu^*)(W, q), \sup(\lambda^* \cap \mu^*)(V, q)\}.
\end{aligned}
$$

Therefore by (1), (2) and Proposition 3.5, we say that $\lambda^* \cap \mu^*$ is a hyper Q-fuzzy HX subgroup of υ. □

Corollary 3.7. The intersection of any collection hyper Q-fuzzy normal HX subgroup is a hyper Q-fuzzy normal HX subgroup of υ.

Proof. Straightforward. □

Definition 3.8. Let μ^* be a hyper Q-fuzzy HX subgroup of υ. Then μ^* is called hyper Q-fuzzy normal HX subgroup of υ if $\inf \mu^*(WV, q) = \inf \mu^*(VW, q)$ and $\sup \lambda^*(WV, q) = \sup \lambda^*(VW, q)$ for all $W, V \in \upsilon, q \in Q$.

Theorem 3.9. Let μ^* be a hyper Q-fuzzy HX subgroup of υ, then the following are equivalent.

1. μ^* be a hyper Q-fuzzy HX normal subgroup of υ.
2. $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V, q), \sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V, q)$.
3. $\inf \mu^*(WV, q) = \inf \mu^*(VW, q)$ and $\sup \mu^*(WV, q) = \sup \mu^*(VW, q)$, for all $W, V \in \upsilon, q \in Q$.

Proof. Straightforward. □

Theorem 3.10. If λ^*, μ^* are two hyper Q-fuzzy normal $HX-$subgroups of υ, then $\lambda^* \cap \mu^*$ is hyper Q-fuzzy normal $HX-$ subgroup of υ.

Proof. Since λ^*, μ^* are two hyper Q-fuzzy HX subgroup of υ, by Theorem 3.6 $\lambda^* \cap \mu^*$ is a hyper Q-fuzzy HX subgroup of υ. Let $W, V \in \upsilon, q \in Q$, then

$$
\begin{aligned}
\inf(\lambda^* \cap \mu^*)(WV, q) &= \min\{\inf \lambda^*(WV, q), \inf \mu^*(WV, q)\} \\
&= \min\{\inf \lambda^*(VW, q), \inf \mu^*(VW, q)\} \\
&= \inf(\lambda^* \cap \mu^*)(VW, q).
\end{aligned}
$$

And

$$
\begin{aligned}
\sup(\lambda^* \cap \mu^*)(WV, q) &= \min\{\sup \lambda^*(WV, q), \sup \mu^*(WV, q)\} \\
&= \min\{\sup \lambda^*(VW, q), \sup \mu^*(VW, q)\} \\
&= \sup(\lambda^* \cap \mu^*)(VW, q).
\end{aligned}
$$

Therefore $(\lambda^* \cap \mu^*)$ is a hyper Q-fuzzy normal HX subgroup of υ. □

Corollary 3.11. The intersection of any collection hyper Q-fuzzy normal HX subgroup is a hyper Q-fuzzy normal HX subgroup of υ.

Proof. Straightforward. □

4. Level subset of hyper Q-fuzzy normal HX subgroup and self conjecate hyper Q-fuzzy HX subgroup

Definition 4.1. Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ. Such that for any $\alpha \in [0,1]$ we define the set $L(\mu^*, \alpha) = \{V \in \upsilon, q \in Q; \inf \mu^*(V,q) \geq \alpha$ and $\sup \lambda^*(V,q) \geq \alpha\}$ is called the level subset of μ^*.

Theorem 4.2. Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ. Then for $\alpha \in [0,1]$ such that $\inf \mu^*(U,q) \geq \alpha$ and $\sup \mu^*(U,q) \geq \alpha, L(\mu^*, \alpha)$ is normal sub HX group of υ.

Proof. For all $W, V \in L(\mu^*, \alpha)$ we have $\inf \mu^*(W,q) \geq \alpha$ and $\inf \lambda^*(V,q) \geq \alpha$. Now $\inf \mu^*(WV^{-1}, q) \geq \min\{\inf \mu^*(W,q), \inf \mu^*(V,q)\} \geq \min\{\alpha, \alpha\} = \alpha$. Thus $\inf \mu^*(WV^{-1}, q) \geq \alpha$ and $\sup \mu^*(W,q) \geq \alpha$ and $\sup \lambda^*(V,q) \geq \alpha$. Now $\sup \mu^*(WV^{-1}, q) \geq \min\{\sup \mu^*(W,q),$
$\sup \mu^*(V,q)\} \geq \min\{\alpha, \alpha\} = \alpha$. Thus $\sup \mu^*(WV^{-1}, q) \geq \alpha$ and we get $WV^{-1} \in L(\mu^*, \alpha)$. Hence $L(\mu^*, \alpha)$ is a sub HX group of υ.

For all $V \in L(\mu^*, \alpha), W \in \upsilon, q \in Q$ and $\alpha \in [0,1]$ such that $\inf \mu^*(U,q) \geq \alpha$ and $\sup \mu^*(U,q) \geq \alpha$ we have $\inf \mu^*(V,q) \geq \alpha$ and $\sup \mu^*(V,q) \geq \alpha$. Since μ^* be a hyper Q-fuzzy normal HX subgroup of υ, then $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q) \geq \alpha$ and $\sup \lambda^*(WVW^{-1}, q) = \sup \lambda^*(V,q) \geq \alpha$ for all $W, V \in \upsilon, q \in Q$, therefore $WVW^{-1} \in L(\mu^*, \alpha)$. Hence $L(\mu^*, \alpha)$ is a normal sub HX group of υ. □

Theorem 4.3. Let υ be a HX group and μ^* be a hyper Q-fuzzy subset of υ such that $L(\mu^*, \alpha)$ is a normal sub HX group of υ. For $\alpha \in [0,1] \inf \mu^*(U,q) \geq \alpha$ and $\sup \mu^*(U,q) \geq \alpha$, then μ^* is a hyper Q-fuzzy normal HX subgroup of υ.

Proof. Let $W, V \in \upsilon$ and $\mu^*(W,q) = \alpha_1$ this means that $\inf \mu^*(W,q) = \sup \mu^*(W,q) = \alpha_1, \mu^*(V,q) = \alpha_2$ this means that $\inf \mu^*(V,q) = \sup \mu^*(V,q) = \alpha_2$.

Suppose that $\alpha_1 > \alpha_2$, then $W, V \in L(\mu^*, \alpha)$ as $L(\mu^*, \alpha)$ is a subgroup of $\upsilon, WV^{-1} \in L(\mu^*, \alpha)$ hence $\inf \mu^*(WV^{-1}, q) \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} \geq \min\{\inf \mu^*(W,q),$ $\inf \mu^*(V,q)\}$. Thus $Inf \mu^*(WV^{-1}, q) \geq \min\{\inf \mu^*(W,q), \inf \mu^*(V,q)\}$ and $\sup \mu^*(WV^{-1}, q) \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} \geq \min\{\sup \mu^*(W,q), \sup \mu^*(V,q)\}$ then $\sup \mu^*(WV^{-1}, q) \geq \min\{\sup \mu^*(W,q), \sup \mu^*(V,q)\}$. Therefore μ^* is a hyper Q-fuzzy HX subgroup of υ. Now for any $\alpha \in [0,1] L(\mu^*, \alpha) \neq \phi$ and $L(\mu^*, \alpha)$ is normal sub HX group. Then, we have $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q), \sup \mu^*(WVW^{-1},$ $q) = \sup \mu^*(V,q)$ for all $W, V \in \upsilon, q \in Q$. Otherwise, if there exists W_0 or $V_0 \in \upsilon$ and $q \in Q$ such that $\inf \mu^*(W_0 V_0 W_0^{-1}, q) = \inf \mu^*(V_0, q), \sup \lambda^*(W_0 V_0 W_0^{-1}, q) = \sup \lambda^*(V_0, q)$, take $\alpha_0 = 0.5[\inf \mu^*(V_0, q) + \inf \mu^*(W_0 V_0 W_0^{-1}, q)] = 0.5[\sup \mu^*(V_0, q)$ $+ Sup \mu^*(W_0 V_0 W_0^{-1}, q)]$. Evidently $\alpha_0 \in [0,1]$, we can infer that $\inf \mu^*(V_0, q) > \alpha_0, \sup \mu^*(V_0, q) > \alpha_0$ and $\inf \mu^*(W_0 V_0 W_0^{-1}, q) < \alpha_0$ and $\sup \mu^*(W_0 V_0 W_0^{-1}, q) < \alpha_0$. Consequently, we have $V_0 \in L(\mu^*, \alpha_0)$ and $W_0 V_0 W_0^{-1} \notin L(\mu^*, \alpha)$, this contradicts that $L(\mu^*, \alpha_0)$ is normal sub HX group. Hence, we get $\inf \mu^*(V_0, q) = \inf \mu^*(W_0 V_0 W_0^{-1}, q), \sup \mu^*(V_0, q) = \sup \mu^*(W_0 V_0 W_0^{-1}, q)$ for all $W, V \in \upsilon, q \in Q$. Thus, μ^* is a hyper Q-fuzzy normal HX subgroup of υ. □

Theorem 4.4. If μ^* is a hyper Q-fuzzy normal HX subgroup of υ. If two level normal sub HX groups $L(\mu^*, \alpha_1), L(\mu^*, \alpha_2)$ for $\alpha_1, \alpha_2 \in [0,1]$ and that $\inf \mu^*(U,q) \geq \alpha_1, \alpha_2, \sup \mu^*(U,q) \geq \alpha_1, \alpha_2$ with $\alpha_1 > \alpha_2$ of μ^* are equal if and only if there is no $A \in \upsilon$ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$.

Proof. $(\Rightarrow)$ Let $L(\mu^*, \alpha_1) = L(\mu^*, \alpha_2)$, suppose there exists $W \in \upsilon$ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$ then $L(\mu^*, \alpha_1) \subseteq L(\mu^*, \alpha_2)$. Thus $W \in L(\mu^*, \alpha_1)$ but $W \notin L(\mu^*, \alpha_2)$, which is contradicts the assumption that $L(\mu^*, \alpha_1) = L(\mu^*, \alpha_2)$. Hence there is no μ^* in υ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$.

$(\Leftarrow)$ Suppose that there is no $W \in \upsilon$ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$. Thus $L(\mu^*, \alpha_2) \subseteq L(\mu^*, \alpha_1)$. Let $W \in L(\mu^*, \alpha_2)$ and there is no W in υ such that $\alpha_2 < \mu^*(W,q) \leq \alpha_1$. Hence $W \in L(\mu^*, \alpha_2)$ and $L(\mu^*, \alpha_1) \subseteq L\mu^*, \alpha_2)$ thus $L(\mu^*, \alpha_1) = L(\mu^*, \alpha_2)$. □

Corollary 4.5. A level subset $L(\mu^*, \alpha); \alpha \in [0,1]$ is normal sub HX group of υ, if μ^* be a hyper Q-fuzzy normal HX subgroup of υ.

Proof. Since μ^* is a hyper Q-fuzzy normal HX subgroup of υ and the Level subsets $L(\mu^*, \alpha); \alpha \in [0,1]$ is normal sub HX group of υ, let $W \in \upsilon$ and $B \in L(\mu^*, \alpha)$, then $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q) \geq \alpha$ and $\sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V,q) \geq \alpha$ since μ^* is a hyper Q-fuzzy normal HX subgroup of υ. That is, $\inf \mu^*(WVW^{-1}, q) \geq \alpha, \sup \mu^*(WVW^{-1}, q) \geq \alpha$ therefore $WVW^{-1} \in L(\mu^*, \alpha)$, hence $L(\mu^*, \alpha)$ is normal HX subgroup of υ. □

Definition 4.6. Let υ be a HX group and μ^* is a hyper Q-fuzzy normal HX subgroup of υ, let $N(\mu^*) = \{W \in \upsilon, \inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q)$ and $\sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V,q)$ for all $V \in \upsilon\}$. Then $N(\mu^*)$ is called the hyper Q-fuzzy normalizer of μ^*.

Proposition 4.7. Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ. Then

1. $N(\mu^*)$ is a sub HX group of υ.
2. μ^* is a hyper Q-fuzzy normal HX subgroup of υ iff $N(\mu^*) = \upsilon$.
3. μ^* is a hyper Q-fuzzy normal HX subgroup of $N(\mu^*)$.

Proof. 1. Let $W, V \in N(\mu^*)$ then $\inf \mu^*(WZW^{-1}, q) = \inf \mu^*(Z,q)$ and $\sup \mu^*(WZW^{-1}, q) = \sup \mu^*(Z,q)$ for all $Z \in \upsilon$.

Now $\inf \mu^*((WV)Z(WV)^{-1}, q) = \inf \mu^*(WVZV^{-1}W^{-1}, q) = \inf \mu^*(VZV^{-1}, q) = \inf \mu^*(Z,q)$ and $\sup \mu^*((WV)Z(WV)^{-1}, q) = \sup \mu^*(WVZV^{-1}W^{-1}, q) = \sup \mu^*(VZV^{-1}, q) = Sup\mu^*(Z,q)$ therefore, we get $\mu^*((WV)Z(WV)^{-1}, q) = \mu^*(Z,q)$ and thus $WVN(\mu^*)$. Hence $N(\mu^*)$ is a sub HX group of υ.

2. $(\Rightarrow)$ Let μ^* is a hyper Q-fuzzy normal HX subgroup of υ, clearly $N(\mu^*) \subseteq \upsilon$. Let $W \in \upsilon$, then $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V,q)$ and $\sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V,q)$ then $W \in N(\mu^*)$ and we get $\upsilon \subseteq N(\mu^*)$, hence $N(\mu^*) = \upsilon$.

($\Leftarrow$) Let $N(\mu^*) = \upsilon$. Clearly, $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V, q)$ and $\sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V, q)$ for all $W, V \in \upsilon$ and hence μ^* is a hyper Q-fuzzy normal HX subgroup of υ.

3. Clear from 2.

□

Theorem 4.8. Let μ^* is a hyper Q-fuzzy HX subgroup of υ and $W \in \upsilon$. Then the hyper Q-fuzzy subset $\lambda^* : \upsilon \times Q \to \wp([0,1])$ defined by $\lambda^*(V, q) = \lambda^*(W^{-1}VW, q)$ for all $V \in \upsilon$ is a hyper Q-fuzzy $HX-$subgroup of υ.

Proof. Let $W, Z \in \upsilon$, then for all $V \in \upsilon$

$$\begin{aligned}\inf \lambda^*(VZ^{-1}, q) &= \inf \mu^*(W^{-1}VZ^{-1}W, q)\\ &= \inf \mu^*(W^{-1}VWW^{-1}Z^{-1}W, q)\\ &= \inf \mu^*((W^{-1}VW)(W^{-1}ZW)^{-1}, q)\\ &\geq \min\{\inf \mu^*(W^{-1}VW, q), \inf \mu^*(W^{-1}ZW, q)\}\\ &= \min\{\inf \lambda^*(V, q), \inf \lambda^*(Z, q)\}.\end{aligned}$$

Also

$$\begin{aligned}\sup \lambda^*(VZ^{-1}, q) &= \sup \mu^*(W^{-1}VZ^{-1}W, q)\\ &= \sup \mu^*(W^{-1}VWW^{-1}Z^{-1}W, q)\\ &= \sup \mu^*((W^{-1}VW)(W^{-1}ZW)^{-1}, q)\\ &\geq \min\{\sup \mu^*(W^{-1}VW, q), \sup \mu^*(W^{-1}ZW, q)\}\\ &= \min\{\sup \lambda^*(V, q), \inf \lambda^*(Z, q)\}.\end{aligned}$$

Therefore μ^* is a hyper Q-fuzzy $HX-$subgroup of υ. □

Definition 4.9. Let μ^* and δ^* be two hyper Q-fuzzy HX subgroup of υ. We say that δ^* is conjugate to μ^* if for some $W \in \upsilon$ we have that $\inf \delta^*(V, q) = \inf \mu^*(W^{-1}VW, q)$ and $\sup \delta^*(V, q) = \sup \mu^*(W^{-1}VW, q)$ for all $V \in \upsilon, q \in Q$.

Definition 4.10. A hyper Q-fuzzy HX subgroup μ^* of υ is said to be self conjugate hyper Q-fuzzy HX subgroup if for all $W, Z \in \upsilon$, we have $\inf \mu^*(Z, q) = \inf \mu^*(W^{-1}ZW, q)$ and $\sup \mu^*(Z, q) = \sup \mu^*(W^{-1}ZW, q)$.

Theorem 4.11. A hyper Q-fuzzy HX subgroup μ^* of υ is normal if and only if μ^* is self conjugate hyper Q-fuzzy HX subgroup.

Proof. ($\Rightarrow$) Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ, then $\inf \mu^*(WV, q) = \inf \mu^*(VW, q)$ and $\sup \mu^*(WV, q) = \sup \mu^*(VW, q)$, thus by Theorem 3.9 we have $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V, q), \sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V, q)$ for all $W, V \in \upsilon, q \in Q$. So, μ^* is a self conjugate hyper Q-fuzzy HX subgroup.

($\Leftarrow$) Let μ^* be a self conjugate hyper Q-fuzzy HX subgroup.

Then $\inf \mu^*(WVW^{-1}, q) = \inf \mu^*(V, q), \sup \mu^*(WVW^{-1}, q) = \sup \mu^*(V, q)$ for all $W, V \in \upsilon, q \in Q$. So, μ^* is a hyper Q-fuzzy HX subgroup of υ. □

Theorem 4.12. Let μ^* be a hyper Q-fuzzy normal HX subgroup of υ and $\lambda^* : N(\mu^*) \to \wp([0,1])$ is defined by $\lambda^*(W, q) = \mu^*(W, q)$ for all $W \in N(\mu^*)$ and $q \in Q$. Then λ^* is a hyper Q-fuzzy normal HX subgroup of $N(\mu^*)$.

Proof. Since μ^* be a hyper Q-fuzzy normal HX subgroup of υ and we have proved that $N(\mu^*)$ is a HX subgroup of υ. Then μ^* be a hyper Q-fuzzy HX subgroup of $N(\mu^*)$. Hence λ^* is a hyper Q-fuzzy HX subgroup of $N(\mu^*)$. We have to prove λ^* is normal, since $N(\mu^*)$ is a HX subgroup of υ, then $W, V \in N(\mu^*)$ thus $W^{-1}VW \in N(\mu^*)$ also,

$$\begin{aligned} \inf \lambda^*(W^{-1}VW, q) &= \inf \mu^*(W^{-1}VW, q)(\text{ since } W^{-1}VW \in N(\mu^*)) \\ &= \inf \mu^*(V, q)(\text{ since } V \in N(\mu^*)) \\ &= \inf \lambda^*(V, q)(\text{ since } V \in N(\mu^*)). \end{aligned}$$

Also

$$\begin{aligned} \sup \lambda^*(W^{-1}VW, q) &= \sup \mu^*(W^{-1}VW, q)(\text{ since } W^{-1}VW \in N(\mu^*)) \\ &= \sup \mu^*(V, q)(\text{ since } V \in N(\mu^*)) \\ &= \sup \lambda^*(V, q)(\text{ since } V \in N(\mu^*)). \end{aligned}$$

□

Conclusion

In this research,the authors presented hyper Q-fuzzy HX subgroup of a HX group, hyper Q-fuzzy normal HX subgroup of a HX group, level sub HX group and conjugate are well defined and the authors discussed some of their conceptes. In the future, we can extend this concept to cosets and its types, homomorphism and in more areas of HX. On the other side, we will use the topics in [21, 22, 23, 24] to generalize on the hyper Q-fuzzy set concept.

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