Mixed monotone property and tripled fixed point theorems in partially ordered bipolar-metric spaces

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Abstract. In this paper, we obtain the existence and unique tripled fixed point theorems by using weak contractivity type and mixed monotone type mapping in a Bipolar metric space endowed with partial order. Also, an example which supports our main result is given.
Keywords: bipolar metric spaces, partially ordered set, mixed monotone property, tripled fixed point.

1. Introduction

In 1922, S. Banach [1] introduced the concept of Banach contraction principle. It is the most celebrated fixed point result in nonlinear analysis. The concept of mixed monotone mapping has been introduced by Bhaskar and Lakshmikantham [2] and established some coupled fixed point results for mixed monotone

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mappings. Subsequently to improve many authors have established coupled fixed point results for mixed monotone see ([3]-[7]). In 2011, V. Berinde and M. Borcut [8] initiated concept of triple fixed point and established some fixed point results for contractive mappings in partially ordered metric spaces. Afterward many investigators have established triples fixed point results for different spaces see ([9]-[11]).

Very recently, in 2016 Mutlu and Gürdal [12] introduced the notion of Bipolar metric spaces, which is one of the generalizations metric spaces. Also they investigated some fixed point and coupled fixed point results on this space, see ([12], [13]). Later, we proved some fixed point theorems in our earlier papers ([14]-[17]).

In this paper, we prove tripled fixed point results in bipolar metric spaces. Also, we gave an example which supports our main result.

2. Preliminaries

First we recall some basic definitions and examples.

Definition 2.1 ([12])**.** *Let A and B be two non-empty sets. Suppose that $d : A \times B \to [0, \infty)$ is a mapping satisfying the following properties :*

(B_0) *If $d(a, b) = 0$ then $a = b$ for all $(a, b) \in A \times B$,*

(B_1) *If $a = b$ then $d(a, b) = 0$, for all $(a, b) \in A \times B$,*

(B_2) *If $d(a, b) = d(b, a)$, for all $a, b \in A \cap B$.*

(B_3) *If $d(a_1, b_2) \leq d(a_1, b_1) + d(a_2, b_1) + d(a_2, b_2)$ for all $a_1, a_2 \in A$, $b_1, b_2 \in B$.*

Then the mapping d is called a bipolar-metric on the pair (A, B) and the triple (A, B, d) is called a bipolar-metric space.

Example 2.2 ([12])**.** Let $A = (1, \infty)$ and $B = [-1, 1]$. Define $d : A \times B \to [0, +\infty)$ as $d(a, b) = |a^2 - b^2|$ for all $(a, b) \in (A, B)$, then the triple (A, B, d) is a bipolar-metric space.

Definition 2.3 ([12])**.** *Assume (A_1, B_1) and (A_2, B_2) as two pairs of sets.*

The function $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be covariant map if $F(A_1) \subseteq A_2$ and $F(B_1) \subseteq B_2$ and denote $F : (A_1, B_1) \rightrightarrows (A_2, B_2)$.

The mapping $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be contravariant map if $F(A_1) \subseteq B_2$ and $F(B_1) \subseteq A_2$ and denote $F : (A_1, B_1) \leftrightarrows (A_2, B_2)$.

In particular, if d_1 and d_2 are bipolar metrics in (A_1, B_1) and (A_2, B_2) respectively, then some times we use the notations $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and $F : (A_1, B_1, d_1) \leftrightarrows (A_2, B_2, d_2)$.

Definition 2.4 ([12])**.** *Let (A, B, d) be a bipolar metric space. A point $v \in A \cup B$ is said to be left point if $v \in A$, a right point if $v \in B$ and a central point if both.*

Similarly, a sequence $\{a_n\}$ *on the set* A *and a sequence* $\{b_n\}$ *on the set* B *are called a left and right sequence respectively.*

In a bipolar metric space, sequence is the simple term for left or right sequence.

A sequence $\{v_n\}$ *is convergent to a point* v *if and only if* $\{v_n\}$ *is a left sequence,* v *is a right point and* $\lim_{n\to\infty} d(v_n, v) = 0$*; or* $\{v_n\}$ *is a right sequence,* v *is a left point and* $\lim_{n\to\infty} d(v, v_n) = 0$.

A bisequence $(\{a_n\}, \{b_n\})$ *on* (A, B, d) *is sequence on the set* $A \times B$*. If the sequence* $\{a_n\}$ *and* $\{b_n\}$ *are convergent, then the bisequence* $(\{a_n\}, \{b_n\})$ *is said to be convergent.* $(\{a_n\}, \{b_n\})$ *is Cauchy sequence , if* $\lim_{n,m\to\infty} d(a_n, b_m) = 0$.

A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 2.5 ([12])**.** *Let* (A_1, B_1, d_1) *and* (A_2, B_2, d_2) *be bipolar metric spaces.*

(*i*) *A map* $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ *is called left-continuous at a point* $a_0 \in A_1$*, if for every* $\epsilon > 0$*, there is a* $\delta > 0$ *such that* $d_1(a_0, b) < \delta$ *implies that* $d_2(F(a_0), F(b)) < \epsilon$ *for all* $b \in B_1$.

(*ii*) *A map* $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ *is called right-continuous at a point* $b_0 \in B_1$*, if for every* $\epsilon > 0$*, there is a* $\delta > 0$ *such that* $d_1(a, b_0) < \delta$ *implies that* $d_2(F(a), F(b_0)) < \epsilon$ *for all* $a \in A_1$.

(*iii*) *A map* F *is called continuous, if it left continuous at each point* $a \in A_1$ *and right continuous at each point* $b \in B_1$.

(*iv*) *A contravariant map* $F : (A_1, B_1, d_1) \leftrightarrows (A_2, B_2, d_2)$ *is continuous if and only if it is continuous as a covariant map* $F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$.

It can be seen from the definition (2.4) that a covariant or a contravariant map F *from* (A_1, B_1, d_1) *to* (A_2, B_2, d_2) *is continuous if and only if* $(u_n) \to v$ *on* (A_1, B_1, d_1) *implies* $F((u_n)) \to F(v)$ *on* (A_2, B_2, d_2).

3. Main results

Definition 3.1. *Let* $(A, B, \leq)$ *be a partial ordered set and* $F : (A, B) \rightrightarrows (A, B)$ *be a covariant mapping, we say that* F *is non-decreasing with respect to* $\leq$ *if* $a, b \in A \cup B$*,* $a \leq b$ *implies* $F(a) \leq F(b)$*, and similarly, a non-increasing mapping is defined.*

Definition 3.2. *Let* $(A, B, \leq)$ *be a partially ordered set and* $F : (A^3, B^3) \rightrightarrows (A, B)$ *be a covariant map. The map* F *has the mixed monotone property, if* $F(a, b, c)$ *is monotone non-decreasing in* a *and* c *and is monotone non-increasing in* b*, that is, for any* $a, b, c \in A \cup B$.

$$a_1, a_2 \in A \cup B, \quad a_1 \leq a_2 \Rightarrow F(a_1, b, c) \leq F(a_2, b, c)$$

$$b_1, b_2 \in A \cup B, \quad b_1 \le b_2 \Rightarrow F(a, b_1, c) \ge F(a, b_2, c)$$

and

$$c_1, c_2 \in A \cup B, \quad c_1 \le c_2 \Rightarrow F(a, b, c_1) \le F(a, b, c_2)$$

Definition 3.3. *Let $F : (A^3, B^3) \rightrightarrows (A, B)$ be a covariant map, an element $(a, b, c) \in A^3 \cup B^3$ is called tripled fixed point of F if*

$$F(a, b, c) = a, \quad F(b, a, b) = b \quad and \quad F(c, b, a) = c.$$

Let $(A, B, \le)$ be a partially ordered set and d be a bipolar metric on (A, B) such that (A, B, d) is complete bipolar metric space. Moreover, we endow the product space (A^3, B^3) with the following partial order: For

$$(a, b, c), (p, q, r) \in A^3 \cup B^3, \quad (p, q, r) \le (a, b, c) \Leftrightarrow a \ge p, \quad b \le q, \quad c \ge r.$$

We begin with the following theorem that achieves the existence of a fixed point results for a mapping F on the product space (A^3, B^3).

Theorem 3.4. *Let $(A, B, \le)$ be a partially ordered set and suppose that d is bipolar metric on (A, B) such that (A, B, d) is a complete bipolar metric spaces. Let the covariant map $F : (A^3, B^3) \rightrightarrows (A, B)$ is a continuous mapping having the mixed monotone property on (A, B) and μ , λ, κ be a non-negative constants with the condition*

$$d\left(F(l, m, n), F(r, s, t)\right) \le \mu d\left(l, r\right) + \lambda d\left(m, s\right) + \kappa d\left(n, t\right) \tag{3.1}$$

$\forall \;\; l, m, n \in A \;\; r, s, t \in B$ *with* $l \ge r, \quad m \le s, n \ge t$ *and* $\mu + \lambda + \kappa < 1$.
If there exist $(l_0, m_0, n_0) \in A^3 \cup B^3$ *such that*

$$l_0 \le F(l_0, m_0, n_0), \quad m_0 \ge F(m_o, l_0, m_0) \text{ and } n_0 \le F(n_0, m_0, l_0).$$

Then there exist $(l, m, n) \in A^3 \cup B^3$ *such that The mapping* $F : A^3 \cup B^3 \to A \cup B$ *has*

$$F(l, m, n) = l, \qquad F(m, l, m) = m \quad and \quad F(n, m, l) = n$$

Proof. Let $l_0, m_0, n_0 \in A$ and $r_0, s_0, t_0 \in B$, we choose an elements $l_1, m_1, n_1 \in A$ and $r_1, s_1, t_1 \in B$ such that

$$l_0 \le F(l_0, m_0, n_0)) = l_1, \quad m_0 \ge F(m_o, l_0, m_0) = m_1, \text{ and } n_0 \le F(n_0, m_0, l_0) = n_1,$$

and also

$$r_0 \le F(r_0, s_0, t_0) = r_1, \quad s_0 \ge F(s_o, r_0, s_0) = s_1, \text{ and } t_0 \le F(t_0, s_0, r_0) = t_1$$

similarly, we take

$$l_2 = F(l_1, m_1, n_1), \quad m_2 = F(m_1, l_1, m_1), \text{ and } n_2 = F(n_1, m_1, l_1)$$

$$r_2 = F(r_1, s_1, t_1), \quad s_2 = F(s_1, r_1, s_1) \text{ and } t_2 = F(t_1, s_1, r_1)$$

we denote

$$F^2(l_0, m_0, n_0) = F\left(F(l_0, m_0, n_0), F(m_0, l_0, m_0), F(n_0, m_0, l_0)\right) = F(l_1, m_1, n_1) = l_2,$$
$$F^2(m_0, l_0, m_0) = F\left(F(m_0, l_0, m_0), F(l_0, m_0, n_0), F(m_0, l_0, m_0)\right) = F(m_1, l_1, m_1) = m_2,$$
$$F^2(n_0, m_0, l_0) = F\left(F(n_0, m_0, l_0), F(m_0, l_0, m_0), F(l_0, m_0, n_0)\right) = F(n_1, m_1, l_1) = n_2,$$
$$F^2(r_0, s_0, t_0) = F\left(F(r_0, s_0, t_0), F(s_0, r_0, s_0), F(t_0, s_0, r_0)\right) = F(r_1, s_1, t_1) = r_2,$$
$$F^2(s_0, r_0, s_0) = F\left(F(s_0, r_0, s_0), F(r_0, s_0, t_0), F(s_0, r_0, s_0)\right) = F(s_1, r_1, s_1) = s_2,$$
$$F^2(t_0, s_0, r_0) = F\left(F(t_0, s_0, r_0), F(s_0, r_0, s_0), F(r_0, s_0, t_0)\right) = F(t_1, s_1, r_1) = t_2,$$

for $n \geq 1$, we denote

$$\begin{aligned} l_n &= F(l_{n-1}, m_{n-1}, n_{n-1}) \quad & r_n &= F(r_{n-1}, s_{n-1}, t_{n-1}) \\ m_n &= F(m_{n-1}, l_{n-1}, m_{n-1}) \quad & s_n &= F(s_{n-1}, r_{n-1}, s_{n-1}) \\ n_n &= F(n_{n-1}, m_{n-1}, l_{n-1}) \quad & t_n &= F(t_{n-1}, s_{n-1}, r_{n-1}). \end{aligned} \tag{3.2}$$

Due to the mixed monotone property, We can obviously verify that

$$l_0 \leq F(l_0, m_0, n_0) = l_1 \leq F(l_1, m_1, n_1) = l_2 \leq \cdots \leq F(l_{n-1}, m_{n-1}, n_{n-1}) \leq \ldots$$
$$m_0 \geq F(m_o, l_0, m_0) = m_1 \geq F(m_1, l_1, m_1) = m_2 \geq \cdots \geq F(m_{n-1}, l_{n-1}, m_{n-1}) \geq \ldots$$
$$n_0 \leq F(n_0, m_0, l_0) = n_1 \leq F(n_1, m_1, l_1) = n_2 \leq \cdots \leq F(n_{n-1}, m_{n-1}, l_{n-1}) \leq \ldots$$
$$r_0 \leq F(r_0, s_0, t_o) = r_1 \leq F(r_1, s_1, t_1) = r_2 \leq \cdots \leq F(r_{n-1}, s_{n-1}, t_{n-1}) \leq \ldots$$
$$s_0 \geq F(s_o, r_0, s_o) = s_1 \geq F(s_1, r_1, s_1) = s_2 \geq \cdots \geq F(s_{n-1}, r_{n-1}, s_{n-1}) \geq \ldots$$
$$t_0 \leq F(t_0, s_0, r_o) = t_1 \leq F(t_1, s_1, r_1) = t_2 \leq \cdots \leq F(t_{n-1}, s_{n-1}, r_{n-1}) \leq \ldots$$

thus, we obtain three bisequences satisfying the following conditions

$$l_0 \leq l_1 \leq l_2 \leq \cdots \leq l_n \leq \ldots, \qquad r_0 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq \ldots$$

$$m_0 \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq \ldots, \qquad s_0 \geq s_1 \geq s_2 \geq \cdots \geq s_n \geq \ldots$$

$$n_0 \leq n_1 \leq n_2 \leq \cdots \leq n_n \leq \ldots, \qquad t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq \ldots$$

to simplify the writing, denote

$$D_n^{X_1} = d(l_{n-1}, r_n) \quad D_n^{Y_1} = d(m_{n-1}, s_n) \quad D_n^{Z_1} = d(n_{n-1}, t_n)$$

$$D_n^{X_2} = d(l_n, r_{n-1}) \quad D_n^{Y_2} = d(m_n, s_{n-1}) \quad D_n^{Z_2} = d(n_n, t_{n-1})$$

$$D_{n-1}^{X_3} = d(l_{n-1}, r_{n-1}) \quad D_{n-1}^{Y_3} = d(m_{n-1}, s_{n-1}) \quad D_{n-1}^{Z_3} = d(n_{n-1}, t_{n-1}).$$

Then by (3.1), we get

$$\begin{aligned} D_2^{X_1} = d(l_1, r_2) \quad &= d\left(F(l_0, m_0, n_0), F(r_1, s_1, t_1)\right) \\ &\leq \mu d\left(l_0, r_1\right) + \lambda d\left(m_0, s_1\right) + \kappa d\left(n_0, t_1\right) \\ &\leq \mu D_1^{X_1} + \lambda D_1^{Y_1} + \kappa D_1^{Z_1}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} D_2^{Y_1} = d(m_1, s_2) &= d\left(F(m_0, l_0, m_0), F(s_1, r_1, s_1)\right) \\ &\leq \mu d\left(m_0, s_1\right) + \lambda d\left(l_0, r_1\right) + \kappa d\left(m_0, s_1\right) \\ &\leq \mu D_1^{Y_1} + \lambda D_1^{X_1} + \kappa D_1^{Y_1} \\ &= \lambda D_1^{X_1} + (\mu + \kappa) D_1^{Y_1} + 0 D_1^{Z_1} \end{aligned}$$

$$\begin{aligned} D_2^{Z_1} = d(n_1, t_2) &= d\left(F(n_0, m_0, l_0), F(t_1, s_1, r_1)\right) \\ &\leq \mu d\left(n_0, t_1\right) + \lambda d\left(m_0, s_1\right) + \kappa d\left(l_0, r_1\right) \\ &\leq \mu D_1^{Z_1} + \lambda D_1^{Y_1} + \kappa D_1^{X_1} \end{aligned}$$

and

$$\begin{aligned} D_3^{X_1} = d(l_2, r_3) &= d\left(F(l_1, m_1, n_1), F(r_2, s_2, t_2)\right) \\ &\leq \mu d\left(l_1, r_2\right) + \lambda d\left(m_1, s_2\right) + \kappa d\left(n_1, t_2\right) \\ &\leq \mu D_2^{X_1} + \lambda D_2^{Y_1} + \kappa D_2^{Z_1} \\ &\leq (\mu^2 + \lambda^2 + \kappa^2) D_1^{X_1} + (2\mu\lambda + 2\lambda\kappa) D_1^{Y_1} + 2\mu\kappa D_1^{Z_1} \end{aligned}$$

$$\begin{aligned} D_3^{Y_1} = d(m_2, s_3) &= d\left(F(m_1, l_1, m_1), F(s_2, r_2, s_2)\right) \\ &\leq \mu d\left(m_1, s_2\right) + \lambda d\left(l_1, r_2\right) + \kappa d\left(m_1, s_2\right) \\ &\leq \mu D_2^{Y_1} + \lambda D_2^{X_1} + \kappa D_2^{Y_1} \\ &\leq \lambda D_2^{X_1} + (\mu + \kappa) D_2^{Y_1} \\ &\leq \lambda\mu D_1^{X_1} + \lambda^2 D_1^{Y_1} + \lambda\kappa D_1^{Z_1}) + (\lambda\mu + \lambda\kappa) D_1^{X_1} \\ &\quad + (\mu + \kappa)^2 D_1^{Y_1} + 0 D_1^{Z_1}) \\ &\leq (2\lambda\mu + \lambda\kappa) D_1^{X_1} + (\lambda^2 + (\mu + \kappa)^2) D_1^{Y_1} + \lambda\kappa D_1^{Z_1} \end{aligned}$$

$$\begin{aligned} D_3^{Z_1} = d(n_2, t_3) &= d\left(F(n_1, m_1, l_1), F(t_2, s_2, r_2)\right) \\ &\leq \mu d\left(n_1, t_2\right) + \lambda d\left(m_1, s_2\right) + \kappa d\left(l_1, r_2\right) \\ &\leq \mu D_2^{Z_1} + \lambda D_2^{Y_1} + \kappa D_2^{X_1} \\ &\leq \mu(\mu D_1^{Z_1} + \lambda D_1^{Y_1} + \kappa D_1^{X_1}) + \lambda(\lambda D_1^{X_1} + (\mu + \kappa) D_1^{Y_1} + 0 D_1^{Z_1}) \\ &\quad + \kappa(\mu D_1^{X_1} + \lambda D_1^{Y_1} + \kappa D_1^{Z_1}) \\ &\leq \mu^2 D_1^{Z_1} + \mu\lambda D_1^{Y_1} + \mu\kappa D_1^{X_1} + \lambda^2 D_1^{X_1} + (\lambda\mu + \lambda\kappa) D_1^{Y_1} + 0 D_1^{Z_1}) \\ &\quad + \kappa\mu D_1^{X_1} + \kappa\lambda D_1^{Y_1} + \kappa^2 D_1^{Z_1} \\ &\leq (2\mu\kappa + \lambda^2) D_1^{X_1} + (2\mu\lambda + 2\lambda\kappa) D_1^{Y_1} + (\mu^2 + \kappa^2) D_1^{Z_1}. \end{aligned}$$

In order to simplify writing we also consider the matrix

$$X = \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu + \kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} = \begin{pmatrix} i_1 & j_1 & k_1 \\ l_1 & m_1 & n_1 \\ p_1 & j_1 & q_1 \end{pmatrix}$$

and further denote

$$X^2 = \begin{pmatrix} \mu^2 + \lambda^2 + \kappa^2 & 2\mu\lambda + 2\lambda\kappa & 2\mu\kappa \\ 2\lambda\mu & \lambda^2 + (\mu + \kappa)^2 & \lambda\kappa \\ 2\mu\kappa + \lambda^2 & 2\mu\lambda + 2\lambda\kappa & \mu^2 + \kappa^2 \end{pmatrix} = \begin{pmatrix} i_2 & j_2 & k_2 \\ l_2 & m_2 & n_2 \\ p_2 & j_2 & q_2 \end{pmatrix}$$

where $i_2+j_2+k_2=l_2+m_2+n_2=p_2+j_2+q_2=(\mu+\lambda+\kappa)^2<\mu+\lambda+\kappa<1$. Now we prove by induction

$$X^n=\begin{pmatrix} i_n & j_n & k_n \\ l_n & m_n & n_n \\ p_n & j_n & q_n \end{pmatrix}$$

where

$$i_n+j_n+k_n=l_n+m_n+n_n=p_n+j_n+q_n=(\mu+\lambda+\kappa)^n<\mu+\lambda+\kappa<1. \tag{3.3}$$

Indeed, if we assume that (3.3) is true for n, then since

$$\begin{aligned} X^{n+1}=X^n.X &= \begin{pmatrix} i_n & j_n & k_n \\ l_n & m_n & n_n \\ p_n & j_n & q_n \end{pmatrix}.\begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu+\kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} \\ &= \begin{pmatrix} i_n\mu+j_n\lambda+k_n\kappa & i_n\lambda+j_n(\mu+\kappa)+k_n\lambda & i_n\kappa+k_n\mu \\ l_n\mu+m_n\lambda+n_n\kappa & l_n\lambda+m_n(\mu+\kappa)+n_n\lambda & l_n\kappa+n_n\mu \\ p_n\mu+j_n\lambda+q_n\kappa & p_n\lambda+j_n(\mu+\kappa)+q_n\lambda & p_n\kappa+q_n\mu \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned} i_{n+1}+j_{n+1}+k_{n+1} &= i_n\mu+j_n\lambda+k_n\kappa+i_n\lambda+j_n(\mu+\kappa)+k_n\lambda+i_n\kappa+k_n\mu \\ &= (\mu+\lambda+\kappa)i_n+(\mu+\lambda+\kappa)j_n+(\mu+\lambda+\kappa)k_n \\ &= (i_n+j_n+k_n)(\mu+\lambda+\kappa)=(\mu+\lambda+\kappa)^n(\mu+\lambda+\kappa) \\ &= (\mu+\lambda+\kappa)^{n+1}<\mu+\lambda+\kappa<1. \end{aligned}$$

Similarly, we can prove that $l_{n+1}+m_{n+1}+n_{n+1}=p_{n+1}+j_{n+1}+q_{n+1}=(\mu+\lambda+\kappa)^{n+1}<\mu+\lambda+\kappa<1$. Therefore, we get

$$\begin{pmatrix} D_{n+1}^{X_1} \\ D_{n+1}^{Y_1} \\ D_{n+1}^{Z_1} \end{pmatrix} \le \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu+\kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix}.\begin{pmatrix} D_1^{X_1} \\ D_1^{Y_1} \\ D_1^{Z_1} \end{pmatrix}$$

that is

$$\begin{aligned} D_{n+1}^{X_1} &\le i_nD_1^{X_1}+j_nD_1^{Y_1}+k_nD_1^{Z_1} \\ D_{n+1}^{Y_1} &\le l_nD_1^{X_1}+m_nD_1^{Y_1}+n_nD_1^{Z_1} \\ D_{n+1}^{Z_1} &\le p_nD_1^{X_1}+j_nD_1^{Y_1}+q_nD_1^{Z_1}. \end{aligned} \tag{3.4}$$

On the other hand

$$\begin{aligned} D_2^{X_2}=d(l_2,r_1) &= d\left(F(l_1,m_1,n_1),F(r_0,s_0,t_0)\right) \\ &\le \mu d\left(l_1,r_0\right)+\lambda d\left(m_1,s_0\right)+\kappa d\left(n_1,t_0\right) \\ &\le \mu D_1^{X_2}+\lambda D_1^{Y_2}+\kappa D_1^{Z_2}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} D_2^{Y_2} = d(m_2, s_1) &= d\left(F(m_1, l_1, m_1), F(s_0, r_0, s_0)\right) \\ &\le \mu d\left(m_1, s_0\right) + \lambda d\left(l_1, r_0\right) + \kappa d\left(m_1, s_0\right) \\ &\le \mu D_1^{Y_2} + \lambda D_1^{X_2} + \kappa D_1^{Y_2} \\ &= \lambda D_1^{X_2} + (\mu+\kappa) D_1^{Y_2} + 0 D_1^{Z_2} \end{aligned}$$

$$\begin{aligned} D_2^{Z_2} = d(n_2, t_1) &= d\left(F(n_1, m_1, l_1), F(t_0, s_0, r_0)\right) \\ &\le \mu d\left(n_1, t_0\right) + \lambda d\left(m_1, s_0\right) + \kappa d\left(l_1, r_0\right) \\ &\le \mu D_1^{Z_2} + \lambda D_1^{Y_2} + \kappa D_1^{X_2} \end{aligned}$$

and

$$\begin{aligned} D_3^{X_2} = d(l_3, r_2) &= d\left(F(l_2, m_2, n_2), F(r_1, s_1, t_1)\right) \\ &\le \mu d\left(l_2, r_1\right) + \lambda d\left(m_2, s_1\right) + \kappa d\left(n_2, t_1\right) \\ &\le \mu D_2^{X_2} + \lambda D_2^{Y_2} + \kappa D_2^{Z_2} \\ &\le (\mu^2+\lambda^2+\kappa^2) D_1^{X_2} + (2\mu\lambda + 2\lambda\kappa) D_1^{Y_2} + 2\mu\kappa D_1^{Z_2} \end{aligned}$$

$$\begin{aligned} D_3^{Y_2} = d(m_3, s_2) &= d\left(F(m_2, l_2, m_2), F(s_1, r_1, s_1)\right) \\ &\le \mu d\left(m_2, s_1\right) + \lambda d\left(l_2, r_1\right) + \kappa d\left(m_2, s_1\right) \\ &\le \mu D_2^{Y_2} + \lambda D_2^{X_2} + \kappa D_2^{Y_2} \\ &\le \lambda D_2^{X_2} + (\mu+\kappa) D_2^{Y_2} \\ &\le \lambda\mu D_1^{X_2} + \lambda^2 D_1^{Y_2} + \lambda\kappa D_1^{Z_2}) + (\lambda\mu + \lambda\kappa) D_1^{X_2} \\ &\quad + (\mu+\kappa)^2 D_1^{Y_2} + 0 D_1^{Z_2}) \\ &\le (2\lambda\mu + \lambda\kappa) D_1^{X_2} + (\lambda^2 + (\mu+\kappa)^2) D_1^{Y_2} + \lambda\kappa D_1^{Z_2} \end{aligned}$$

$$\begin{aligned} D_3^{Z_2} = d(n_3, t_2) &= d\left(F(n_2, m_2, l_2), F(t_1, s_1, r_1)\right) \\ &\le \mu d\left(n_2, t_1\right) + \lambda d\left(m_2, s_1\right) + \kappa d\left(l_2, r_1\right) \\ &\le \mu D_2^{Z_2} + \lambda D_2^{Y_2} + \kappa D_2^{X_2} \\ &\le \mu(\mu D_1^{Z_2} + \lambda D_1^{Y_2} + \kappa D_1^{X_2}) + \lambda(\lambda D_1^{X_2} + (\mu+\kappa) D_1^{Y_2} + 0 D_1^{Z_1}) \\ &\quad + \kappa(\mu D_1^{X_2} + \lambda D_1^{Y_2} + \kappa D_1^{Z_2}) \\ &\le \mu^2 D_1^{Z_2} + \mu\lambda D_1^{Y_2} + \mu\kappa D_1^{X_2} + \lambda^2 D_1^{X_2} + (\lambda\mu + \lambda\kappa) D_1^{Y_2} + 0 D_1^{Z_2}) \\ &\quad + \kappa\mu D_1^{X_2} + \kappa\lambda D_1^{Y_2} + \kappa^2 D_1^{Z_2} \\ &\le (2\mu\kappa + \lambda^2) D_1^{X_2} + (2\mu\lambda + 2\lambda\kappa) D_1^{Y_2} + (\mu^2+\kappa^2) D_1^{Z_2}. \end{aligned}$$

In order to simplify writing we also consider the matrix

$$Y = \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu+\kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} = \begin{pmatrix} i_1 & j_1 & k_1 \\ l_1 & m_1 & n_1 \\ p_1 & j_1 & q_1 \end{pmatrix}$$

and further denote

$$Y^2 = \begin{pmatrix} \mu^2+\lambda^2+\kappa^2 & 2\mu\lambda + 2\lambda\kappa & 2\mu\kappa \\ 2\lambda\mu & \lambda^2 + (\mu+\kappa)^2 & \lambda\kappa \\ 2\mu\kappa + \lambda^2 & 2\mu\lambda + 2\lambda\kappa & \mu^2+\kappa^2 \end{pmatrix} = \begin{pmatrix} i_2 & j_2 & k_2 \\ l_2 & m_2 & n_2 \\ p_2 & j_2 & q_2 \end{pmatrix}$$

where $i_2+j_2+k_2 = l_2+m_2+n_2 = p_2+j_2+q_2 = (\mu+\lambda+\kappa)^2 < \mu+\lambda+\kappa < 1$. Now we prove by induction

$$Y^n = \begin{pmatrix} i_n & j_n & k_n \\ l_n & m_n & n_n \\ p_n & j_n & q_n \end{pmatrix}$$

where

$$i_n+j_n+k_n = l_n+m_n+n_n = p_n+j_n+q_n = (\mu+\lambda+\kappa)^n < \mu+\lambda+\kappa < 1. \tag{3.5}$$

Indeed, if we assume that (3.5) is true for n, then since

$$\begin{aligned} Y^{n+1} = Y^n.Y &= \begin{pmatrix} i_n & j_n & k_n \\ l_n & m_n & n_n \\ p_n & j_n & q_n \end{pmatrix} . \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu+\kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} \\ &= \begin{pmatrix} i_n\mu + j_n\lambda + k_n\kappa & i_n\lambda + j_n(\mu+\kappa) + k_n\lambda & i_n\kappa + k_n\mu \\ l_n\mu + m_n\lambda + n_n\kappa & l_n\lambda + m_n(\mu+\kappa) + n_n\lambda & l_n\kappa + n_n\mu \\ p_n\mu + j_n\lambda + q_n\kappa & p_n\lambda + j_n(\mu+\kappa) + q_n\lambda & p_n\kappa + q_n\mu \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned} i_{n+1} + j_{n+1} + k_{n+1} &= i_n\mu + j_n\lambda + k_n\kappa + i_n\lambda + j_n(\mu+\kappa) + k_n\lambda + i_n\kappa + k_n\mu \\ &= (\mu+\lambda+\kappa)i_n + (\mu+\lambda+\kappa)j_n + (\mu+\lambda+\kappa)k_n \\ &= (i_n+j_n+k_n)(\mu+\lambda+\kappa) = (\mu+\lambda+\kappa)^n(\mu+\lambda+\kappa) \\ &= (\mu+\lambda+\kappa)^{n+1} < \mu+\lambda+\kappa < 1. \end{aligned}$$

Similarly, we can prove that $l_{n+1} + m_{n+1} + n_{n+1} = p_{n+1} + j_{n+1} + q_{n+1} = (\mu+\lambda+\kappa)^{n+1} < \mu+\lambda+\kappa < 1$. Therefore, we get

$$\begin{pmatrix} D^{X_2}_{n+1} \\ D^{Y_2}_{n+1} \\ D^{Z_2}_{n+1} \end{pmatrix} \leq \begin{pmatrix} \mu & \lambda & \kappa \\ \lambda & \mu+\kappa & 0 \\ \kappa & \lambda & \mu \end{pmatrix} . \begin{pmatrix} D^{X_2}_1 \\ D^{Y_2}_1 \\ D^{Z_2}_1 \end{pmatrix}$$

that is

$$\begin{aligned} D^{X_2}_{n+1} &\leq i_n D^{X_2}_1 + j_n D^{Y_2}_1 + k_n D^{Z_2}_1 \\ D^{Y_2}_{n+1} &\leq l_n D^{X_2}_1 + m_n D^{Y_2}_1 + n_n D^{Z_2}_1 \\ D^{Z_2}_{n+1} &\leq p_n D^{X_2}_1 + j_n D^{Y_2}_1 + q_n D^{Z_2}_1. \end{aligned} \tag{3.6}$$

Moreover,

$$\begin{aligned} D^{X_3}_{n+1} &\leq i_n D^{X_3}_1 + j_n D^{Y_3}_1 + k_n D^{Z_3}_1 \\ D^{Y_3}_{n+1} &\leq l_n D^{X_3}_1 + m_n D^{Y_3}_1 + n_n D^{Z_3}_1 \\ D^{Z_3}_{n+1} &\leq p_n D^{X_3}_1 + j_n D^{Y_3}_1 + q_n D^{Z_3}_1. \end{aligned} \tag{3.7}$$

Using the property (B_3), we obtain

$$\begin{aligned}
&d(l_n, r_m) \leq d(l_n, r_{n+1}) + d(l_{n+1}, r_{n+1}) + \cdots d(l_{m-1}, r_m)\\
&d(m_n, s_m) \leq d(m_n, s_{n+1}) + d(m_{n+1}, s_{n+1}) + \cdots d(m_{m-1}, s_m)\\
&d(n_n, t_m) \leq d(n_n, t_{n+1}) + d(n_{n+1}, t_{n+1}) + \cdots d(n_{m-1}, t_m)
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&d(l_m, r_n) \leq d(l_m, r_{m-1}) + d(l_{m-1}, r_{m-1}) + \cdots d(l_{n+1}, r_n)\\
&d(m_m, s_n) \leq d(m_m, s_{m-1}) + d(m_{m-1}, s_{m-1}) + \cdots d(m_{n+1}, s_n)\\
&d(n_m, t_n) \leq d(n_m, t_{m-1}) + d(n_{m-1}, t_{m-1}) + \cdots d(n_{n+1}, t_n)
\end{aligned} \tag{3.9}$$

for each $n, m \in N$ with $n < m$. Then from (3.4), (3.6), (3.7), (3.8)and (3.9), we have

$$\begin{aligned}
&d(l_n, r_m) + d(m_n, s_m) + d(n_n, t_m)\\
&\leq (d(l_n, r_{n+1}) + d(m_n, s_{n+1}) + d(n_n, t_{n+1}))\\
&+ (d(l_{n+1}, r_{n+1}) + d(m_{n+1}, s_{n+1}) + d(n_{n+1}, t_{n+1})) + \cdots\\
&+ (d(l_{m-1}, r_{m-1}) + d(m_{m-1}, s_{m-1}) + d(n_{m-1}, t_{m-1}))\\
&+ (d(l_{m-1}, r_m) + d(m_{m-1}, s_m) + d(n_{m-1}, t_m))\\
&\leq D^{X_1}_{n+1} + D^{Y_1}_{n+1} + D^{Z_1}_{n+1} + D^{X_3}_{n+1} + D^{Y_3}_{n+1} + D^{Z_3}_{n+1} + \cdots\\
&+ D^{X_3}_{m-1} + D^{Y_3}_{m-1} + D^{Z_3}_{m-1} + D^{X_1}_{m} + D^{Y_1}_{m} + D^{Z_1}_{m}\\
&\leq ((i_n + i_{n+1} + \cdots + i_{m-1}) + (l_n + l_{n+1} + \cdots + l_{m-1})\\
&+ (p_n + p_{n+1} + \cdots + p_{m-1}))D^{X_1}_1\\
&+ ((j_n + j_{n+1} + \cdots + j_{m-1}) + (m_n + m_{n+1} + \cdots + m_{m-1}))2D^{Y_1}_1\\
&+ ((k_n + k_{n+1} + \cdots + k_{m-1}) + (n_n + n_{n+1} + \cdots + n_{m-1})\\
&+ (q_n + q_{n+1} + \cdots + q_{m-1}))D^{Z_1}_1\\
&+ ((i_n + i_{n+1} + \cdots + i_{m-2}) + (l_n + l_{n+1} + \cdots + l_{m-2})\\
&+ (p_n + p_{n+1} + \cdots + p_{m-2}))D^{X_3}_1\\
&+ ((j_n + j_{n+1} + \cdots + j_{m-2}) + (m_n + m_{n+1} + \cdots + m_{m-2}))2D^{Y_3}_1
\end{aligned}$$

$$\begin{aligned}
&+ ((k_n + k_{n+1} + \cdots + k_{m-2}) + (n_n + n_{n+1} + \cdots + n_{m-2})\\
&+ (q_n + q_{n+1} + \cdots + q_{m-2}))D^{Z_3}_1\\
&\leq (\beta^n + \beta^{n+1} + \cdots + \beta^{m-1})(3D^{X_1}_1 + 4D^{Y_1}_1 + 3D^{Z_1}_1)\\
&+ (\beta^n + \beta^{n+1} + \cdots + \beta^{m-2})(3D^{X_3}_1 + 4D^{Y_3}_1 + 3D^{Z_3}_1)\\
&\leq \beta^n.\frac{1-\beta^{m-n}}{1-\beta}(3D^{X_1}_1 + 4D^{Y_1}_1 + 3D^{Z_1}_1) + \beta^n.\frac{1-\beta^{m-n+1}}{1-\beta}(3D^{X_3}_1 + 4D^{Y_3}_1 + 3D^{Z_3}_1)\\
&\to 0 \text{ as } n, m \to \infty.
\end{aligned}$$

Similarly, we can prove $d(l_m, r_n) + d(m_m, s_n) + d(n_m, t_n) \to 0$ as $n, m \to \infty$ where $\beta = \mu + \lambda + \kappa < 1$. Which shows (l_n, r_n), (m_n, s_n) and (n_n, t_n) are three Cauchy bisequences. Since (A, B, d) is complete bipolar metric spaces, there exist $l, m, n \in A$ and $r, s, t \in B$ with

$$\begin{aligned} &\lim_{n\to\infty} l_n = r \quad \lim_{n\to\infty} m_n = s \quad \lim_{n\to\infty} n_n = t \\ &\lim_{n\to\infty} r_n = l \quad \lim_{n\to\infty} s_n = m \quad \lim_{n\to\infty} t_n = n. \end{aligned} \tag{3.10}$$

Finally, we will prove

$$\begin{aligned} &r = F(l, m, n), \quad s = F(m, n, m) \quad \text{and} \quad t = F(n, m, l). \\ &l = F(r, s, t), \quad m = F(s, r, s) \quad \text{and} \quad n = F(t, s, r). \end{aligned}$$

Let $\epsilon > 0$. Since F is continuous at (l, m, n), or (r, s, t) for a given $\frac{\epsilon}{3} > 0$, there exists a $\delta > 0$ such that $d(l, r)+d(m, s)+d(n, t) < \delta \Rightarrow d(F(l, m, n), F(r, s, t)) < \frac{\epsilon}{3}$. Then by (3.10) it follows that $\xi = \min\left\{\frac{\delta}{3}, \frac{\epsilon}{3}\right\}$, there exist n_0, m_0, p_0 such that, for $n \ge n_0, m \ge m_0, p \ge p_0$, we have

$$\begin{aligned} &d(l_n, r) < \xi \quad d(m_n, s) < \xi \quad d(n_n, t) < \xi \\ &d(l, r_n) < \xi \quad d(m, s_n) < \xi \quad d(n, t_n) < \xi. \end{aligned}$$

Now, put $k_0 = max\{n_0, m_0, p_0\}$. Then, for any integer $n \ge k_0$, we have

$$\begin{aligned} d(F(l, m, n), r) &\le d(F(l, m, n), r_{n+1}) + d(l_{n+1}, r_{n+1}) + d(l_{n+1}, r) \\ &\le d(F(l, m, n), F(r_n, s_n, t_n)) + d(F(l_n, m_n, n_n), F(r_n, s_n, t_n)) \\ &\quad + d(l_{n+1}, r) \\ &\le \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \xi < \epsilon \end{aligned}$$

which implies that $d(F(l, m, n), r) = 0$ that is $F(l, m, n) = r$. Similarly, we can prove that

$$\begin{aligned} &s = F(m, n, m) \quad \text{and} \quad t = F(n, m, l). \\ &l = F(r, s, t), \quad m = F(s, r, s) \quad \text{and} \quad n = F(t, s, r). \end{aligned}$$

On the other hand (3.10), we obtain

$$\begin{aligned} d(l, r) &= d(\lim_{n\to\infty} r_n, \lim_{n\to\infty} l_n) = \lim_{n\to\infty} d(l_n, r_n) = 0 \\ d(m, s) &= d(\lim_{n\to\infty} s_n, \lim_{n\to\infty} m_n) = \lim_{n\to\infty} d(m_n, s_n) = 0 \\ d(n, t) &= d(\lim_{n\to\infty} t_n, \lim_{n\to\infty} n_n) = \lim_{n\to\infty} d(n_n, t_n) = 0. \end{aligned}$$

So $l = r, m = s$ and $n = t$. Therefore, $(l, m, n) \in A^3 \cap B^2$ is a fixed point of F. □

The achieved theorem is still valid for the covariant map F is not necessarily continuous. Instead, we require that underlying bipolar metric space (A, B, d) has an additional postulate. We discuss this in the following result.

Theorem 3.5. *Let $(A, B, \leq)$ be a partially ordered set and suppose that (A, B, d) is complete bipolar metric spaces on (A, B) such that (A, B) has the following postulate:*

(i) *If a non-decreasing bisequence $(\{l_n,\}, \{r_n\}) \to (l, l)$ then $(l_n, l_n) \leq (l, l)$, $\forall$ n;*

(ii) *If a non-increasing bisequence $(\{m_n,\}, \{s_n\}) \to (m, m)$ then $m \leq (m_n, s_n) \leq (m, m)$, $\forall$ n;*

(iii) *If a non-decreasing bisequence $(\{n_n,\}, \{t_n\}) \to (n, n)$ then $(n_n, t_n) \leq (n, n)$, $\forall$ n.*

Let $F : (A^3, B^3) \rightrightarrows (A, B)$ be covariant mapping having the mixed monotone property on (A, B) and μ , λ, κ be a non-negative constants with the condition (3.1) satisfied for each $l \geq r, m \leq s$ and $n \geq t$ and $\mu + \lambda + \kappa < 1$. If there exist $(l_0, m_0, n_0) \in A^3 \cup B^3$ such that

$$l_0 \leq F(l_0, m_0, n_0), \quad m_0 \geq F(m_o, l_0, m_0) \text{ and } n_0 \leq F(n_0, m_0, l_0)$$

then there exist $(l, m, n) \in A^3 \cup B^3$ such that the mapping $F : A^3 \cup B^3 \to A \cup B$ has

$$F(l, m, n) = l, \quad F(m, l, m) = m \quad \text{ and } F(n, m, l) = n$$

Proof. Following the proof of previous Theorem 3.4, we only have to prove $F(l, m, n) = l$, $F(m, l, m) = m$ and $F(n, m, l) = n$. Let $\epsilon > 0$.

Since $\{F^n(l_0, m_0, n_0)\} \to r$, $\{F^n(m_0, l_0, m_0)\} \to s$, $\{F^n(n_0, m_0, l_0)\} \to t$ and $\{F^n(r_0, s_0, t_0)\} \to l$, $\{F^n(s_0, r_0, s_0)\} \to m$, $\{F^n(t_0, s_0, r_0)\} \to n$. Then there exist $n_1, n_2, n_3, n_4, n_5, n_6 \in N$ such that for all $n_1 \geq n, n_2 \geq m, n_3 \geq p$ and $n_4 \geq k, , n_5 \geq q, n_6 \geq w$ every $\epsilon > 0$, we have

$$d(F^n(l_0, m_0, n_0), r) < \tfrac{\epsilon}{4}, \quad d(F^m(m_0, l_0, m_0), s) < \tfrac{\epsilon}{4}, \; d(F^p(n_0, m_0, l_0), t) < \tfrac{\epsilon}{4}$$

$$d\left(l, F^k(r_0, s_0, t_0)\right) < \tfrac{\epsilon}{4}, \quad d(m, F^q(s_0, r_0, s_0)) < \tfrac{\epsilon}{4}, \; d(n, F^w(t_0, s_0, r_0)) < \tfrac{\epsilon}{4}$$

for each $n \geq n_1$ and every $\epsilon > 0$, since $(\{F^n(l_0, m_0, n_0)\}, \{F^n(r_0, s_o, t_0)\})$, $(\{F^n(m_0, l_0, m_0)\}, \{F^n(s_0, r_o, s_0)\})$ and $(\{F^n(n_0, m_0, l_0)\}, \{F^n(t_0, s_o, r_0)\})$ are Cauchy bisequences, we get

$$d(F^n(l_0, m_0, n_0), F^n(r_0, s_o, t_0)) < \tfrac{\epsilon}{4}, \quad d(F^n(m_0, l_0, m_0), F^n(s_0, r_o, s_0)) < \tfrac{\epsilon}{4}$$
$$\text{and } d(F^n(n_0, m_0, l_0), F^n(t_0, s_o, r_0)) < \tfrac{\epsilon}{4}$$

taking $n \in N$, $n \geq \{n_1, n_2, n_3, n_4, n_5, n_6\}$ and using

$$F^n(l_0, m_0, n_0) \leq r, \quad F^n(m_0, l_0, m_0) \geq s \text{ and } F^n(n_0, m_0, l_0) \leq t$$
$$F^n(r_0, s_0, t_0) \leq l, \quad F^n(s_0, r_0, m_0) \geq m \text{ and } F^n(t_0, s_0, r_0) \leq n$$

we obtain

$$\begin{aligned}
&d\left(F\left(l,m,n\right),r\right)\leq d\left(F\left(l,m,n\right),F^{n+1}\left(r_0,s_0,t_0\right)\right)\\
&\quad+d\left(F^{n+1}\left(l_0,m_0,n_0\right),F^{n+1}\left(r_0,s_0,t_0\right)\right)+d\left(F^{n+1}\left(l_0,m_0,n_0\right),r\right)\\
&\leq d\left(F\left(l,m,n\right),F\left(F^{n}\left(r_0,s_0,t_0\right),F^{n}\left(s_0,r_0,s_0\right),F^{n}\left(t_0,s_0,r_0\right)\right)\right)\\
&+d\left(F^{n+1}\left(l_0,m_0,n_0\right),F^{n+1}\left(r_0,s_0,t_0\right)\right)+d\left(F^{n+1}\left(l_0,m_0,n_0\right),r\right)\\
&\leq \mu d\left(l,F^{n}\left(r_0,s_0,t_0\right)\right)+\lambda d\left(m,F^{n}\left(s_0,r_0,s_0\right)\right)+\kappa d\left(n,F^{n}\left(t_0,s_0,r_0\right)\right)\\
&+d\left(F^{n+1}\left(l_0,m_0,n_0\right),F^{n+1}\left(r_0,s_0,t_0\right)\right)+d\left(F^{n+1}\left(l_0,m_0,n_0\right),r\right)\\
&<(\mu+\lambda+\kappa)\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}<\epsilon.
\end{aligned}$$

This implies $d\left(F\left(l,m,n\right),r\right)=0$, hence $F\left(l,m,n\right)=r$. Similarly, we obtain $F\left(m,l,n\right)=s$, $F\left(n,m,l\right)=t$, $F\left(r,s,t\right)=l$, $F\left(s,r,s\right)=m$ and $F\left(t,s,r\right)=n$.

On the other hand,

$$\begin{aligned}
d\left(l,r\right)\quad&=d\left(\lim_{n\to\infty}F^{n}(l_0,m_0,n_0),\lim_{n\to\infty}F^{n}(r_0,s_0,t_0)\right)\\
&=\lim_{n\to\infty}d\left(F^{n}(l_0,m_0,n_0),F^{n}(r_0,s_0,t_0)\right)=0.
\end{aligned}$$

Similarly, we show that $d(m,s)=0$ and $d(n,t)=0$. Therefore, $l=r$, $m=s$ and $n=t$, hence $F\left(l,m,n\right)=l$, $F\left(m,l,m\right)=m$ and $F\left(n,m,l\right)=n$ □

Theorem 3.6. *Adding conditions of Theorem 3.4 to the hypothesis of theorem 3.5, then the mapping $F:A^3\cup B^3\to A\cup B$ has unique tripled fixed point.*

Proof. Let $(l^*,m^*,n^*)\in A^3\cup B^3$ be a another fixed point of F. Then we prove that $d\left(l,l^*\right)+d\left(m,m^*\right)+d\left(n,n^*\right)=0$ where

$$\lim_{n\to\infty}F^{n}\left(l_0,m_0,n_0\right)=l\qquad\lim_{n\to\infty}F^{n}\left(m_0,l_0,m_0\right)=m,\ \lim_{n\to\infty}F^{n}\left(n_0,m_0,l_0\right)=n.$$

If $(l^*,m^*,n^*)\in A^3$ and (l,m,n) is comparable to (l^*,m^*,n^*) with respect to the partial ordering in $\left(A^3,B^3\right)$, then, for every $n\in N$, we have

$$\begin{aligned}
&\left(F^{n}\left(l,m,n\right),F^{n}\left(m,l,m\right),F^{n}\left(n,m,l\right)\right)=\left(l,m,n\right)\text{ is comparable to}\\
&\qquad\left(F^{n}\left(l^*,m^*,n^*\right),F^{n}\left(m^*,l^*,m^*\right),F^{n}\left(n^*,m^*,l^*\right)\right)=\left(l^*,m^*,n^*\right).
\end{aligned}$$

Now we claim that $d(l,l^*)+d(m,m^*)+d(n,n^*)=0$

$$\begin{aligned}
d\left(l^*,l\right)&=d\left(F^{n}\left(l^*,m^*,n^*\right),F^{n}\left(l,m,n\right)\right)\\
&=d\begin{pmatrix}F\left(F^{n-1}\left(l^*,m^*,n^*\right),F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(n^*,m^*,l^*\right)\right),\\ F\left(F^{n-1}\left(l,m,n\right),F^{n-1}\left(m,l,m\right),F^{n-1}\left(n,m,l\right)\right)\end{pmatrix}\\
&\leq\mu d(F^{n-1}\left(l^*,m^*,n^*\right),F^{n-1}\left(l,m,n\right))\\
&+\lambda d(F^{n-1}\left(m^*,l^*,m^*\right),F^{n-1}\left(m,l,m\right))\\
&+\kappa d(F^{n-1}\left(n^*,m^*,l^*\right),F^{n-1}\left(n,m,l\right))\qquad(3.11)
\end{aligned}$$

and

$$
\begin{aligned}
d\left(m^{*}, m\right) &= d\left(F^{n}\left(m^{*}, l^{*}, m^{*}\right), F^{n}(m, l, m)\right) \\
&= d\begin{pmatrix} F\left(F^{n-1}\left(m^{*}, l^{*}, m^{*}\right), F^{n-1}\left(l^{*}, m^{*}, n^{*}\right), F^{n-1}\left(m^{*}, l^{*}, m^{*}\right)\right), \\ F\left(F^{n-1}(m, l, m), F^{n-1}(l, m, n)\right), F^{n-1}(m, l, m) \end{pmatrix} \\
&\leq \mu d(F^{n-1}\left(m^{*}, l^{*}, m^{*}\right), F^{n-1}(m, l, m)) \\
&+ \lambda d(F^{n-1}\left(l^{*}, m^{*}, n^{*}\right), F^{n-1}(l, m, n)) \\
&+ \kappa d(F^{n-1}\left(m^{*}, l^{*}, m^{*}\right), F^{n-1}(m, l, m))
\end{aligned} \tag{3.12}
$$

and also

$$
\begin{aligned}
d\left(n^{*}, n\right) &= d\left(F^{n}\left(n^{*}, m^{*}, l^{*}\right), F^{n}(n, m, l)\right) \\
&= d\begin{pmatrix} F\left(F^{n-1}\left(n^{*}, m^{*}, l^{*}\right), F^{n-1}\left(m^{*}, l^{*}, m^{*}\right), F^{n-1}\left(l^{*}, m^{*}, n^{*}\right)\right), \\ F\left(F^{n-1}(n, m, l), F^{n-1}(m, l, m), F^{n-1}(l, m, n)\right) \end{pmatrix} \\
&\leq \mu d(F^{n-1}\left(n^{*}, m^{*}, l^{*}\right), F^{n-1}(n, m, l)) \\
&+ \lambda d(F^{n-1}\left(m^{*}, l^{*}, m^{*}\right), F^{n-1}(m, l, m)) \\
&+ \kappa d(F^{n-1}\left(l^{*}, m^{*}, n^{*}\right), F^{n-1}(l, m, n))
\end{aligned} \tag{3.13}
$$

for all $n \in N$, combining (3.11), (3.12) and (3.13)

$$
\begin{aligned}
\begin{pmatrix} d\left(l^{*}, l\right) \\ +d\left(m^{*}, m\right) \\ +d\left(m^{*}, m\right) \end{pmatrix} &\leq (\mu+\lambda+\kappa)\begin{Bmatrix} d(F^{n-1}\left(l^{*}, m^{*}, n^{*}\right), F^{n-1}(l, m, n)) \\ +d(F^{n-1}\left(m^{*}, l^{*}, m^{*}\right), F^{n-1}(m, l, m)) \\ +d(F^{n-1}\left(n^{*}, m^{*}, l^{*}\right), F^{n-1}(n, m, l)) \end{Bmatrix} \\
&\leq (\mu+\lambda+\kappa)^{2}\begin{Bmatrix} d(F^{n-2}\left(l^{*}, m^{*}, n^{*}\right), F^{n-2}(l, m, n)) \\ +d(F^{n-2}\left(m^{*}, l^{*}, m^{*}\right), F^{n-2}(m, l, m)) \\ +d(F^{n-2}\left(n^{*}, m^{*}, l^{*}\right), F^{n-2}(n, m, l)) \end{Bmatrix} \\
&\vdots \\
&\leq (\mu+\lambda+\kappa)^{n}\begin{Bmatrix} d(F\left(l^{*}, m^{*}, n^{*}\right), F(l, m, n)) \\ +d(F\left(m^{*}, l^{*}, m^{*}\right), F(m, l, m)) \\ +d(F\left(n^{*}, m^{*}, l^{*}\right), F(n, m, l)) \end{Bmatrix} \\
&\leq (\mu+\lambda+\kappa)^{n}(d(l, l^{*})+d(m, m^{*})+d(n, n^{*})).
\end{aligned}
$$

Since $\mu+\lambda+\kappa<1$ which implies $d\left(l^{*}, l\right)+d\left(m^{*}, m\right)+d\left(n^{*}, n\right)=0$. Hence we obtain $l^{*}=l$, $m^{*}=m$ and $n=n^{*}$.

Similarly, If $(l^{*}, m^{*}, n^{*}) \in B^{3}$ and (l, m, n) is comparable to (l^{*}, m^{*}, n^{*}) with respect to the partial ordering in $\left(A^{3}, B^{3}\right)$, then we have $l^{*}=l$, $m^{*}=m$ and $n=n^{*}$.

If $(l^{*}, m^{*}, n^{*}) \in A^{3}$ and (l, m, n) is not comparable to (l^{*}, m^{*}, n^{*}), then there exist two comparable lower or upper bounds (a, b, c) , $(a^{*}, b^{*}, c^{*}) \in A^{3} \cup B^{3}$ of (l, m, n) and (l^{*}, m^{*}, n^{*}). Then, for all $n \in N$, $(F^{n}(a, b, c), F^{n}(b, a, b), F^{n}(c, b, b))=(a, b, c)$ and $(F^{n}(a^{*}, b^{*}, c^{*}), F^{n}(b^{*}, a^{*}, b^{*}), F^{n}(c^{*}, b^{*}, a^{*}))=(a^{*}, b^{*}, c^{*})$ is comparable to $(F^{n}(l, m, n), F^{n}(m, l, m), F^{n}(n, m, l),)=(l, m, n)$ and $(F^{n}(l^{*}, m^{*}, n^{*}),$

$F^n(m^*, l^*, m^*), F^n(n^*, m^*, l^*)) = (l^*, m^*, n^*)$. Then we have

$$\begin{aligned}
d\left(\begin{pmatrix} l \\ m \\ n \end{pmatrix} \begin{pmatrix} l^* \\ m^* \\ n^* \end{pmatrix}\right) &= d\left(\begin{pmatrix} F^n(l,m,n) \\ F^(m,n,m) \\ F^n(n,m,l) \end{pmatrix} \begin{pmatrix} F^n(l^*,m^*,n^*) \\ F^(m^*,n^*,m^*) \\ F^n(n^*,m^*,l^*) \end{pmatrix}\right) \\
&\leq d\left(\begin{pmatrix} F^n(l,m,n) \\ F^(m,n,m) \\ F^n(n,m,l) \end{pmatrix} \begin{pmatrix} F^n(a,b,c) \\ F^(b,a,b) \\ F^n(c,b,a) \end{pmatrix}\right) \\
&+ d\left(\begin{pmatrix} F^n(a^*,b^*,c^*) \\ F^(b^*,a^*,b^*) \\ F^n(c^*,b^*,a^*) \end{pmatrix} \begin{pmatrix} F^n(a,b,c) \\ F^(b,a,b) \\ F^n(c,b,a) \end{pmatrix}\right) \\
&+ d\left(\begin{pmatrix} F^n(a^*,b^*,c^*) \\ F^(b^*,a^*,b^*) \\ F^n(c^*,b^*,a^*) \end{pmatrix} \begin{pmatrix} F^n(l^*,m^*,n^*) \\ F^(m^*,l^*,m^*) \\ F^n(n^*,m^*,l^*) \end{pmatrix}\right) \\
&\leq (\mu+\lambda+\kappa)^n(d(l,a)+d(m,b)+d(n,c)) \\
&+ (\mu+\lambda+\kappa)^n(d(a^*,a)+d(b^*,b)+d(c^*,c)) \\
&+ (\mu+\lambda+\kappa)^n(d(a^*,l^*)+d(b^*,m^*)+d(c^*,n^*)) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

so that

$$d\left(\begin{pmatrix} l \\ m \\ n \end{pmatrix} \begin{pmatrix} l^* \\ m^* \\ n^* \end{pmatrix}\right) = 0$$

which implies $l^* = l$, $m^* = m$ and $n^* = n$. Similarly, If $(l^*, m^*, n^*) \in B^3$ and (l, m, n) is incomparable to (l^*, m^*, n^*) with respect to the partial ordering in $\left(A^3, B^3\right)$, then we have $l^* = l$, $m^* = m$ and $n^* = n$. Therefore $(l, m, n) \in A^3 \cap B^3$. Hence (l, m, n) is unique tripled fixed point of F. □

Example 3.7. Let $A = \{U_m(R)/U_m(R)$ is upper triangular matrices over R$\}$ and $B = \{L_m(R)/L_m(R)$ is lower triangular matrices over R $\}$ with the bipolar metric

$$d(P,Q) = \sum_{i,j=1}^{m} |p_{ij} - q_{ij}|,$$

for all $P = (p_{ij})_{m\times m} \in U_m(R)$ and $Q = (q_{ij})_{m\times m} \in L_m(R)$. On the set (A, B), we consider the following relation :

$$P, Q \in A \cup B, P \preceq Q \Leftrightarrow p_{ij} \leq q_{ij},$$

where $\leq$ is usual ordering. Then clearly, (A, B, d) is a complete bipolar metric space and $(A, B, \preceq)$ is a partially ordered set, and (A, B) has the property as in Theorem (3.4). Let $F : \left(A^3, B^3\right) \rightrightarrows (A, B)$ be defined as

$$\begin{aligned}
F(P,Q,R) &= \left(\tfrac{3p_{ij}+6q_{ij}-2r_{ij}+9i_{ij}}{12}\right)_{m\times m}, \\
&\forall \quad (P = (p_{ij})_{m\times m}, Q = (q_{ij})_{m\times m}, R = (r_{ij})_{m\times m}) \in A^3 \cup B^3.
\end{aligned}$$

Then obviously, F has the mixed monotone property, taking

$$\left\{ \begin{array}{ll} (P=(p_{ij})_{m\times m}, Q=(q_{ij})_{m\times m}, R=(r_{ij})_{m\times m}) & \in A^3\cup B^3 \\ (S=(s_{ij})_{m\times m}, T=(t_{ij})_{m\times m}, U=(u_{ij})_{m\times m}) & \end{array} \right\}$$

with $P\succeq S$, $Q\preceq T$ and $R\succeq U$ that is $p_{ij}\geq s_{ij}$, $q_{ij}\leq t_{ij}$ and $r_{ij}\geq u_{ij}$, we have

$$d(F(P,Q,R),F(S,T,U)) = d(\frac{3p_{ij}+6q_{ij}-2r_{ij}+9i_{ij}}{12},\frac{3s_{ij}+6t_{ij}-2u_{ij}+9i_{ij}}{12})$$
$$= \frac{1}{12}\sum_{i,j=1}^{m} |(3p_{ij}+6q_{ij}-2r_{ij}+9i_{ij})-(3s_{ij}+6t_{ij}-2u_{ij}+9i_{ij})|$$
$$\leq \frac{1}{12}\left(\sum_{i,j=1}^{m}|3p_{ij}-3s_{ij}|+\sum_{i,j=1}^{m}|6q_{ij}-6t_{ij}|+\sum_{i,j=1}^{m}|2r_{ij}-2u_{ij}|\right)$$
$$\leq \frac{1}{12}\left(3\sum_{i,j=1}^{m}|p_{ij}-s_{ij}|+6\sum_{i,j=1}^{m}|q_{ij}-t_{ij}|+2\sum_{i,j=1}^{m}|r_{ij}-u_{ij}|\right)$$
$$\leq \frac{1}{12}\left(3d(P,S)+6d(Q,T)+2d(R,U)\right)$$
$$\leq \frac{3}{12}d(P,S)+\frac{6}{12}d(Q,T)+\frac{2}{12}d(R,U).$$

It is easy to verified that F satisfies Theorem (3.4) and Theorem (3.6) with $\mu=\frac{1}{4}$, $\lambda=\frac{1}{2}$ and $\kappa=\frac{1}{6}$ and $(\frac{9}{5}I,\frac{9}{5}I,\frac{9}{5}I)$ is the unique tripled fixed point of F. Note that in this case we have $P=Q=R$.

Definition 3.8. *Let $F:(A\times B\times A, B\times A\times B)\rightrightarrows(A,B)$ be a covariant map, an element $(a,p,b)\in A\times B\times A$ is called tripled fixed point of F if*

$$F(a,p,b)=a, F(p,a,p)=p \text{ and } F(b,p,a)=b.$$

Theorem 3.9. *Let $F:(A\times B\times A, B\times A\times B)\rightrightarrows(A,B)$ be a covariant map. If F is a continuous mapping having the mixed monotone property on (A,B) and μ , λ, κ be a non-negative constants with the condition*

$$d\left(F(l,r,m),F(s,n,t)\right)\leq \mu d\left(l,s\right)+\lambda d\left(n,r\right)+\kappa d\left(m,t\right), \tag{3.14}$$

$\forall l,m,n\in Ar,s,t\in B$ with $l\geq s$, $n\leq r$, and $m\geq t$ with $\mu+\lambda+\kappa<1$. If there exist $(l_0,r_0,m_0)\in(A\times B\times A)\cup(B\times A\times B)$ such that

$$l_0\leq F(l_0,r_0,m_0), r_0\geq F(r_o,l_0,r_0) \text{ and } m_0\leq F(m_0,r_0,l_0),$$

then there exists $(l,r,m)\in(A\times B\times A)\cup(B\times A\times B)$ such that the mapping $F:(A\times B\times A)\cup(B\times A\times B)\to A\cup B$ has

$$F(l,r,m)=l, F(r,l,r)=r \text{ and } F(m,r,l)=l.$$

4. Conclusions

In the present research, we introduced and proved tripled fixed point theorems by using weak contractive type and mixed monotone type mappings, defined on a partially ordered bipolar metric space and gave suitable example that support our main result.

Acknowledgments

The authors are very thankful to the reviewers and editors for valuable comments, remarks and suggestions for improving the content of the paper.

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Accepted: 14.06.2018