Numerical solution of cubic free undamped Duffing oscillator equation using continuous implicit hybrid method

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Abstract. In this paper, we investigate the solution of the cubic free undamped Duffing oscillator equation based on the implicit hybrid method. We investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of the proposed method. In addition, we study the zero stability, the order, and the error constant of the block method which is generated from the proposed method. Numerical results are presented to show the efficiency of the proposed method.

Keywords: cubic free undamped Duffing oscillator equation, implicit hybrid method, consistency, zero stable, convergence.

1. Introduction

In this paper, we study the cubic free undamped Duffing oscillator [1] of the form

$$y''(t) + ay(t) + by^3(t) = 0, 0 < t < T$$

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subject to

$$y(0) = \alpha_0, y'(0) = 0$$

where α_0, a, b are constants. Several methods were used to solve this problem numerically such as homtopy perturbation method [2], the modified homtopy perturbation method [3], variational iteration method [4-8], Adomian decomposition method [9], artificial parameter decomposition [10], He's parameter expanding method [11]. More methods can be found in [16-30].

In addition, many analytical solutions for Duffing problem were developed using many methods such as method of multiple scales [12], the Krylov-Bogolubov method [13], the straightforward expansion [14], and Linstedt-Poincare method [13].

We can rewrite the above system in the form

$$y''(t) = g(y), 0 \leq t \leq T \tag{1}$$

subject to

$$y(0) = \alpha_0, y'(0) = 0. \tag{2}$$

In this paper, we use a numerical method based on the implicit hybrid method. We used collocation and interpolation techniques on the power series approximate solution. The proposed method has high order method which is very accurate comparing with other method such as the one-step methods. This method is cheaper than the multi-step methods. We investigate the consistency, zero stable, convergence, the order, error constant, and region of absolute stability of the proposed method. In addition, we study the zero stability, the order, and the error constant of the block method which is generated from the proposed method. Numerical results are presented to show the efficiency of the proposed method. More details about this method can be found in [23]. In the next section, we describe the proposed method. In section 3, we analyze the proposed method. Finally, we draw some conclusions.

2. Method of solution

In this section, we derive the proposed method. Approximate the solution of Eqn. (1) by

$$y(t) = \sum_{i=0}^{5} a_i t^i. \tag{3}$$

Then, the first derivative of the solution of Eqn. (3) is given by

$$y'(t) = \sum_{i=1}^{5} i a_i t^{i-1}. \tag{4}$$

Let$\{t_0 = 0, t_1 = \Delta, \ldots, t_M = M\Delta = T\}$ be a uniform partition of $[0, T]$ where $t_i = i\Delta, i = 0 : M$ and $\Delta = \frac{T}{M}$. Interpolate Eqn. (3) at $t_{n+\frac{1}{3}}, t_{n+\frac{2}{3}}$ and collocate Eqn. (4) at $t_{n+\frac{j}{3}}, j = 0, 1, 2, 3$, to get the following linear system

$$\begin{pmatrix} 1 & t_{n+\frac{1}{3}} & t^2_{n+\frac{1}{3}} & t^3_{n+\frac{1}{3}} & t^4_{n+\frac{1}{3}} & t^5_{n+\frac{1}{3}} \\ 1 & t_{n+\frac{2}{3}} & t^2_{n+\frac{2}{3}} & t^3_{n+\frac{2}{3}} & t^4_{n+\frac{2}{3}} & t^5_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 \\ 0 & 0 & 2 & 6t_{n+\frac{1}{3}} & 12t^2_{n+\frac{1}{3}} & 20t^3_{n+\frac{1}{3}} \\ 0 & 0 & 2 & 6t_{n+\frac{2}{3}} & 12t^2_{n+\frac{2}{3}} & 20t^3_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6t_{n+1} & 12t^2_{n+1} & 20t^3_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix} \tag{5}$$

Let

$$t_{n+\frac{2}{3}} = t - \Delta s, t_{n+1} = t - \Delta s + \frac{\Delta}{3},$$

$$t_{n+\frac{1}{3}} = t - \Delta s - \frac{\Delta}{3}, t_n = t - \Delta s - \frac{2\Delta}{3}.$$

Then, using the above change of variables and solving System (5), we get

$$\begin{aligned} a_0(s) &= -3s, \\ a_1(s) &= 1 + 3s, \\ a_2(s) &= -\frac{\Delta^2 s(7 - 90s^2 + 243s^4)}{1080}, \\ a_3(s) &= \frac{1}{360}\Delta^2 s(22 - 180s^2 + 135s^3 + 243s^4), \\ a_4(s) &= \frac{1}{360}\Delta^2 s(43 + 180s + 90s^2 - 270s^3 - 243s^4), \\ a_5(s) &= \frac{\Delta^2 s(-8 + 180s^2 + 405s^3 + 243s^4)}{\Delta^2 s(-8 + 180s^2 + 405s^3 + 243s^4)}. \end{aligned}$$

When $t = t_{n+1}, t_{n+\frac{2}{3}} = t_{n+1} - \Delta s$. Thus,

$$s = \frac{t_{n+1} - t_{n+\frac{2}{3}}}{\Delta} = \frac{\Delta}{3\Delta} = \frac{1}{3}.$$

Similarly, when $t = t_{n+\frac{2}{3}}, t_{n+\frac{1}{3}}, t_n, s = 0, -1/3, -2/3$, respectively. Thus, at $s = 1/3, -2/3$, Eqn. (3) becomes

$$\begin{aligned} y_{n+1} &= -y_{n+\frac{1}{3}} + 2t_{n+\frac{2}{3}} + \Delta^2\left(\frac{-f_{n+\frac{1}{3}}}{108} + \frac{5t_{n+\frac{2}{3}}}{54} + \frac{f_{n+1}}{108}\right), \\ y_n &= 2y_{n+\frac{1}{3}} - y_{n+\frac{2}{3}} + \Delta^2\left(\frac{f_n}{108} + \frac{5t_{n+\frac{1}{3}}}{54} + \frac{f_{n+\frac{2}{3}}}{108}\right). \end{aligned} \tag{6}$$

Using the change of variable $t_{n+\frac{2}{3}} = t - \Delta s$, we have

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{1}{\Delta}\frac{dy}{ds}.$$

Hence,

$$\frac{dy}{ds} = \frac{1}{\Delta}\left(\sum_{i=1}^{2} a'_{i-1}(s) y_{n+\frac{1}{3}} + \sum_{i=3}^{6} a'_{i-1}(s) f_{n+\frac{i-3}{3}}\right). \tag{7}$$

At $s = 1/3, 0, -1/3, -2/3$, Eqn. (7) implies that

$$\begin{aligned}
y'_{n+1} &= \frac{-3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{f_n}{135} - \frac{f_{n+\frac{1}{3}}}{120} + \frac{23 f_{n+\frac{2}{3}}}{60} + \frac{127 f_{n+1}}{1080}\right),\\
y'_{n+\frac{2}{3}} &= \frac{-3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{-7 f_n}{1080} + \frac{11 f_{n+\frac{1}{3}}}{180} + \frac{43 f_{n+\frac{2}{3}}}{360} - \frac{f_{n+1}}{135}\right),\\
y'_{n+\frac{1}{3}} &= \frac{-3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{f_n}{135} - \frac{43 f_{n+\frac{1}{3}}}{360} - \frac{11 f_{n+\frac{2}{3}}}{180} + \frac{7 f_{n+1}}{1080}\right),\\
y'_{n} &= \frac{-3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{-127 f_n}{1080} - \frac{23 f_{n+\frac{1}{3}}}{60} + \frac{f_{n+\frac{2}{3}}}{120} - \frac{f_{n+1}}{135}\right).
\end{aligned} \tag{8}$$

Let

$$A_1 = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ \frac{3}{\Delta} & \frac{-3}{\Delta} & 0 & 0 & 0 & 1 \\ \frac{3}{\Delta} & \frac{-3}{\Delta} & 0 & 0 & 1 & 0 \\ \frac{3}{\Delta} & \frac{-3}{\Delta} & 0 & 1 & 0 & 0 \\ \frac{3}{\Delta} & \frac{-3}{\Delta} & 0 & 0 & 0 & 0 \end{pmatrix}, Y_{1,n} = \begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{1}{3}} \\ y_{n+1} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{2}{3}} \\ y'_{n+1} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}, Y_{2,n} = \begin{pmatrix} y_n \\ y'_n \end{pmatrix}, F_{1,n} = (f_n), F_{2,n} = \begin{pmatrix} f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 \\ \frac{\Delta^2}{108} \\ \frac{\Delta}{135} \\ \frac{-7\Delta}{1080} \\ \frac{\Delta}{135} \\ \frac{-127\Delta}{1080} \end{pmatrix}, A_4 = \begin{pmatrix} \frac{\Delta^2}{108} & \frac{5\Delta^2}{54} & \frac{\Delta^2}{108} \\ \frac{5\Delta^2}{54} & \frac{\Delta^2}{108} & 0 \\ \frac{-\Delta}{120} & \frac{23\Delta}{60} & \frac{127\Delta}{1080} \\ \frac{11\Delta}{180} & \frac{43\Delta}{360} & \frac{-\Delta}{135} \\ \frac{-43\Delta}{360} & \frac{-11\Delta}{180} & \frac{7\Delta}{1080} \\ \frac{-23\Delta}{60} & \frac{\Delta}{120} & \frac{-\Delta}{135} \end{pmatrix}.$$

Then, Systems (6) and (8) can be written in the matrix form as

$$A_1 Y_{1,n} = A_2 Y_{2,n} + A_3 F_{1,n} + A_4 F_{2,n}. \tag{9}$$

Multiply both sides of Eqn. (9) by A_1^{-1} to get

$$B_1 Y_{1,n} = B_2 Y_{2,n} + B_3 F_{1,n} + B_2 F_{2,n} \tag{10}$$

where $B_1 = I_{15}$,

$$B_2 = \begin{pmatrix} 1 & \frac{1}{3} \\ 1 & \frac{2}{3} \\ 1 & 1 \\ 0 & \frac{1}{\Delta} \\ 0 & \frac{1}{\Delta} \\ 0 & \frac{1}{\Delta} \end{pmatrix}, B_3 = \begin{pmatrix} \frac{97\Delta^2}{3240} \\ \frac{28\Delta^2}{405} \\ \frac{13\Delta^2}{120} \\ 0 \\ 0 \\ 0 \end{pmatrix}, B_4 = \begin{pmatrix} \frac{19\Delta^2}{540} & \frac{-13\Delta^2}{1080} & \frac{\Delta^2}{405} \\ \frac{22\Delta^2}{135} & \frac{-2\Delta^2}{135} & \frac{2\Delta^2}{405} \\ \frac{3\Delta^2}{10} & \frac{3\Delta^2}{40} & \frac{\Delta^2}{60} \\ \frac{19\Delta}{72} & \frac{-5\Delta}{72} & \frac{\Delta}{72} \\ \frac{4\Delta}{9} & \frac{\Delta}{9} & 0 \\ \frac{3\Delta}{8} & \frac{3\Delta}{8} & \frac{\Delta}{8} \end{pmatrix}.$$

Then, we solve System (10) iteratively.

3. Analysis of the proposed method

In this section, we investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of main equation

$$y_{n+1} = -y_{n+\frac{1}{3}} + 2t_{n+\frac{2}{3}} + \Delta^2 \left(\frac{-f_{n+\frac{1}{3}}}{108} + \frac{5t_{n+\frac{2}{3}}}{54} + \frac{f_{n+1}}{108} \right). \tag{11}$$

In addition, we study the zero stability, the order, and the error constant of the block method (10). The first and second characteristic functions are given by

$$\tau_1(z) = z^{1/3} - 2z^{2/3} + z$$

and

$$\tau_2(z) = \frac{z^{1/3}}{108} + \frac{5z^{2/3}}{54} + \frac{z}{108}.$$

Then,

$$\tau_1(1) = 0,$$
$$\tau_1'(1) = 0,$$
$$\tau_1''(1) - 2!\tau_2(1) = 0.$$

The roots of $\tau_1(z)$ for which $|z| = 1$ and $z \neq 1$ are simple. Thus, Eqn. (11) is consistent and zero stable. Therefore, it is convergent. To find the region of absolute stability, let

$$\mu(z) = \frac{\tau_1(z)}{\tau_2(z)}, z = e^{i\theta}, \theta \in [0, 2\pi].$$

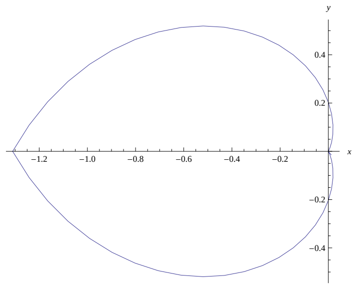

Figure 1: Region of absolute stability

Then, the interval of absolute stability is $(-1.31029, 0)$ and the region of absolute stability is given in Figure 1.

$$\bar{B_2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\det\left(sB_1 - \bar{B_2}\right) = (s-1)s^5.$$

Since the roots of the above equation which has modulus 1 is simple, the block method is zero stable as $\Delta \to 0$. Using the Taylor series, Eqn. (11) becomes

$$y_{n+1} + y_{n+\frac{1}{3}} - 2t_{n+\frac{2}{3}} - \Delta^2\left(\frac{-f_{n+\frac{1}{3}}}{108} + \frac{5t_{n+\frac{2}{3}}}{54} + \frac{f_{n+1}}{108}\right) = \frac{-\Delta^6 y_n^{(6)}}{174960} + \cdots.$$

Thus, the order of Eqn. (11) is 4 and the error constant is $-0.000005715\Delta^6 y_n^{(6)}$. Similarly, the Taylor expansion of System (10) is give as

$$B_1Y_{1,n} - B_2Y_{2,n} - B_3F_{1,n} - B_2F_{2,n} = \begin{pmatrix} -0.0000200047\Delta^6 y_n^{(6)} \\ -0.00004572473\Delta^6 y_n^{(6)} \\ -0.000077160493\Delta^6 y_n^{(6)} \\ -0.0001085962505\Delta^6 y_n^{(6)} \\ -0.0000457247370\Delta^6 y_n^{(6)} \\ -0.0001543209876\Delta^6 y_n^{(6)} \end{pmatrix} + ...$$

Thus, the block method (10) has the following order $(4,4,4,4,4,4)^t$ with error constant

$$\begin{pmatrix} -0.0000200047\Delta^6 y_n^{(6)} \\ -0.00004572473\Delta^6 y_n^{(6)} \\ -0.000077160493\Delta^6 y_n^{(6)} \\ -0.0001085962505\Delta^6 y_n^{(6)} \\ -0.0000457247370\Delta^6 y_n^{(6)} \\ -0.0001543209876\Delta^6 y_n^{(6)} \end{pmatrix}.$$

4. Numerical results

In this section, we present some of our numerical results to show the efficiency of the proposed method which is described in Section 2. Consider the following cubic free undamped Duffing oscillator equation of the form

$$y''(t) + ay(t) + by^3(t) = 0, 0 < t < T \tag{12}$$

subject to

$$y(0) = \alpha_0, y'(0) = 0. \tag{13}$$

The exact solution of Problem (12)-(13) is not Known. Therefore, the numerical solutions have determined by built in file of MATHEMATICA based on the fully explicit Runge-Kutta method and this solution is used as the standard or reference for comparison. In Tables 1 and 2, we compare our results with HPM [24] , MHPM [25], SHPM [24], and numerical solution for $\alpha_0 = 1, a = 1$, and $b = 1$ and for $\alpha_0 = 0.75, a = 1.5$, and $b = 1.5$, respectively

t	HPM	MHPM	SHPM	Present results	Numerical results
0.5	0.762476	0.768902	0.768766	0.768802	0.768802
1.0	0.176929	0.233741	0.233680	0.233692	0.233692
2.0	−1.055110	−0.891260	−0.859323	−0.859349	−0.859349
3.5	−0.461650	−0.079433	−0.093034	−0.093013	−0.093013
5.0	2.049041	−0.079433	0.947107	0.947130	0.947130

Table 1: The approximate solution for $\alpha_0 = 1, a = 1$, and $b = 1$.

t	HPM	MHPM	SHPM	Present results	Numerical results
1	0.056288	0.080176	0.080519	0.080527	0.080527
2	−0.808192	−0.739174	−0.729000	−0.729018	−0.729018
3	−0.339208	−0.239413	−0.238620	−0.238626	−0.238626
4	0.891267	0.706827	0.667953	0.668022	0.668022
5	0.893003	0.395315	0.387550	0.387551	0.387551

Table 2: The approximate solution for $\alpha_0 = 0.75, a = 1.5$, and $b = 1.5$.

5. Conclusion

We compare our results with HPM [26] , MHPM [27], SHPM [26], and numerical solution for $\alpha_0 = 1, a = 1$,and $b = 1$ and for $\alpha_0 = 1, a = 1$,and $b = 1$. From Tables 5.1 and 5.2, we see that our results agree exceptionally well with the numerical results and give more accurate results than HPM, MHPM, and SHPM.

The results in this paper confirm that the proposed method is a powerful and efficient method for solving nonlinear differential equations in different fields of sciences and engineering. In addition, we investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of the proposed method. Also, we study the zero stability, order, and the error constant of the block method which is generated from the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of the paper.

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