On counting by a pair of Fibonacci generated countable sets of irrationals

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Abstract. A Fibonacci sequence representation of any rational number of the real line R is used to construct a pair of phi-nary (in relation to the Golden ratio φ)unique countable sets of irrational numbers. These sets, which have a potential for novel applications, are further employed in the proof that R should be populated "much more with irrational numbers than with rational numbers".

Keywords: Fibonacci numbers, irrational numbers, rational numbers, multiplicative decomposition.

1. Introduction

It is well known that the set of rationals Q in the set of real numbers R is infinite and countable, while the set of irrationals $Q^c = R\backslash Q$ is infinite but uncountable, and are equally dense, [1, p. 36], as can be deduced from the following famous results, in which $k \in \mathbb{N} = \{1, 2, 3, ...\}$ and $\gamma \in \Gamma \subset Q^c$,where Γ is an uncountable set.

Theorem 1 ([2, p. 14], Density of rational numbers)**.** *There is a countable set* $A = \{r_k\}_{k=1}^{\infty} \subset [Q \cap (a,\ b)]$ *between any two distinct real numbers a,b* $\in R$.

Corollary 1 (*Density of irrational numbers*)**.** *There is a countable family of uncountable sets,*$\{\{\mathcal{S}_{\gamma k}\}_{\gamma\in\Gamma}\}_{k=1}^{\infty} \triangleq \{\mathcal{S}_{\gamma k}\} \subset [Q^c \cap (a, b)]$, *between any two distinct real numbers* $a, b \in R$.

The proofs of these results are given for the interested reader in the Appendix. The corollary and its proof indicate, moreover, that "Q^c is however more populated than Q". And our major aim in this note is to construct Fibonacci-generated, see e.g. [3, p. 143], countable phinary sets of irrationals for a related multiplicative decomposition principle of rationals. These phi-nary sets, which have a potential for novel applications, are demonstrated, in section 2, to be asymptotically rationally distinct (ARD), then used to assert Corollary 1, without invoking Theorem 1.

2. Main result

The double subscript in the set family $\{\{\mathcal{S}_{\gamma k}\}_{\gamma\in\Gamma}\}_{k=1}^{\infty}$ emphasizes the fact that its existence is tied to the existence of $\mathfrak{A} = \{\rho_k\}_{k=1}^{\infty} \subset Q \cap (a,b)$.This being a fact, and not a restriction, should by no means affect the reality of the next remark.

Remark 1. The ratio $\frac{\#\,\mathfrak{A}}{\#\{\{S_{\gamma k}\}_{\gamma}\}_{k=1}^{\infty}} = \frac{\aleph_0}{\aleph_0\,\aleph_1} = \frac{\aleph_0}{\aleph_1}$ does not have a quantitative meaning. However, in terms of the Lebesgue measure μ, the previous ratio reduces to $\frac{\mu(\mathfrak{A}\,)}{\mu(\{\{S_{\gamma k}\}_{\gamma}\}_{k=1}^{\infty})} = \frac{0}{\infty} = 0$, for any $(a,b) \subset R$, and this can imply that Q^c should be more populated than Q.

Let $p, q \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$, and $q \neq 0$, $r = \frac{p}{q} \in Q$, and $s, z \in Q^c$. Rather trivially $rs, \frac{s}{r}, \frac{r}{s} \in Q^c$, while $r \pm s$, $sz \in R$, facts that can easily be proved by contradiction in assuming the opposite to be true. Equally simple, is the proof of

$$r = s\,z. \tag{1}$$

Incidentally Dedekind cuts of the Q set can be used, see e.g. [4, p. 38], as a basis for the construction of R. Hence the multiplicative decomposability of r into two irrational numbers, (1), happens to represent a sort of reversed Dedekind construction.

As for $R^+ = [0,\infty)$ and $r = \frac{p}{q} \in [R^+ \cap Q]$, $p \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and $q \in \mathbb{N}$ happen to generate a sequence $< r_k >$, arranged viz $0 \le r_1 \le r_2 \le r_3 \le .....$, in a countable set $\{r_k\}_{k=1}^{\infty} \subset [R^+ \cap Q]$, for which $\triangle r_k = r_{k+1} - r_k$ is not constant with varying k, but has a fractal structure, as indicated by the Minkowski lattice [5]. Being essential for the completeness of the R^+ set, the irrational numbers are in fact defined by Dedekind, [4, p. 38], in 1901 as the cut which is created in order to fill the gaps produced by the r_k members of this structure.

Indeed, since s, in (1), can be varied within a certain set $\{s_\gamma\} = \mathcal{F} \subset Q^c$, which could be countable, $\boldsymbol{G}$, or uncountable , $\boldsymbol{U}$. Then z varies, for a fixed r, as $z_\gamma = \frac{p}{q\,s_\gamma} \in \{z_\gamma\} = \hbar \subset Q^c$,with $\hbar$ similarly countable as $\mathcal{F}$, i.e. with corresponding countable $\boldsymbol{H}$ and uncountable $\boldsymbol{V}$.

As an example of this, if the s irrationals are chosen, not from rule-generated sequences, to be, e.g. $s_1 = 1 - \sqrt{2}$, $s_2 = e^3$, $s_3 = \pi^2$, ...etc, then the corresponding, via (1), z irrationals will also be not rule-generated, namely : $z_1 = \frac{p}{q(1-\sqrt{2})}$, $z_2 = \frac{p}{qe^3}$, $z_3 = \frac{p}{q\pi^2}$,... Needless to say here that π and e are transcendental irrational numbers, while φ is an algebraic irrational number. Also π^n , e^n , and φ^n ,$\forall n \in \mathbb{N}$, are all irrationals. Therefore, possible examples of rule-generated $\boldsymbol{G}$ for s_n , over $(0,\infty) \cap Q^c$could be

$$\begin{gathered} \{n\pi\}_{n=1}^{\infty}, \{ne\}_{n=1}^{\infty}, \left\{n\frac{e}{\pi}\right\}_{n=1}^{\infty}, \left\{n\frac{\pi}{e}\right\}_{n=1}^{\infty}, \{n\varphi\}_{n=1}^{\infty}, \quad \text{or;} \\ \{\pi^n\}_{n=1}^{\infty}, \{e^n\}_{n=1}^{\infty}, \left\{\left(\frac{\pi}{e}\right)^n\right\}_{n=1}^{\infty}, \{\varphi^n\}_{n=1}^{\infty}, \left\{\frac{\pi}{e}\varphi^n\right\}_{n=1}^{\infty}, \left\{\frac{e}{\pi}\varphi^n\right\}_{n=1}^{\infty}. \end{gathered} \tag{2}$$

While over $(0,1)\cap Q^c$, a rule-generated $\boldsymbol{G}$ could be

$$\{\pi^{-n}\}_{n=1}^{\infty}, \{e^{-n}\}_{n=1}^{\infty}, \left\{\left(\frac{\pi}{e}\right)^{-n}\right\}_{n=1}^{\infty},$$
$$\{\varphi^{-n}\}_{n=1}^{\infty}, \left\{\frac{\pi}{e}\varphi^{-n}\right\}_{n=1}^{\infty}, \left\{\frac{e}{\pi}\varphi^{-n}\right\}_{n=1}^{\infty}. \tag{3}$$

In a similar fashion the associated $\boldsymbol{H}=\{z_n=\frac{p}{qs_n}\}_{n=1}^{\infty}$ could be defined.

A basic inconvenience in the suggested $\boldsymbol{G}'$s, of (2-3), is their restricted applicability to certain domains and the possibility for their asymptotic behavior to be uncertain, [5-7]. Hence the construction of a universal and computationally robust $\boldsymbol{G}$, that could be blended with the (p,q) pair, in a $\frac{p}{q}$ representation, should be of some value for several applications.

Unlike $\boldsymbol{G}$ and $\boldsymbol{H}$, the uncountable $\boldsymbol{U}=\{s_\gamma\}$ and $\boldsymbol{V}=\{z_\gamma=\frac{p}{qs_\gamma}\}$, with $\#\,\boldsymbol{U}=\#\,\boldsymbol{V}=\aleph_1$, can exist in a variety of rather chaotic fashions that may, or may not, follow any specific rule.

So, in order to develop a nonqualitative (or precise) approach to the analysis of the relative population of Q and Q^c, we shall resort, when generating the $\boldsymbol{G}$ and $\boldsymbol{H}$ sets, to the sequence $\langle F_n\rangle$ of Fibonacci numbers F_n,

Table 1: Some Fibonacci numbers

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$< F_n >$	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
F_n	-3	2	-1	1	0	1	1	2	3	5	8	13	21

which is an example of a complete sequence in the sense of Zeckendorf [8], and defined recursively viz $F_n=F_{n-1}+F_{n-2}$, with $F_{-n}=(-1)^{n+1}F_n$ when $n\in\mathbb{Z}$.

This sequence, though aperiodic [9], is structurally periodic (or semi-periodic) as every third number in it is even. Also, any three consecutive Fibonacci numbers are pairwise coprime, [9, 10, p. 33], and every F_n that is prime must have a prime index n; with the exception of $F_4=3$. Furthermore, a characteristic feature of the spiral, defined by $\langle F_n\rangle$, is that $\lim_{n\to\infty}\frac{F_n}{F_{n-1}}=\varphi$. Defined via $\varphi=[1;1,1,1,1,....]=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+....}}}}=\sum_{k=1}^{\infty}\varphi^{-k}$, is the Golden ratio, which is a famous irrational number 1.618033988..., satisfying the remarkably unique properties

$$\varphi-1=\varphi^{-1},\quad \varphi+1=\varphi^2, \tag{4}$$

properties that enter in the structure of Binet's formula

$$F_n=\frac{\varphi^n-(-\varphi)^{-n}}{\sqrt{5}}. \tag{5}$$

Definition 1. An ordered countable set of irrational numbers $\boldsymbol{B} = \{z_\gamma\}_{\gamma\in\Gamma}$ is asymptotically rationally distinct (ARD) if $\lim_{\gamma\to\infty} z_\gamma = \rho \in Q$.

Now if $f(n)$ is a function of integers, then $\lfloor f(n)\rfloor$ is the floor function for $f(n)$, i.e. the largest integer less than or equal to n. Obviously $n - \lfloor n\rfloor = 0$, and $N^n - \lfloor N^n\rfloor = 0, \forall n, N \in \mathbb{N}$, are rational distinctivity properties for the sequences $< n >$ and $< N^n >$, respectively.

φ is also special in several other ways. The powers $\varphi^n = \sqrt{5}F_n + (1-\varphi)^n$, which satisfy $\lim_{n\to\infty} \varphi^n = \sqrt{5}F_n$, though are all irrationals, they however lie unexpectedly [6-7] close to integers: for instance $\varphi^{11} = 199.005$ is unusually close to 199. Moreover, it happens, [6], rather puzzlingly, that

$$\lim_{n\to\infty} (\varphi^n - \lfloor\varphi^n\rfloor) = 0, \tag{6}$$

despite the fact that $\varphi^n - \lfloor\varphi^n\rfloor \neq 0$, for all finite n.

An explanation of the reason for the puzzle (6), representing an asymptotic rational distinctivity for the sequence $< \varphi^n >$, is provided by the lemma to follow.

Remark 2. Throughout this paper we need to distinguish between asymptotic computational infinity ∞, as a very large (but finite as a must) rational number and mathematical infinity $\bowtie$,which is neither rational or irrational, but irrational (as a must) when conceived as $\bowtie = \frac{P}{0}$, $p \in \mathbb{N}$.

Accordingly, $\varphi^{\bowtie} = \bowtie \in Q^c$, despite the computational fact (6), which stipulates that

$$\varphi^{\infty} = \infty \in Q. \tag{7}$$

Lemma 1. *An ordered countable set of irrational numbers* $B = \{z_n\}_{n=1}^{\infty}$ *is asymptotically rationally distinct (ARD), i.e.* $\lim_{n\to\infty} z_n = \rho \in Q$, *if*

(i) $\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = s \in Q^c$,

(ii) $\lim_{n\to\infty} (z_n - z_{n-1}) = s^* \in Q^c$, *or* $= 0$,

(iii) *such that* $\frac{s^*}{s-1} \in Q$.

Proof. Satisfaction of (i) means that

$$\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = s \in Q^c, \tag{8}$$

while satisfaction of (ii) can be expressed as

$$\lim_{n\to\infty} (z_n - z_{n-1}) = \lim_{n\to\infty} z_{n-1}\left(\frac{z_n}{z_{n\ -1}} - 1\right) = s^* \in Q^c. \tag{9}$$

Now in view of (8), relation (9) is the same as

$$\lim_{n\to\infty} (z_n - z_{n-1}) = (s-1)\lim_{n\to\infty} z_{n-1} = s^*. \tag{10}$$

Furthermore, as $(s-1)=s^o \in Q^c$, then (10), by (iii) becomes $\lim_{n\to\infty} z_{n-1} = \frac{s^*}{s^o} = \rho \in Q$, which is the same as the required result $\lim_{n\to\infty} z_n = \rho \in Q$; and here the proof ends.

Corollary 2. *An ordered countable set of irrational numbers* $B=\{z_n\}_{n=1}^{\infty}$ *is asymptotically rationally distinct (ARD), when*

(a) $s^*=0$, *if (i), of Lemma 1, is violated.*

(b) if both (i), and (ii), of Lemma 1, are simultaneously violated, provided that $s \neq 1$.

Proof. By direct substitution of the stated conditions in Lemma 1.

Example 1. Test the ARD for The phi-nary set $\{\varphi^n\}_{n=1}^{\infty}$ whose sequence $\langle \varphi^n\ ,\ n \in \mathbb{N}\rangle$ diverges as an expanding Golden spiral.

Clearly ,

(i) $\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = \lim_{n\to\infty} \frac{\varphi^n}{\varphi^{n-1}} = \varphi = s \in Q^c$.

(ii) $\lim_{n\to\infty} (z_n - z_{n-1}) = (\varphi-1) \quad \lim_{n\to\infty} \varphi^{n-1}$.

Additionally, since $\lim_{n\to\infty} \varphi^{n-1} = \lim_{n\to\infty} \lfloor \varphi^{n-1} \rfloor$ is rational, then $\lim_{n\to\infty} (z_n - z_{n-1}) = s^* \in Q^c$. Furthermore $\frac{s^*}{s-1} = \frac{(\varphi-1)}{(\varphi-1)} \lim_{n\to\infty} \lfloor \varphi^{n-1} \rfloor = \rho \in Q$; and (iii) is satisfied. Hence this set is ARD. Interestingly, this turns out to to be consistent with (6).

Example 2. Test the ARD for The phi-nary set $\{\varphi^{-n}\}_{n=1}^{\infty}$ whose sequence $\langle \varphi^{-n}\ ,\ n \in \mathbb{N}\rangle$ converges to 0 as a contracting Golden spiral.

Here correspondingly,

(i) $\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = \lim_{n\to\infty} \frac{\varphi^{-n}}{\varphi^{-n+1}} = \varphi^{-1} = s \in Q^c$.

(ii) $\lim_{n\to\infty} (z_n - z_{n\ -1}) = (1-\varphi) \quad \lim_{n\to\infty} \varphi^{-n} = 0 = s^* \in Q$.

(iii) $\frac{s^*}{s-1} = \frac{(1-\varphi)}{(\varphi^{-1}-1)} \lim_{n\to\infty} \varphi^{-n} = -\varphi \lim_{n\to\infty} \varphi^{-n} = 0 = \rho \in Q$.

Therefore this set is also ARD.

Example 3. Test the ARD for $\boldsymbol{B} = \left\{ z_n = (n+\sqrt{2}) \right\}_{n=1}^{\infty}$ of algebraic irrationals.

Clearly $\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = \lim_{n\to\infty} \frac{n+\sqrt{2}}{n-1+\sqrt{2}} = 1 \notin Q^c$; violation of (i). Here Corollary 2 can be applied. Moreover, $\lim_{n\to\infty} (z_n - z_{n-1}) = \lim_{n\to\infty} [n+\sqrt{2} - (n-1+\sqrt{2})] = 1 \notin Q^c$ or 0; violation of (ii). Then $\frac{s^*}{s-1} = \frac{1}{0} = \bowtie \notin Q$; violation of (iii). Therefore this $\boldsymbol{B}$ in not ARD.

In addition to the demonstrated ARD behavior of $\varphi^n$and φ^{-n}, it is rather straightforward, [3, p. 214], to prove that

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2} = \varphi F_n + F_{n-1}, \tag{11}$$

$$\varphi^{-n} = \varphi^{2-n} - \varphi^{1-n} = (-1)^n [F_{n+1} - \varphi F_n]. \tag{12}$$

Remark 3. There are two canonical countable sets $\boldsymbol{C}$ and $\boldsymbol{D}$ in $[R^+ \cap Q^c]$, and removing them from the set $[R^+ \cap Q^c]$ does not alter the cardinality of this set.

Proof. According to the decomposition principle (1), each $r = \frac{p}{q} = sz \in [R^+ \cap Q]$ defines countable $\boldsymbol{C}$ and $\boldsymbol{D} \subset [R^+ \cap Q^c]$. Moreover as $\boldsymbol{C} \sim \boldsymbol{D} \sim [R^+ \cap Q]$, then $\# \boldsymbol{C} = \# \boldsymbol{D} = \aleph_0$. Also $\boldsymbol{C} \precsim [R^+ \cap Q^c]$ and $\boldsymbol{D} \precsim [R^+ \cap Q^c]$, then since $R^+ \cap Q^c$ is not denumerable, then $\boldsymbol{C} \prec [R^+ \cap Q^c]$ and $\boldsymbol{D} \prec [R^+ \cap Q^c]$.

Now when $\boldsymbol{C} \cup \boldsymbol{D}$ is taken out from $R^+ \cap Q^c$ to form $R^+ \cap Q$, the set

$$R^+ \cap Q^c \setminus (\boldsymbol{C} \cup \boldsymbol{D}), \tag{13}$$

remains inside $R^+ \cap Q^c$ and

$$\#[R^+ \cap Q^c] \setminus (\boldsymbol{C} \cup \boldsymbol{D}) = \#[R^+ \cap Q^c] = \aleph_1. \tag{14}$$

Therefore in the least sense $\# [R^+ \cap Q^c] = 2 \# [R^+ \cap Q] = 2\aleph_0$, but the previous process can be repeated an infinite number of times with the same result. However $\aleph_0 = \infty$, and if the inequality $2\dot{\infty} > \infty$ has any meaning (actually it does not), then we may claim that $\# [R^+ \cap Q^c] > \infty(2\aleph_0) = \infty\aleph_0$.

This holds regardless of the fact that $\frac{\# [R^+ \cap Q^c]}{\# [R^+ \cap Q]} = \infty\frac{\aleph_0}{\aleph_0} = \frac{\aleph_1}{\aleph_0}$, does not have a quantitative meaning. Hence since removal of $\boldsymbol{C}$ and $\boldsymbol{D}$ can indefinitely be repeated from $R^+ \cap Q^c$, without affecting the cardinality of the remaining set, we may say that "$R^+ \cap Q^c$ should be much more populated than $R^+ \cap Q$". Finally, since $R^+ = [R^+ \cap Q] \cup [R^+ \cap Q^c]$, then there exists much more irrational numbers in R^+ than rational numbers. Here the proof ends.

The only weakness in the previous proof of Remark 3, is that $\boldsymbol{C}$ and $\boldsymbol{D}$ are only qualitative or symbolic in nature, i.e. without a rule of generation, and with rather uncertain asymptotic behavior. Hence in a move towards stregthening this proof, it is quite desirable to redefine these sets in a rule-generated and in a computationally ARD fashion.

Remark 4. The arguments used in Remark 3 can also be applied on $R^- \cap Q^c$, then for $R \cap Q^c = Q^c$.

Theorem 3 (Fibonacci generated countable sets of irrational factors of r)**.** *The two irrational factors of $r \in [R^+ \cap Q]$ form a pair of phi-nary, Fibonacci-generated for r, countable ARD sets $\boldsymbol{C}^*$ and $\boldsymbol{D}^*$.*

Proof. Any two positive integers that stand in $r = \frac{p}{q}$, $p \in \mathbb{Z}^+$, $q \in \mathbb{N}$ can be represented in terms of Fibonacci numbers viz $p = p(m,i) = [F_m - i] \in (F_{m-1}, F_m]$, $0 \leq i \leq F_{m-2} - 1, m \geq 0, m \neq 2$, yielding

$$\mathbb{Z}^+ = \left\{ \{F_m - i\}_{i=0}^{F_{m-2}-1} \right\}_{m=0, m\neq 2}^{\infty}$$

and $q = p(n,j) = [F_n - j] \in (F_{n-1}, F_n],\ 0 \le j \le F_{n-2} - 1, n \ge 1, n \ne 2$, yielding $\mathbb{N} = \left\{ \{F_n - j\}_{j=0}^{F_{n-2}-1} \right\}_{n=1, n\neq 2}^{\infty}$.

It should be noted that $m \ne 2$ and $n \ne 2$ in the previous relations happen to be reflection of the remarkable fact that $F_1 = F_2 = 1$.

Therefore $r_k = r(m,n,i,j) = \frac{p(m,i)}{q(n,j)} = \frac{[Fm-i];m\in\mathbb{Z}^+\setminus\{2\},0\le i\le F_{m-2}-1}{[F_n-j];n\in\mathbb{N}\setminus\{2\},0\le j\le F_{n-2}-1} \ge 0$.

Apparently, different p's may have the same m (that defines the closest F_m), but different i's. Also different p's may have different m's but the same i. Regardless of that, $p(m,i) = [F_m - i]$ should be unique, i.e. exists one and only one p that can be defined by the (m,i) pair. This follows directly from the uniqueness of $F_m, \forall m \ne 2$.

Make then use of (5) in this r_k to rewrite it as $r_k = \frac{\varphi^m - (1-\varphi)^{-m} - i\sqrt{5}}{\varphi^n - (1-\varphi)^{-n} - j\sqrt{5}}$, which in view of (4) decomposes to

$$r_k = \frac{\pi n}{em(i+j)} \frac{[1-(-1)^m(1-\varphi^{-1})^m]/i - \varphi^{-m}\sqrt{5}}{[1-(-1)^n(1-\varphi^{-1})^n]/j - \varphi^{-n}\sqrt{5}} \ \frac{e}{\pi} \frac{m}{n}(i+j)\frac{i}{j}\varphi^{m-n}. \tag{15}$$

By virtue of (1) , it is possible to write

$$\begin{aligned} \wp_{m-n,i,j}\ (\varphi) &= \frac{e}{\pi}\frac{m}{n}(i+j)\frac{i}{j}\ \varphi^{m-n} \\ &= \frac{e}{\pi}\frac{m}{n}(i+j)\frac{i}{j}\begin{cases} \varphi F_{m-n} + F_{m-n-1}, & m-n>0 \\ (-1)^{n-m}[F_{n-m+1} - \varphi F_{n-m}], & m-n<0. \end{cases} \end{aligned} \tag{16}$$

Since distinct $r = \frac{p}{q}$'s may have the same $\frac{i}{j}\ \varphi^{m-n}$, to entail a virtual contradiction. However, the multiplication of $\frac{i}{j}\ \varphi^{n-m}$ by $\frac{n}{m}(i+j)$, in (16), is meant to guarantee uniqueness of $\wp_k(\varphi) = \wp_{m-n,i,j}\ (\varphi)$.A uniqueness that follows from the fact that the system of equations:

$\varphi^{m-n} = \alpha, \frac{m}{n} = \beta, \frac{i}{j} = \sigma,\ i+j = \theta$, with $\begin{cases} \alpha > 1, & \beta > 1, m-n>0, \\ \alpha < 1, & \beta < 1, m-n<0 \end{cases}$, has always the unique solution vector $(\frac{\beta ln\alpha}{(\beta-1)ln\varphi}, \frac{ln\alpha}{(\beta-1)ln\varphi}, \frac{\gamma\theta}{1+\sigma}, \frac{\sigma}{1+\sigma}) = (m,n,i,j) \triangleq r_k$.

Now as this $\wp_{m-n,i,j}\ (\varphi) \in \boldsymbol{C}^*$ is an irrational factor of r_k , and $\boldsymbol{C}^* \sim [R^+ \cap\ Q]$, then

$$\frac{\pi n}{em(i+j)} \frac{[1-(-1)^m(1-\varphi^{-1})^m]/i\ - \varphi^{-m}\sqrt{5}}{[1-(-1)^n(1-\varphi^{-1})^n]/j - \varphi^{-n}\sqrt{5}} = \Im_{m,n,i,j}(\varphi), \tag{17}$$

can only be an irrational belonging in some other $\boldsymbol{D}^* \sim [R^+ \cap Q]$, and distinct from $\boldsymbol{C}^*$.

Hence, the multiplicative decomposition principle (1) writes in phi-nary factors as

$$\begin{aligned} & r_k = \ \wp_{m-n,i,j}\ (\varphi)\ \Im_{m,n,i,j}\ (\varphi), \\ & (m,n,i,j)\ \triangleq \frac{p}{q} = r_k \in \{r_k\}_{k=1}^{\infty} \subset [R^+ \cap\ Q]. \end{aligned} \tag{18}$$

The proof that $\boldsymbol{C}^*$ is ARD is trivial, and unlike the the proof of the ARD of $\boldsymbol{D}^*$,which is indirect. Moreover, $\wp_k = \wp_{m-n,i,j}(\varphi) \in \{\wp_k\}_{k=1}^{\infty} = \boldsymbol{C}^*$ and $\Im_k = \Im_{m,n,i,j}\ (\varphi) \in \{\Im_k\}_{k=1}^{\infty} = \boldsymbol{D}^*$, establishing that both $\boldsymbol{C}^*$and $\boldsymbol{D}^*$ are Fibonacci-generated for $\{r_k\}_{k=1}^{\infty}$, countable irrational phinary ARD sets. In particular, $\wp_k = \wp_{m-n,i,j}(\varphi) \in \boldsymbol{C}^*$ is Fibonacci-generated uniquely for every r_k via the vector $(\frac{n}{m}, (n-m), \frac{i}{j}, (i+j))$. Also $\Im_k = \Im_{m,n,i,j}\ (\varphi) \in \boldsymbol{D}^*$ corresponds, to the same r_k , via its m, n, i, j. Hence the pair of irrational numbers $(\wp_k\ ,\ \Im_k\ ) = (\wp_{m-n,i,j}(\varphi)\ ,\ \Im_{m,n,i,j}\ (\varphi))$ uniquely represents any rational number r_k via the Fibonacci-based rules (16)-(17).

Here the proof ends.

Corollary 3. *The set* $[\boldsymbol{C}^* U \boldsymbol{D}^*]$ *is a is a unique countable set of irrationals on* R^+.

Proof. Each element $\wp_k = \wp_{m-n,i,j}(\varphi) \in \boldsymbol{C}^*$and $\Im_k = \Im_{m,n,i,j}(\varphi)\ \in \boldsymbol{D}^*$ is , by virtue of (16) and (17), respectively unique. So are the pertaining $\boldsymbol{C}^*$ and $\boldsymbol{D}^*$ sets.

Result 1. Remark 3 holds, when each of the $\boldsymbol{C}$ and $\boldsymbol{D}$ sets, in its proof, is respectively replaced by $\boldsymbol{C}^*$ and $\boldsymbol{D}^*$ of Theorem 3.

As $\boldsymbol{C}^* \cup \boldsymbol{D}^*$ may preserve some of the aperiodicity (or semi-periodicity) features, [10, p.33], of the Fibonacci sequence, then Result 1 should not be the only possible application for this set. One can envisage, e.g., some role to be played by $\boldsymbol{C}^* \cup \boldsymbol{D}^*$, in some modifications to Fibonacci search algorithms [11-12], to be investigated in the future. Specifically, in certain optimization problems, a stable (resonance free) optimal frequency, that may be sought by such algorithms, may not be admissible in the Q set, [5]. Such situations invoke a modification of the Fibonacci to "irrational-Fibonacci" search algorithm, to yield a stable optimal frequency in $\boldsymbol{C}^* \cup \boldsymbol{D}^*$.

3. Conclusion

It is worthwhile to remark that this work has nothing to do with Fibonacci coding [7-8], or with base φ representations [13] of whole numbers. The reported result of this note illustrates, nevertheless, that the Fibonacci generated countable irrational $\boldsymbol{C}^*$ and $\boldsymbol{D}^*$ sets can improve arguments of the proofs of some related real analysis results and may have potential application in some irrational-Fibonacci search algorithms. It also poses a question on its possible relation to the subject of completeness of the $\langle F_n \rangle$ sequence. Zeckendorf completeness means that every positive integer, like p or q, can be written in a unique way as a sum of one or more distinct F_n's, in such a way that the sum does not include any two consecutive F_n's. Our Theorem 3 demonstrates that by using $\langle F_n \rangle$ sequence representations, any r_k rational is multiplicatively decomposable to two unique irrationals, $\wp_k \in \boldsymbol{C}^*$ and $\Im_k \in \boldsymbol{D}^*$. The tenuous relationship between this result and completeness is perhaps one between additive and multiplicative decomposability, but not of the same objects though.

4. Appendix

Q and Q^c are both equally dense, as this word has a precise mathematical meaning that they share. Density here is a topological concept, and has nothing to do with cardinality #; see e.g. [1]. As the word "more dense"sounds meaningless in topology, one way to measure how relatively populated Q and Q^c are, is the fact that Q has a Lebesgue measure 0, whereas Q^c has full measure in any interval $(a, b) \subset R$.Hence if one picks at random a number x from $(0, 1)$, the probability of $x = r \in A = \{r_k\}_{k=1}^{\infty} \subset Q$ is zero, while the probability of $x = s \in B = \{s_k\} \subset Q^c$ is one. Remarkably, this happens to take place while the countable A and uncountable B are both infinite sets and both are equally dense.

This may sound physically paradoxical, if density is conceived conventionally as a "number of points per unit length". The paradox is resolvable however, if one simulates rational numbers by a tiny amount of a solute of small molecules, dissolved in a pond of a solvent of large molecules. Topologically, the species density should be some ratio like

$$\frac{\text{Molecular size}}{\text{Average distance between any two molecules}},$$

for each of the two species. Such a density can be the same for the solvent (B) and solute (A), despite the infinitely larger relative population of the solvent molecules in B.

Indeed B appears to be more populated that A, as can be deduced from Theorem 1 and its corollary.

Proof of Theorem 1. Reapply the same classical proof, see e.g. [2], of this theorem on the existence of a rational number between any two real numbers $a,b \in R$. This invokes the Archimedian property of R to start with $0 < \frac{1}{n_k} < b - a$, $n_k \in N = \{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$.

Take then $m_* = \frac{\min m}{n_k} \geq b$, $m \in \mathbb{N}$, to arrive at $a < r_k = \frac{m_* - 1}{n_k} < b$, $r_k \in A = \{r_k\}_{k=1}^{\infty} \subset Q \cap (a, b)$. Here the proof ends.

Proof of Corollary 1. Reconsider the open interval (a, b) of Theorem 1 as $(\frac{a}{z_\gamma}, \frac{b}{z_\gamma})$, $\gamma \in \Gamma$, with an arbitrary $z_\gamma \in \mathfrak{B} \subset Q^c \cap (a, b)$, not necessarily countable, with $\#\mathfrak{B} = \aleph_1$. This theorem stipulates the existence of $\rho_k \in \mathfrak{A} = \{\rho_k\}_{k=1}^{\infty} \subset Q \cap (a, b)$ satisfying $\frac{a}{z_\gamma} < \rho_k < \frac{b}{z_\gamma}$, $\rho_k \in \mathfrak{A} = \{\rho_k\}_{k=1}^{\infty} \subset Q \cap (a, b)$, with $\#\mathfrak{A} = \aleph_0$, i.e. $a < s_{\gamma k} = z_\gamma\, \rho_k < b$, and $s_{\gamma k} \triangleq (z_\gamma\, , \rho_k) \in \mathfrak{B} \times \mathfrak{A}$.Now since $z_\gamma\, \rho_k$ is irrational, then $s_{jk} \in \{\{s_{\gamma k}\}_{\gamma \in \Gamma}\}_{k=1}^{\infty} \subset Q^c \cap (a, b)$, which is the required result.

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