Fuzzy order relative to fuzzy B-algebras

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Abstract. Let $(X; *, 0)$ be a B-algebra and μ a fuzzy B-algebra defined on X. For each $x \in X$, the fuzzy order of x, denoted by $FO^B_\mu(x)$, is the least positive integer n such that $\mu(x^n) = \mu(0)$. Also, the order of an element $x \in X$, denoted by $|x|_B$, is the least positive integer n such that $x^n = 0$. In this paper, some properties of $FO^B_\mu(x)$ are established. Also, the relationship between $FO^B_\mu(x)$ and the order $|x|_B$ of x is also explored.

Keywords: B-algebra, fuzzy B-algebras, order of an element, fuzzy order.

1. Introduction

Neggers and Kim [8] first introduced the notion of *B-algebras* in 2002. It is an algebra $(X; *, 0)$ of type $(2, 0)$ such that the following axioms are satisfied for all $x, y, z \in X : (I) x * x = 0, (II) x * 0 = x$, and $(III) (x * y) * z = x * (z * (0 * y))$. Meanwhile, the concept of *subalgebra* of a B-algebra was introduced in [7]. It is a nonempty subset A of X such that $x * y \in A$ whenever $x, y \in A$. It was proven (see [2,6]) that the class of B-algebras and the class of groups coincide; that is, one can view a B-algebra as a group and vice versa. Because of this, new researches were published to further explore the properties of a B-algebra when viewed as a group. One such research is the concept of *cyclic B-algebras* which was introduced by Gonzaga and Vilela [4] in 2015. Endam and Teves [3] meanwhile provided more properties of cyclic B-algebras and went to introduce the concept of "order of an element" of a B-algebra similar to that in groups. On the other hand, Zadeh [9] initiated the notion of *fuzzy sets*. It is a generalization of an ordinary set in which each element is associated with a *membership value* within the interval $[0, 1]$. That is, it is a function $\mu : A \to [0, 1]$ where A is any

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nonempty (crisp) set. Since it generalizes an ordinary set, various researches in fuzzy sets were established in an attempt to generalize some concepts in modern algebra. An example of this is the notion of *fuzzy B-algebra* which was introduced by Jun, Roh, and Kim [5]. It is a fuzzy set μ defined on a B-algebra $(X;*,0)$ such that the inequality $\mu(x*y) \geq \min\{\mu(x),\mu(y)\}$ holds for all $x,y \in X$. Ahn and Bang [1] meanwhile explored the properties of fuzzy B-algebras with respect to the subalgebras of the underlying B-algebra. Continuing this research trend, this paper will introduce the concept of fuzzy order of an element of a B-algebra as a generalization of the concept introduced in [3]. Some of its properties are extensively investigated, and its relationship to the ordinary"order" is also explored.

2. Preliminaries

Throughout this paper, X will denote the B-algebra $(X;*,0)$ unless otherwise specified. Recall the following property from [8]:

Lemma 2.1. *If X is a B-algebra, then $x = 0*(0*x)$ for all $x \in X$.*

Let $n \in \mathbb{Z}^+$ and $x \in X$. In [4], the exponential expression x^n is defined as follows: $x^n = x*(0* \prod^{n-1} x)$ where the expression $0* \prod^{n-1} x$ is equal to $[\ldots[(0*x)*x]*\ldots]*x$ such that x occurs $n-1$ times. By convention, $x^1 = x$, while $x^0 = 0$ is taken from [8]. Also, the notations $-x$ and x^{-n} are equal to $0*x$ and $(-x)^n$, respectively. Using these observations, the following results are obtained:

Corollary 2.2. *For all $x \in X$ and for all $m,n \in \mathbb{Z}$, $x^m * x^n = x^{m-n}$.*

Corollary 2.3. *For all $x \in X$ and for all $n \in \mathbb{Z}$,*

i. $x^{-n} = (-x)^n = -(x^n) = 0*x^n$;

ii. $(x^{-n})^{-1} = (x^{-1})^{-n} = x^n$.

Proposition 2.4. *Let $x \in X$. Then for all $m,n \in \mathbb{Z}$, $(x^m)^n = x^{mn}$.*

For a subset $A \subseteq X$, the *cyclic subalgebra* $\langle A\rangle_B$ of X genereted by A is the intersection of all subalgebras of X containing A [4]. If A is a singleton set, that is, $A = a$, then $\langle A\rangle_B = \langle a\rangle_B$, the cyclic subalgebra of X containing a. If there is an $x \in X$ such that $\langle x\rangle_B = X$, then X is a *cyclic B-algebra*.

The following propositions hold for all fuzzy B-algebra defined on a B-algebra, see [5]:

Proposition 2.5. *Every fuzzy B-algebra μ defined on X satisfies the inequality $\mu(0) \geq \mu(x)$ for all $x \in X$.*

Proposition 2.6. *If a fuzzy set μ defined on X is a fuzzy B-algebra, then $\mu(-x) \geq \mu(x)$ for all $x \in X$.*

Proposition 2.6 is strengthen in the following lemma:

Lemma 2.7. *If a fuzzy set μ defined on X is a fuzzy B-algebra, then $\mu(-x) = \mu(x)$ for all $x \in X$.*

Proof. It remains to show that $\mu(-x) \leq \mu(x)$ for all $x \in X$. Indeed, $\mu(x) = \mu(0 * (0 * x)) \geq \min\{\mu(0), \mu(0 * x)\} = \mu(0 * x) = \mu(-x)$ by Lemma 2.1 and Proposition 2.5. □

Let μ be a fuzzy B-algebra defined on a B-algebra X. For all $t \in [0, 1]$, the subset $\mu_t = \{x \in X | \mu(x) \geq t\}$ of X is called the *t-level subsets* of μ[1]. For the construction of examples in this paper, consider the following theorem [1]:

Theorem 2.8. *Let X be a B-algebra and μ be a fuzzy set on X such that every t-level subsets of X is a subalgebra of X, where $0 \leq t \leq \mu(0)$. Then μ is a fuzzy B-algebra defined on X.*

The following lemma will be used extensively in this study.

Lemma 2.9. *Let X be a B-algebra and μ a fuzzy B-algebra defined on X. Then for all $x \in X$ and for all $n \in \mathbb{Z}$, $\mu(x^n) \geq \mu(x)$.*

Proof. Let $x \in X$ and $n \in \mathbb{Z}$. If $n = 0$, then $\mu(x^0) = \mu(0) \geq \mu(x)$ by Proposition 2.5. Suppose $n > 0$. Applying (1) repeatedly and Lemma 2.7,

$$
\begin{aligned}
\mu(x^n) &= \mu(x * (0 * \prod^{n-1} x)) \\
&\geq \min\{\mu(x), \mu(0 * \prod^{n-1} x)\} \\
&= \min\{\mu(x), \mu[[\ldots[(0 * \underbrace{x) * x] * \ldots] * x]}_{n-1}\} \\
&\geq \min\{\mu(x), \min\{\mu[[\ldots[(0 * \underbrace{x) * x] * \ldots] * x]}_{n-2}, \mu(x)\}\} \\
&\geq \min\{\mu(x), \min\{\min\{\mu[[\ldots[(0 * \underbrace{x) * x] * \ldots] * x]}_{n-3}, \mu(x)\}, \mu(x)\}\} \\
&= \min\{\mu(x), \min\{\mu[[\ldots[(0 * \underbrace{x) * x] * \ldots] * x]}_{n-3}, \mu(x)\}\} \\
&\vdots \\
&= \min\{\mu(x), \min\{\mu(0 * x), \mu(x)\}\} \\
&= \min\{\mu(x), \min\{\mu(-x), \mu(x)\}\} \\
&= \min\{\mu(x), \min\{\mu(x), \mu(x)\}\} \\
&= \min\{\mu(x), \mu(x)\} \\
&= \mu(x).
\end{aligned}
$$

If $n < 0$, then $-n > 0$. Thus, $\mu(x^n) = \mu(-(x^{-n})) = \mu(x^{-n} \geq \mu(x)$ by Corollary 2.3 and Lemma 2.7. □

3. Main results

In [3], the order of an element $x \in X$, denoted by $|x|_B$, is the least positive integer n such that $x^n = 0$.

Definition 3.1. Let μ be a fuzzy B-algebra defined on a B-algebra X and $x \in X$. The *fuzzy order* of x with respect to μ, denoted by $FO_\mu^B(x)$, is the least positive integer n such that $\mu(x^n) = \mu(0)$. If there is such least n satisfying the equality, then x is of μ-finite order; otherwise, x is of μ-infinite order.

By Proposition 2.5, for all $x \in X$, $\mu(0) \geq \mu(x^n)$ for any fuzzy B-algebra μ defined on X. Thus, the following remark is valid.

Remark 3.2. Let $x \in X$ where X is a B-algebra and $n \in \mathbb{Z}^+$. Then $n = FO_\mu^B(x)$ if and only if n is the least positive integer such that $\mu(x^n) \geq \mu(0)$.

The following example shows that the fuzzy order of an element is not equal to its order and fuzzy orders of two elements are equal but their respective orders are not.

Example 3.3. Consider the B-algebra $X = \{0, 1, 2, 3\}$ with the following table of operation [5], and μ defined on X:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

$$\mu(x) = \begin{cases} 0.9, & \text{if } x \in \{0, 2\}, \\ 0.3, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy B-algebra, $FO_\mu^B(0) = FO_\mu^B(2) = 1$ and $FO_\mu^B(1) = FO_\mu^B(3) = 2$. But $|2|_B = 2 \neq 1 = FO_\mu^B(2)$ and $|0|_B = 1 \neq 2 = |2|_B$.

On the other hand, the following example shows that the orders of two elements are equal but their fuzzy orders are not.

Example 3.4. Consider the Klein-4 group $V = \{e, a, b, ab\}$ with binary operation "$\cdot$" defined in the Cayley table below:

$\cdot$	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

Then by routine calculations, $\mathbf{V} = (V; \cdot, e)$ is a B-algebra (this is a special case where the operation "$\cdot$" of a group and the operation "$*$" of its derived B-algebra coincide). Define the fuzzy set μ for all $x \in V$, where $t \in (0,1]$ by

$$\mu(x) = \begin{cases} t, & \text{if } x \in \{e, ab\}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the only level subsets of μ are $\{e, ab\}$ and V. Since $\{e, ab\} = \langle ab \rangle$ in view of the group V, $\{e, ab\} = \langle ab \rangle_B$ is a subalgebra of $\mathbf{V}$ which is cyclic. Therefore by Theorem 2.8, μ is a fuzzy B-algebra defined on $\mathbf{V}$. Now, $|a|_B = |ab|_B = 2$ since $\langle a \rangle_B = \{e, a\}$. But then, $FO_\mu^B(a) = 2 \neq 1 = FO_\mu^B(ab)$.

In view of Examples 3.3 and 3.4, the following remarks hold.

Remark 3.5. Let μ be a fuzzy B-algebra defined on X and $x, y \in X$.

a) The equality $|x|_B = FO_\mu^B(x)$ may not hold in general.

b) The equality $|x|_B = |y|_B$ does not imply that $FO_\mu^B(x) = FO_\mu^B(y)$, and vice versa.

Let μ be a fuzzy B-algebra defined on X. If $x \in X$ is of finite order (that is, $|x|_B < +\infty$), then x is also of μ-finite order. To see this, note that if $|x|_B = n$, then $\mu(x^n) = \mu(0)$. Thus, $FO_\mu^B(x) \leq n$. Observe that the inequality is still possible because the definition of μ may vary along the interval $[0,1]$. That is, there might be an $r \in \mathbb{Z}^+$ with $r < n$ such that $\mu(x^r) = \mu(0)$. If there is no such r, however, then it is guaranteed that $FO_\mu^B(x) = n$.

The following proposition gives the idea that any element of a B-algebra that is of infinite order may still be of μ-finite order.

Proposition 3.6. *Let $x \in X$ be of infinite order. Then there is a fuzzy B-algebra defined on X such that x is of μ-finite order.*

Proof. Consider the fuzzy set $\mu : X \to [0,1]$ defined for all $y \in X$, where $r_1, r_2 \in [0,1]$ and $n \in \mathbb{Z}^+$ with $r_1 \geq r_2$ as follows:

$$\mu(y) = \begin{cases} r_1, & \text{if } y \in \langle x^n \rangle, \\ r_2, & \text{otherwise.} \end{cases}$$

Then we see that $\langle x^n \rangle$ and X are the only level subsets of X, which are all subalgebras of X. By Theorem2.8, μ is a fuzzy B-algebra defined on X. Now, observe that $\mu(x^n) = r_1 = \mu(0)$ since $0 \in \langle x^n \rangle$. Hence, $FO_\mu^B(x) = n$. $\square$

A natural question after observing Examples 3.3 and 3.4 is the following: what then is the relationship between $FO_\mu^B(x)$ and $|x|_B$ for any element x of a B-algebra X and for any fuzzy B-algebra μ defined on X. In Example 3.3, $FO_\mu^B(1)$ divides $|1|_B$. This observation is generalized in the next result.

Theorem 3.7. *Let μ be a fuzzy B-algebra defined on X, $x \in X$ such that x is of μ-finite order and $m \in \mathbb{Z}^+$. Then $\mu(x^m) = \mu(0)$ if and only if $FO_\mu^B(x)$ divides m. In particular, $FO_\mu^B(x)$ divides $|x|_B$.*

Proof. Set $FO_\mu^B(x) = n$. Suppose $m \in \mathbb{Z}^+$ with $n|m$. Then $m = nq$ for some $q \in \mathbb{Z}^+$. By Proposition 2.4, Lemma 2.9 and Proposition 2.5, $\mu(x^m) = \mu(x^{nq}) = \mu((x^n)^q) \geq \mu(x^n) = \mu(0) \geq (x^m)$. Hence, $\mu(x^m) = \mu(0)$. Conversely, let $m \in \mathbb{Z}^+$ such that $\mu(x^m) = \mu(0)$. By the Division Algorithm (applied to m and n), there are unique $q, r \in \mathbb{Z}$ with $0 \leq r < n$ such that $m = nq + r$. By Corollary 2.2, Proposition 2.4, Lemma 2.9 and Proposition 2.5,

$$\begin{aligned}
\mu(x^r) = \mu(x^{m-nq}) = \mu(x^m * x^{nq}) &\geq \min\{\mu(x^m), \mu(x^{nq})\} \\
&\geq \min\{\mu(0), \mu((x^n)^q)\} \\
&= \mu((x^n)^q) \\
&\geq \mu(x^n) \\
&= \mu(0) \\
&\geq \mu(x^r).
\end{aligned}$$

Thus, $\mu(x^r) = \mu(0)$. But this contradicts to the minimality of n. Hence, $r = 0$ and so $m = nq$. This shows that $n|m$. The particular statement holds since $\mu(x^m) = \mu(0)$ whenever $|x|_B = m < +\infty$ and, in the extended real number system, $+\infty$ is divisible by both $+\infty$ and a positive real number. □

The next results further characterize the concept of fuzzy order of an element of a B-algebra with respect to a well-defined fuzzy B-algebra.

Proposition 3.8. *Let μ be a fuzzy B-algebra defined on X. Then $FO_\mu^B(x) = FO_\mu^B(-x)$ for all $x \in X$.*

Proof. If $FO_\mu^B(x) = +\infty$, then $\mu(x^m) \neq \mu(0)$ for all $m \in \mathbb{Z}^+$. By Corollary 2.3(i) and Lemma 2.7, $\mu((-x)^m) = \mu(-(x^m)) = \mu(x^m) \neq \mu(0)$ for all $m \in \mathbb{Z}^+$. Hence, $FO_\mu^B(-x) = +\infty$. Suppose now that $FO_\mu^B(x) = n$ where $n \in \mathbb{Z}^+$. Then n is the least positive integer such that $\mu(x^n) = \mu(0)$. Again by Corollary 2.3(i) and Lemma 2.7, $\mu((-x)^n) = \mu(-(x^n)) = \mu(x^n) = \mu(0)$. Let $r \in \mathbb{Z}^+$ such that $\mu((-x)^r) = \mu(0)$ but $r < n$. Then by Corollary 2.3,

$$\mu(x^r) = \mu(-(x^{-r})) = \mu(x^{-r}) = \mu((-x)^r) = \mu(0),$$

a contradiction to the minimality of n. Therefore, $FO_\mu^B(-x) = n$. □

Theorem 3.9. *Let μ be a fuzzy B-algebra defined on X. Let $x \in X$ with $FO_\mu^B(x) = n < +\infty$. Then $FO_\mu^B(x^m) = \frac{n}{gcd(m,n)}$ for all $m \in \mathbb{Z}^+$.*

Proof. Set $FO_\mu^B(x^m) = t$ and $d = gcd(m, n)$. Then $d|m$ and so $m = dk$ for some $k \in \mathbb{Z}^+$. By Proposition 2.5, Corrolary 2.3, and Lemma 2.9, $\mu(0) \geq \mu((x^m)^{\frac{n}{d}}) =$

$\mu((x^{dk})^{\frac{n}{d}}) = \mu(x^{nk}) = \mu((x^n)^k) \geq (x^n) = \mu(0)$. Thus, $\mu((x^m)^{\frac{n}{d}}) = \mu(0)$. By Theorem 3.7, $t|\frac{n}{d}$. Also, $d = gcd(m,n)$ implies there exist $p, q \in \mathbb{Z}$ such that $d = mp + nq$. Now, by Proposition 2.5, Corollary 2.2, Proposition 2.4, and Lemmas 2.7 and 2.9,

$$\begin{aligned}
\mu(0) \geq \mu(x^{td}) = \mu(x^{t(mp+nq)}) &= \mu(x^{tmp+tnq}) \\
&= \mu(x^{mtp} * x^{-tnq}) \\
&\geq \min\{\mu(x^{mtp}), \mu(x^{-ntq})\} \\
&= \mu\{(((x^m)^t)^p), \mu(-(x^n)^{tq})\} \\
&\geq \min\{\mu(((x^m)^t)), \mu((x^n)^{tq})\} \\
&\geq \min\{\mu(0), \mu(x^n)\} \\
&= \min\{\mu(0), \mu(0)\} \\
&= \mu(0).
\end{aligned}$$

So, $\mu(x^t d) = \mu(0)$. This means that $n|td$ by Theorem 3.7. Hence, $ns = td$ for some $s \in \mathbb{Z}^+$, that is, $(\frac{n}{d})s = t$. This implies that $\frac{n}{d}|t$. Consequently, $t = \frac{n}{d}$. □

Proposition 3.10. *Let μ be a fuzzy B-algebra defined on X and $x \in X$ which is of μ-finite order. If m is an integer such that m and $FO_\mu^B(x)$ are relatively prime, then $\mu(x^m) = \mu(x)$.*

Proof. Set $FO_\mu^B(x) = n$ and suppose $m \in \mathbb{Z}$ such that $gcd(m,n) = 1$. Then there exist $r, s \in \mathbb{Z}$ such that $nr + ms = 1$. By Corollary 2.2, Proposition 2.4, and Lemma 2.7, Proposition 2.5, and Lemma 2.9,

$$\begin{aligned}
\mu(x) = \mu(x^{nr+ms}) = \mu(x^{nr} * x^{-ms}) &\geq \min\{\mu(x^{nr}), \mu(x^{-ms})\} \\
&\geq \min\{\mu((x^n)^r), \mu(-(x^m)^s\} \\
&\geq \min\{\mu(x^n), \mu((x^m)^s)\} \\
&\geq \min\{\mu(0), \mu(x^m)\} \\
&= \mu(x^m) \\
&= \mu(x).
\end{aligned}$$

Therefore, $\mu(x^m) = \mu(x)$. □

Theorem 3.11. *Let μ be a fuzzy B-algebra defined on X. Suppose $FO_\mu^B(x) = n < +\infty$ where $x \in X$ and $i, j \in \mathbb{Z}$. If $i \equiv j(modn)$, then $FO_\mu^B(x^i) = FO_\mu^B(x^j)$.*

Proof. Set $FO_\mu^B(x^i) = s$ and $FO_\mu^B(x^j) = t$. Now, $i \cong j(modn)$ implies $nk = i - j$ for some $k \in Z$. By Proposition 2.5, Corollary 2.2, Proposition 2.4, and

Lemmas 2.7 and 2.9,

$$\begin{aligned}\mu(0) \geq \mu((x^i)^t) = \mu((x^{j+nk})^t) = \mu(x^{jt+nkt}) &= \mu(x^{jt}) * x^{-nkt}) \\ &\geq \min\{\mu(x^{jt}), \mu(x^{-nkt}\} \\ &= \min\{\mu((x^j)^t), \mu(-(x^{nkt}))\} \\ &= \min\{\mu(0), \mu(x^{nkt})\} \\ &= \mu((x^n)^{kt}) \\ &\geq \mu(x^n) \\ &= \mu(0).\end{aligned}$$

Thus, $\mu((x^i)^t) = \mu(0)$. By Theorem 3.7, $s|t$. Also by Proposition 2.5, Corollary 2.2, Proposition 2.4, and Lemma 2.9,

$$\begin{aligned}\mu(0) \geq \mu((x^j)^s) = \mu((x^{i-nk})^s) = \mu(x^{it-nks}) &= \mu(x^{is} * x^{nks}) \\ &\geq \min\{\mu(x^{is}), \mu(x^{nks}\} \\ &= \min\{\mu((x^i)^s), \mu((x^n)^{ks})\} \\ &\geq \min\{\mu(0), \mu(x^n)\} \\ &= \min\{\mu(0), \mu(0)\} \\ &= \mu(0).\end{aligned}$$

So, $\mu((x^j)^s) = \mu(0)$. By Theorem 3.7, $t|s$. Consequently, $s = t$. □

References

[1] S. S. Ahn and K. Bang, *On Fuzzy subalgebras in B-algebras,* Commun. Korean Math. Soc., 18 (2003), 429-437.

[2] P. J. Allen, J. Neggers, and H. S. Kim, *B-algebras and groups*, Scientiae Mathematicae Japonicae Online, 9 (2003), 159-165.

[3] J. C. Endam and R. C. Teves, *Some properties of cyclic B-algebras*, International Mathematical Forum, 11 (2016), 387-394.

[4] N. C. Gonzaga and J. P. Vilela, *On cyclic B-algebras*, Applied Mathematical Sciences, 9 (2015), 5507-5522.

[5] Y. B. Jun, E. H. Roh, and H. S. Kim, *On fuzzy B-algebras*, Czechoslovak Mathematical Journal, 52 (2002), 375-384.

[6] M. Kondo and Y. B. Jun, *The class of B-algebras Coincides with the Class of Groups*, Scientiae Mathematicae Japonicae Online, 7 (2002), 175-177.

[7] J. Neggers and H. S. Kim, *A fundamental theorem of B-homomorphism for B-algebras,* Int. Math. Journal, 2 (2002), 207-214.

[8] J. Neggers and H. S. Kim, *On B-algebras*, Math. Vesnik, 54 (2002), 21-29.

[9] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 87-96.

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