On certain properties of hyperbolically convex functions

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Abstract. The aim of this paper is to prove that the envelope of hyperbolically convex functions is hyperbolically convex function. Furthermore, we study the standard functional operations of hyperbolically convex functions and introduce a class $BH[a, b]$ of functions representable as the difference of two hyperbolically convex functions.
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1. Introduction

One of the powerful properties of functions, which play a very important role in many areas of mathematics both pure and applied, is convexity. An arbitrary function f defined on an interval I is said to be convex if each point on the chord between $(u, f(u))$ and $(v, f(v))$ is above the graph of f for any $u, v \in I$. In fact, there are families of real functions $\{F(x)\}$ which are not topologically equivalent to the family $\{L(x)\}$ of all non vertical line segments terminating on $x = u$ and $x = v$. So that, there are many activities concerning to generalize the notion of a convex function to other classes of functions. But many properties of convex functions are satisfied for these general functions. Generalized convex

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functions were first defined and systematically investigated by Beckenbach [2] and studied furthermore by Beckenbach and Bing [3]. More generally, let $\{F(x)\}$ be a family of real functions $F(x)$ defined in an interval I, such that for given points $p_1 : (u_1, v_1)$ and $p_2 : (u_2, v_2)$, $u_1, u_2 \in I$ with $u_1 < u_2$, there is a unique member of $\{F(x)\}$ through p_1 and p_2. Functions $f(x)$ dominated by $\{F(x)\}$ are said to be convex relative to $\{F(x)\}$. In this work, we concern with one of these generalizations in the sense of Beckenbach by replacing the family of linear functions with a family of hyperbolic functions,

$$H(x) = A \cosh px + B \sinh px,$$

where A, B arbitrary constants and p is a fixed positive constant.

The class of hyperbolically convex functions is closed under addition but it is not closed under subtraction and scalar multiplication . We may consider the class of functions representable as the difference of two hyperbolically convex functions. This larger class is closed under addition, substraction and scalar multiplication, thus forming a linear space.

2. Definitions and preliminary results

In this section, we introduce the basic definitions and results which will be used later. For more informations see [1], [4], [5], [6].

Definition 2.1. *A function $f : I \to \mathbb{R}$ is said to be sub H-function on I or hyperbolically convex function, if for any arbitrary closed subinterval $[u, v]$ of I the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the function, determined by the equation $H(x) = H(x, u, v, f) = A \cosh px + B \sinh px$; $p > 0$ where A and B are chosen such that $H(u) = f(u)$, and $H(v) = f(v)$.*

Equivalently, for all $x \in [u, v]$

$$f(x) \le H(x) = \frac{f(u) \sinh p(v - x) + f(v) \sinh p(x - u)}{\sinh p(v - u)}. \tag{2.1}$$

Remark 2.1. The hyperbolically convex functions possess a number of properties analogous to those of convex functions. For example: If $f : I \to \mathbb{R}$ is hyperbolically convex function, then for any $u, v \in I$, the inequality $f(x) \ge H(x)$ holds outside the interval $[u, v]$.

Note that, from the above definitions $H(x)$ majorizes the function f(x) and $H_{ij}(x)$ denote the member satisfying $H_{ij}(x_i) = f(x_i), H_{ij}(x_j) = f(x_j), (a < x_i < x_j < b)$.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function. Then f is hyperbolically convex function on I if and only if $f''(x) - p^2 f(x) \ge o$ for all x in I.*

Definition 2.2. *Let a function $f : I \to \mathbb{R}$ be hyperbolically convex function*

$$S_u(x) = A\cosh px + B\sinh px$$

is said to be supporting function for $f(x)$ at the point $u \in (a,b)$ if

1. $S_u(u) = f(u)$
2. $S_u(x) \le f(x), \forall x \in I.$

That is, if $f(x)$ and $S_u(x)$ agree at $x = u$ the graph of $f(x)$ does not lie under the support curve.

Proposition 2.1. *If $f : I \to \mathbb{R}$ is a differentiable hyperbolically convex function, then the supporting function for $f(x)$ at the point $u \in I$ has the form*

$$S_u(x) = f(u)\cosh p(x-u) + \frac{f'(u)}{p}\sinh p(x-u).$$

Theorem 2.2. *If $f : I \to \mathbb{R}$ is differentiable function. Then f is convex if and only if f' is increasing.*

Theorem 2.3. *If $f : (a,b) \to \mathbb{R}$ and $g : (a,b) \to \mathbb{R}$ are both non-negative, increasing (decreasing), and convex, then $h(x) = f(x)g(x)$ also preserve these three properties.*

Definition 2.3. *Let $f : I \to \mathbb{R}$, f is said to satisfy a Lipschitz condition if there exists a constant $K > 0$ such that for every $x, y \in I$ we have*

$$|f(x) - f(y)| \le K|x-y|.$$

Theorem 2.4. *A function f that is F ρ-convex satisfies a Lipschitz condition in every compact subinterval J of (a,b), and thus is absolutely continuous and has a derivative almost everywhere that is bounded in J.*

Definition 2.4. *Let $f : [a,b] \to \mathbb{R}$ be a function and $\Pi = \{x_0, x_1, x_2, ..., x_n\}$ a partition of $[a,b]$. We denote $V_\Pi(f) = \Sigma_{k=0}^{n-1}|f(x_{k+1}) - f(x_k)|$ and set*

$$V_a^b(f) = \sup_{\Pi} V_\Pi(f),$$

where the supremum is taken over all partitions of $[a,b]$. A function f is said to be of bounded variation on $[a,b]$ if $V_a^b(f)$ is finite. If f is of bounded variation on $[a,b]$ we write $f \in V[a,b]$.

Theorem 2.5. *If $f : [a,b] \to \mathbb{R}$ satisfies a Lipschitz condition on $[a,b]$ with constant K, then $f \in V[a,b]$ and $V_a^b(f) \le K(b-a)$.*

3. Results

Definition 3.1. *Let $f_\alpha : I \to \mathbb{R}$ be an arbitrary family of hyperbolically convex functions majorized by f, i.e $f_\alpha \leq f$, $g(x)$ is called* ***envelope of*** *$f(x)$ if*

$$g(x) = \textit{envelopef(x)} = \sup_\alpha f_\alpha(x).$$

Lemma 3.1. *Let $f_\alpha : I \to \mathbb{R}$ be an arbitrary bounded family of hyperbolically convex functions and $f(x) = \sup_\alpha f_\alpha(x)$ then f is hyperbolically convex function.*

Proof. Let $x \in [a, b] \subseteq I$, since $f_\alpha(x)$ is hyperbolically convex function. Then,

$$\begin{aligned} f(x) &= \sup_\alpha f_\alpha(x) \\ &\leq \sup_\alpha [\frac{f_\alpha(a)\sinh p(b-x) + f_\alpha(b)\sinh p(x-a)}{\sinh p(b-a)}] \\ &\leq [\frac{\sup_\alpha f_\alpha(a)\sinh p(b-x) + \sup_\alpha f_\alpha(b)\sinh p(x-a)}{\sinh p(b-a)}]. \end{aligned}$$

Hence,

$$f(x) \leq [\frac{f(a)\sinh p(b-x) + f(b)\sinh p(x-a)}{\sinh p(b-a)}].$$

Then, f is hyperbolically convex function.

Theorem 3.1. *Let $f_\alpha : [a, b] \to \mathbb{R}$ be an arbitrary bounded family of hyperbolically convex functions majorized by f if $f_\alpha(x) \leq f(x)$ and $g(x) = \sup_\alpha f_\alpha(x)$, where the supremum is taken over all hyperbolically convex functions then*

1. *g is hyperbolically convex function.*
2. *If f is hyperbolically convex function, then $f = g$.*
3. *g is continuous function.*

Proof.

1. g is hyperbolically convex function.
It is obvious from Lemma 3.1.
2. If f is hyperbolically convex function, then $f = g$.
Since, $f(x)$ is hyperbolically convex function $f(x) \leq \sup_\alpha f_\alpha(x) = g(x)$, then

$$f(x) \leq g(x). \tag{3.1}$$

Since $f_\alpha(x) \leq f(x)$, then $f(x)$ is an upper bound.
Since $f_\alpha(x) \leq \sup_\alpha f_\alpha(x)$, then $\sup f_\alpha(x) \leq f(x)$. Thus,

$$g(x) \leq f(x). \tag{3.2}$$

Hence, from 3.1 and 3.2 we get $f(x) = g(x)$.

3. g is continuous function.

We shall show that $g(x)$ is continuous at an arbitrary x_o, with $a < x_o < b$.

Let $a < x_1 < x_o < x_2 < b$. Since $g(x)$ is hyperbolically convex function, for $x_1 < x_o - h < x_o + h < x_2$. Then,

$$H_{o2}(x_o - h) \leq g(x_o - h) \leq H_{1o}(x_o - h),$$

$$H_{1o}(x_o + h) \leq g(x_o + h) \leq H_{o2}(x_o + h), \tag{3.3}$$

$$-H_{1o}(x_o - h) \leq -g(x_o - h) \leq -H_{o2}(x_o - h), \tag{3.4}$$

Hence, from 3.3 and 3.4 we get $H_{1o}(x_o+h)-H_{1o}(x_o-h) \leq g(x_o+h)-g(x_o-h) \leq H_{o2}(x_o+h)-H_{o2}(x_o-h)$, $-|H_{1o}(x_o+h)-H_{1o}(x_o-h)| \leq H_{1o}(x_o+h)-H_{1o}(x_o-h) \leq g(x_o+h)-g(x_o-h) \leq H_{o2}(x_o+h)-H_{o2}(x_o-h) \leq |H_{o2}(x_o+h)-H_{o2}(x_o-h)|$, $-|H_{1o}(x_o+h)-H_{1o}(x_o-h)| \leq g(x_o+h)-g(x_o-h) \leq |H_{o2}(x_o+h)-H_{o2}(x_o-h)|$, $|g(x_o+h)-g(x_o-h)| \leq \max[|H_{1o}(x_o+h)-H_{1o}(x_o-h)|, |H_{o2}(x_o+h)-H_{o2}(x_o-h)|] < \epsilon$, $\forall \epsilon > o \exists \delta > o$ s.t $|h| < \delta \to |g(x_o + h) - g(x_o - h)| < \epsilon$. Hence, $g(x)$ is continuous.

Theorem 3.2. *If $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are hyperbolically convex functions and $\alpha \geq 0$ then $f + g$ and αf are hyperbolically convex functions.*

Proof. Since f, g are hyperbolically convex function, then $\forall\ x \in [u, v] \subseteq I$,

$$\begin{aligned} f(x) &\leq \frac{f(u)\sinh p(v-x) + f(v)\sinh p(x-u)}{\sinh p(v-u)}, \\ g(x) &\leq \frac{g(u)\sinh p(v-x) + g(v)\sinh p(x-u)}{\sinh p(v-u)}. \end{aligned}$$

Then,

$$\begin{aligned} f(x) + g(x) &\leq \frac{f(u)\sinh p(v-x) + f(v)\sinh p(x-u)}{\sinh p(v-u)} \\ &+ \frac{g(u)\sinh p(v-x) + g(v)\sinh p(x-u)}{\sinh p(v-u)}, \\ f(x) + g(x) &\leq \frac{(f(u)+g(u))\sinh p(v-x) + (f(v)+g(v))\sinh p(x-u)}{\sinh p(v-u)}. \end{aligned}$$

Thus,

$$(f+g)(x) \leq \frac{(f+g)(u))\sinh p(v-x) + (f+g)(v)\sinh p(x-u)}{\sinh p(v-u)}.$$

Hence, $f + g$ is hyperbolically convex function.

$$\text{Since, } f(x) \leq \frac{f(u)\sinh p(v-x) + f(v)\sinh p(x-u)}{\sinh p(v-u)} \qquad \textit{and} \qquad \alpha \geq o$$

then

$$\alpha f(x) \leq \frac{\alpha f(u)\sinh p(v-x) + \alpha f(v)\sinh p(x-u)}{\sinh p(v-u)}.$$

Hence, αf is hyperbolically convex function.

Remark 3.1. If $\alpha < 0$ then αf is not hyperbolically convex function. For example if $f : [-1,1] \to \mathbb{R}$, $f(x) = x^2$ is hyperbolically convex function. As $2 - x^2 \geq 0$ from theorem 2.1, but $-x^2$ is not hyperbolically convex function. While αf is hyperbolically convex function either for $f(x) = \cosh px$ or $f(x) = \sinh px$ for any $\alpha \in \mathbb{R}$, as shown in the following lemma.

Lemma 3.2. *For any real number m, $m\sinh px$ and $m\cosh px$ are hyperbolically convex functions.*

Proof. Let $x \in [u,v]$, Since

$$\sinh px = \frac{\sinh pu \sinh p(v-x) + \sinh pv \sinh p(x-u)}{\sinh p(v-u)}.$$

Then, $m\sinh px$ is hyperbolically convex function. Similarly, $m\cosh px$.

Theorem 3.3. *Assume that $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ where $range(f) \subseteq J$, let f and g are both non-negative, hyperbolically convex function, two times continuously differentiable functions and g is increasing. If $f'(x) \geq 1$, then the composite function gof is hyperbolically convex function on I.*

Proof. Since g is increasing, then

$$g'(x) \geq o, \qquad \forall x \in J. \tag{3.5}$$

Since f and g are hyperbolically convex functions, then using theorem 2.1,

$$\begin{aligned} &\text{we have} & f''(x) - p^2 f(x) &\geq 0, & \forall x \in I, \qquad (3.6)\\ &\text{and} & g''(x) - p^2 g(x) &\geq 0, & \forall x \in J. \qquad (3.7)\end{aligned}$$

Since,

$$\begin{aligned} h(x) &= g(f(x)) \\ h'(x) &= g'(f(x)).f'(x) \\ h''(x) &= g''(f(x)).(f'(x))^2 + g'(f(x)).f''(x). \end{aligned}$$

Then,

$$\begin{aligned} h''(x) - p^2 h(x) &= g''(f(x)).(f'(x))^2 + g'(f(x)).f''(x) - p^2 g(f(x)) \\ &= (f'(x))^2[g''(f(x) - \frac{p^2}{(f'(x))^2} g(f(x))] + g'(f(x)).f''(x). \end{aligned}$$

Now using 3.5, 3.6, 3.7 and $f'(x) \geq 1$ we conclude that

$$h''(x) - p^2 h(x) \geq 0.$$

Hence, $h(x)$ is hyperbolically convex function.

Theorem 3.4. *If $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are both non-negative, increasing, two times continuously differentiable functions and hyperbolically convex functions then $k(x) = f(x)g(x)$ also exhibits theses three properties.*

Proof. Since f and g are non-negative and increasing functions, then

$$f(x) \geq 0, \ g(x) \geq 0, \tag{3.8}$$

and

$$f'(x) \geq 0, \ g'(x) \geq 0. \tag{3.9}$$

Since f and g are hyperbolically convex functions, then using theorem 2.1,

$$\text{we have} \quad f''(x) - p^2 f(x) \geq 0, \quad \forall x \in I, \tag{3.10}$$

$$\text{and} \quad g''(x) - p^2 g(x) \geq 0, \quad \forall x \in I. \tag{3.11}$$

Since,

$$\begin{aligned} k(x) &= f(x).g(x) \\ k'(x) &= f'(x).g(x) + f(x).g'(x) \\ k''(x) &= f''(x).g(x) + 2f'(x).g'(x) + f(x).g''(x). \end{aligned}$$

Then,

$$\begin{aligned} k''(x) - p^2 k(x) &= f''(x).g(x) + 2f'(x).g'(x) + f(x).g''(x) - p^2 f(x).g(x) \\ &= (f''(x) - p^2 f(x)).g(x) + 2f'(x).g'(x) + f(x).g''(x). \end{aligned}$$

Now using 3.8, 3.9, 3.10 and 3.11, we conclude that

$$k''(x) - p^2 k(x) \geq 0.$$

Hence, $k(x)$ is hyperbolically convex function.

Theorem 3.5. *If (f_n) is a real valued sequence of hyperbolically convex functions converging to a finite limit function f on I, then f is hyperbolically convex function.*

Proof. Let $x \in [a, b] \subseteq I$

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) \\ &\leq \lim_{n\to\infty} \frac{f_n(a)\sinh p(b-x) + f_n(b)\sinh p(x-a)}{\sinh p(b-a)} \\ &= \frac{\lim_{n\to\infty} f_n(a)\sinh p(b-x) + \lim_{n\to\infty} f_n(b)\sinh p(x-a)}{\sinh p(b-a)} \\ &= \frac{f(a)\sinh p(b-x) + f(b)\sinh p(x-a)}{\sinh p(b-a)}. \end{aligned}$$

It follows that f is hyperbolically convex function.

Definition 3.2. *Let $BH[a,b]$ be the class of functions $f : [a,b] \to R$ representable in the form $f = g - h$ where g and h are hyperbolically convex functions on $[a,b]$ and $g'_+(a), g'_-(b), h'_+(a), h'_-(b)$ are all finite.*

Theorem 3.6. *$BH[a,b]$ is closed under addition, subtraction and scalar multiplication.*

Proof. Let $f, g \in BH[a,b]$, $f = f_1 - f_2$, $g = g_1 - g_2$. For addition: $f + g = f_1 - f_2 + g_1 - g_2 = (f_1 + g_1) - (f_2 + g_2)$. Since $f_1 + g_1, f_2 + g_2$ are hyperbolically convex functions, by theorem 3.2, then $f + g \in BH[a,b]$. For subtraction: $f - g = f_1 - f_2 - g_1 + g_2 = (f_1 + g_2) - (f_2 + g_1)$. Since $f_1 + g_2, f_2 + g_1$ are hyperbolically convex functions, by theorem3.2, then $f - g \in BH[a,b]$.

For scalar multirlication:

Case(1): let $\alpha \geq o$, then $\alpha f = \alpha f_1 - \alpha f_2$.

Since $\alpha f_1, \alpha f_2$ are hyperbolically convex functions, by theorem3.2, then $\alpha f \in BH[a,b]$. Case(2): let $\alpha < o$, then $\alpha f = \alpha f_1 - \alpha f_2 = -\alpha f_2 - (-\alpha) f_1$.

Since $-\alpha f_2, -\alpha f_1$ are hyperbolically convex functions, by theorem3.2, then $\alpha f \in BH[a,b]$.

Finally, it is clear that all the previous functions have finite endpoint derivatives.

Corollary 3.1. *BH[a,b] is a linear space.*

Theorem 3.7. *If $f \in BH[a,b]$, then f satisfies lipschitz condition and consequently absolutely continuous on $[a,b]$.*

Proof. Let $f \in BH[a,b], f = f_1 - f_2$. Since f_1 and f_2 are hyperbolically convex functions. Then, from theorem 2.4, f_1, f_2 satisfies lipschitz condition,

$$|f_1(x) - f_1(y)| \leq k|x - y| \qquad \text{and} \qquad |f_2(x) - f_2(y)| \leq m|x - y|.$$

$$\begin{aligned}
f(x) - f(y) &= f_1(x) - f_2(x) - (f_1(y) - f_2(y)) \\
&= f_1(x) - f_1(y) - (f_2(x) - f_2(y)). \\
|f(x) - f(y)| &= |f_1(x) - f_1(y) - (f_2(x) - f_2(y))| \\
&\leq |f_1(x) - f_1(y)| + |-(f_2(x) - f_2(y)| \\
&\leq |f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| \\
&\leq k|x - y| + m|x - y| \\
&= (k + m)|x - y| \\
&= h|x - y|.
\end{aligned}$$

Then

$$|f(x) - f(y)| \leq h|x - y|, \forall x, y \in [a,b].$$

Hence, f satisfies lipschitz condition.

Let $\epsilon > o$, choose $\delta = \frac{\epsilon}{h}$ such that for any collection $\{(x_i, y_i) : i = 1, 2, ..., n\}$ of disjoint open subinterval of $[a, b]$ with $\sum_1^n |x_i - y_i| < \delta$,

$$then \sum_1^n |f(x_i) - f(y_i)| < \sum_1^n h|x_i - y_i| = h \sum_1^n |x_i - y_i| < h\frac{\epsilon}{h} = \epsilon.$$

Hence, f is absolutely continuous on $[a, b]$.

Corollary 3.2. *If $f \in BH[a, b]$, then $V_a^b(f) < \infty$.*

Proof. Let $f \in BH[a, b]$ then, from theorem 3.7, f satisfies lipschitz condition. then, from theorem 2.5, $f \in V[a, b]$. Hence, $V_a^b(f) < \infty$.

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