A new criterion of optimization of the cross multipole coefficients in a modified surface stress operator for the elastic two-dimensional case

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Abstract. The question of non-uniqueness in the integral formulation of an exterior boundary value problem in the elastic two-dimensional case has been resolved using the modified' Green's function technique. In this work, a new criterion of optimality based on the minimization of the norm of the surface stress operator (modified traction operator) is established.

Keywords: cross multipole coefficients, modified traction operator, Green's function, integral equations, linear elasticity.

1. Introduction

The question of non-uniqueness in the integral formulation of an exterior boundary value problem in the elastic two-dimensional case has been resolved using the modified' Green's function technique, where the simple and cross multipole coefficients must satisfy some suitable and mild conditions (2.5) [3]. Some criteria to determine an optimal choice for these multipole coefficients are developed recently; the first criterion is based on the minimization of the norm of the modified integral operator [2, 8], and motivated by enlarging the radius of convergence of the numerical method used (successive approximations). The second criterion is based on the minimization of the norm of the modified Green's function [9], and motivated by the minimization of the norm of the difference between the modified and exact Green's function. In [1], Argyropoulos et al.

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have presented another criterion based on the minimization of the condition number of the boundary integral equations describing the problem. In [6], we have developed a new criterion based on the minimization of the norm of the surface stress operator, or the norm of the modified traction operator using simple multipole coefficients, motivated this time by the minimization of the norm of the difference between the modified and exact kernel of the integral operator. In this work we generalize the same criterion but for the case of cross multipole coefficients.

2. Formulation of the problem

An exterior Neumann boundary value problem in two-dimensional elastic case can be described through a boundary integral equation of the form [3]:

$$\left(\frac{1}{2}I + \bar{K}_0^*\right)(\varphi)(p) = f(p), \; p \in \partial D, \tag{2.1}$$

where f is a Holder continuous density, and the integral operator K_0 is defined as :

$$(K_0\varphi)(p) = \frac{1}{2\pi}\int\limits_{\partial D} T_p G_0(p,\; q)\;\varphi(q)\;ds_q,\; p \in \partial D, \tag{2.2}$$

G_0 is the Green's function (fundamental solution), and T is the surface stress operator.

Using the modified Green's function technique, by introduce a regular solution [3], the modified Green's function is written as:

$$\begin{aligned} &G_1(p,\; Q) \\ &= \frac{i}{4\mu K^2}\sum_{m=0}^{+\infty}\sum_{\sigma=1}^{2}\sum_{l=1}^{2}\left[\begin{array}{c} F_m^{\sigma l}(P)\otimes\hat{F}_m^{\sigma l}(Q) + a_m^{\sigma l}\; F_m^{\sigma l}(P)\otimes F_m^{\sigma l}(Q) \\ +(-1)^{\sigma+l}\, b_m\; F_m^{\sigma l}(P)\otimes F_m^{(3-\sigma)(3-l)}(Q)\end{array}\right]\end{aligned} \tag{2.3}$$

where

$$\begin{aligned} F_m^{\sigma 1}(P) &= \operatorname{grad}\left(H_m^1(kr_p)\;E_m^\sigma(\theta_p)\right), \\ F_m^{\sigma 2}(P) &= rot\left(H_m^1(Kr_p)\;E_m^\sigma(\theta_p)\;\hat{e}_3\right), \end{aligned} \tag{2.4}$$

$\hat{F}_m^{\sigma l}$ are obtained by changing the function of Hankel H_m^1 of the vector Hankel functions into the function of Bessel J_m^1 [3], and

$$E_m^\sigma(\theta_p) = \sqrt{\varepsilon_m}\begin{cases}\cos(m\theta_p),\; \sigma=1\\ \sin(m\theta_p),\; \sigma=2\end{cases}, \text{ with } \varepsilon_m = \begin{cases}1,\; m=0,\\ 2,\; m>0,\end{cases}$$

$a_m^{\sigma l}$ and b_m are the simple and cross multipole coefficients, which must satisfy the following conditions:

$$\bar{b}_m\left(a_m^{\sigma 1}+\frac{1}{2}\right) + b_m\left(\bar{a}_m^{\sigma 2}+\frac{1}{2}\right) = 0, \tag{2.5}$$

and

$$\left|a_m^{\sigma l} + \frac{1}{2}\right|^2 + |b_m|^2 - \frac{1}{4} < 0,\ \forall m = 0 : \infty \text{ and } \forall \sigma,\ l = 1 : 2.$$

3. Main results

3.1 General case

We now consider the question of how to choose the simple and cross multipole coefficients $a_m^{\sigma l}$ and b_m in the modification (2.3) so to minimize $\|T_p G_1\|_{L_2(\partial D)}$. The question is answered by the following theorem.

Theorem 1. *There are uniquely defined simple and cross multipole coefficients $a_m^{\sigma l}$ and b_m which minimize the quantity:*

$$\int\limits_{r_p=A} \|T_p G_1\|^2_{L_2(\partial D)}\ ds_p,\ \forall A \geq \max r_q,\ q \in \partial D. \tag{3.1}$$

These simple and cross multipole coefficients are given by the relations:

$$a_m^{\sigma l} = \frac{\bar{B}_m^{\sigma l} M_{m,1}^{\sigma l} + \bar{\beta}_m^{\sigma l} M_{m,2}^{\sigma l} - A_m^{(3-\sigma)(3-l)} N_{m,1}^{\sigma l} - \alpha_m^{(3-\sigma)(3-l)} N_{m,2}^{\sigma l}}{\Delta_{m,\partial D}^{\sigma}}, \tag{3.2}$$

and

$$(-1)^{\sigma+l}\, b_m = \frac{B_m^{\sigma l} N_{m,1}^{\sigma l} + \beta_m^{\sigma l} N_{m,2}^{\sigma l} - A_m^{\sigma l} M_{m,1}^{\sigma l} - \alpha_m^{\sigma l} M_{m,2}^{\sigma l}}{\Delta_{m,\partial D}^{\sigma}}, \tag{3.3}$$

with

$$M_{m,1}^{\sigma l} = \Delta_{m,A}^{\sigma}\left[B_m^{(3-\sigma)(3-l)} g_m^{(3-\sigma)(3-l)} - A_m^{(3-\sigma)(3-l)} h_m^{\sigma l}\right], \tag{3.4}$$

$$M_{m,2}^{\sigma l} = \Delta_{m,\partial D}^{\sigma}\left[\beta_m^{(3-\sigma)(3-l)} g_m^{(3-\sigma)(3-l)} - \alpha_m^{(3-\sigma)(3-l)} h_m^{\sigma l}\right], \tag{3.5}$$

$$N_{m,1}^{\sigma l} = \Delta_{m,A}^{\sigma}\left[B_m^{(3-\sigma)(3-l)} h_m^{(3-\sigma)(3-l)} - A_m^{(3-\sigma)(3-l)} g_m^{\sigma l}\right], \tag{3.6}$$

$$N_{m,2}^{\sigma l} = \Delta_{m,\partial D}^{\sigma}\left[\beta_m^{(3-\sigma)(3-l)} h_m^{(3-\sigma)(3-l)} - \alpha_m^{(3-\sigma)(3-l)} g_m^{\sigma l}\right], \tag{3.7}$$

$$g_m^{\sigma l} = -\left\langle \bar{\beta}_m^{\sigma l} T \hat{F}_m^{(3-\sigma)(3-l)} + \alpha_m^{\sigma l} T\ \hat{F}_m^{\sigma l},\ T F_m^{\sigma l}\right\rangle, \tag{3.8}$$

$$h_m^{\sigma l} = -\left\langle \bar{\beta}_m^{\sigma l} T \hat{F}_m^{(3-\sigma)(3-l)} + \alpha_m^{\sigma l} T\ \hat{F}_m^{\sigma l},\ T F_m^{(3-\sigma)(3-l)}\right\rangle, \tag{3.9}$$

$$\alpha_m^{\sigma l} = \left\|T F_m^{\sigma l}\right\|_A^2,\ \beta_m^{\sigma l} = -\left\langle T F_m^{\sigma l},\ T F_m^{(3-\sigma)(3-l)}\right\rangle_A, \tag{3.10}$$

$$A_m^{\sigma l} = \left\|T F_m^{\sigma l}\right\|_{\partial D}^2,\ B_m^{\sigma l} = -\left\langle T F_m^{\sigma l},\ T F_m^{(3-\sigma)(3-l)}\right\rangle_{\partial D}, \tag{3.11}$$

$$\Delta_{m,A}^{\sigma} = \left(\alpha_m^{\sigma 1} A_m^{\sigma 1} \alpha_m^{(3-\sigma)2} A_m^{(3-\sigma)2} - \beta_m^{\sigma 1} B_m^{\sigma 1} \bar{\beta}_m^{\sigma l} \bar{B}_m^{\sigma l}\right)_A, \tag{3.12}$$

$$\Delta_{m,\partial D}^{\sigma} = \left(\alpha_m^{\sigma 1} A_m^{\sigma 1} \alpha_m^{(3-\sigma)2} A_m^{(3-\sigma)2} - \beta_m^{\sigma 1} B_m^{\sigma 1} \bar{\beta}_m^{\sigma l} \bar{B}_m^{\sigma l}\right)_{\partial D}, \tag{3.13}$$

where $\langle .,\ .\rangle_{\partial D}$ *and* $\langle .,\ .\rangle_A$ *are the inner product on the boundary* ∂D *and on a circle of radius* A *respectively.*

Proof. Step 1:

We have

$$\int\limits_{r_p=A} \|T_pG_1\|^2_{L_2(\partial D)}\, ds_p = \int\limits_{r_p=A}\int\limits_{\partial D} T_pG_1\,(P,\ q) : \overline{T_pG_1}\,(q,\ P)\, ds_p ds_q \tag{3.14}$$

$$= \sum_{m=0}^{+\infty}\sum_{\sigma+1}^{2} \alpha_m^{\sigma 1}\left[\begin{array}{c} \|\hat{F}_m^{\sigma 1}\|^2_{\partial D} + \bar{a}_m^{\sigma 1}\langle \hat{F}_m^{\sigma 1},\ F_m^{\sigma 1}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{b}_m\langle \hat{F}_m^{\sigma 1},\ F_m^{(3-\sigma)2}\rangle_{\partial D} \\ +a_m^{\sigma 1}\langle F_m^{\sigma 1},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} + a_m^{\sigma 1}\bar{a}_m^{\sigma 1}A_m^{\sigma 1} \\ -(-1)^{\sigma}a_m^{\sigma 1}\bar{b}_m B_m^{\sigma 1} - (-1)^{\sigma}b_m\langle F_m^{(3-\sigma)2},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{a}_m^{\sigma 1}b_m\bar{B}_m^{\sigma 1} + b_m\bar{b}_m A_m^{(3-\sigma)2} \end{array}\right]$$

$$+\ \beta_m^{\sigma 1}\left[\begin{array}{c} \langle \hat{F}_m^{\sigma 1},\ \hat{F}_m^{(3-\sigma)2}\rangle_{\partial D} + \bar{a}_m^{(3-\sigma)2}\langle \hat{F}_m^{\sigma 1},\ F_m^{(3-\sigma)2}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{b}_m\langle \hat{F}_m^{\sigma 1},\ F_m^{\sigma 1}\rangle_{\partial D} + a_m^{\sigma 1}\langle F_m^{\sigma 1},\ \hat{F}_m^{(3-\sigma)2}\rangle_{\partial D} \\ +a_m^{\sigma 1}\bar{a}_m^{(3-\sigma)2}B_m^{\sigma 1} - (-1)^{\sigma}a_m^{\sigma 1}\bar{b}_m A_m^{\sigma 1} \\ -(-1)^{\sigma}b_m\langle F_m^{(3-\sigma)2},\ \hat{F}_m^{(3-\sigma)2}\rangle_{\partial D} - (-1)^{\sigma}\bar{a}_m^{(3-\sigma)2}b_m A_m^{(3-\sigma)2} \\ +b_m\bar{b}_m\bar{B}_m^{\sigma 1} \end{array}\right]$$

$$+\ \overline{\beta}_m^{\sigma 1}\left[\begin{array}{c} \langle\ \hat{F}_m^{(3-\sigma)2},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} + a_m^{(3-\sigma)2}\langle F_m^{(3-\sigma)2},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{b}_m\langle \hat{F}_m^{(3-\sigma)2},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} + a_m^{(3-\sigma)2}\bar{a}_m^{\sigma 1}\bar{B}_m^{\sigma 1} \\ -(-1)^{\sigma}a_m^{(3-\sigma)2}\bar{b}_m A_m^{(3-\sigma)2} - (-1)^{\sigma}b_m\langle F_m^{\sigma 1},\ \hat{F}_m^{\sigma 1}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{a}_m^{\sigma 1}b_m A_m^{\sigma 1} + b_m\bar{b}_m B_m^{\sigma 1} \end{array}\right]$$

$$= \alpha_m^{(3-\sigma)2}\left[\begin{array}{c} \|\hat{F}_m^{(3-\sigma)2}\|^2_{\partial D} + \bar{a}_m^{(3-\sigma)2}\langle \hat{F}_m^{(3-\sigma)2},\ F_m^{(3-\sigma)2}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{b}_m\langle \hat{F}_m^{(3-\sigma)2},\ F_m^{\sigma 1}\rangle_{\partial D} + a_m^{(3-\sigma)2}\bar{a}_m^{(3-\sigma)2}A_m^{(3-\sigma)2} \\ -(-1)^{\sigma}a_m^{(3-\sigma)2}\bar{b}_m B_m^{\sigma 1} - (-1)^{\sigma}b_m\langle F_m^{\sigma 1},\ \hat{F}_m^{(3-\sigma)2}\rangle_{\partial D} \\ -(-1)^{\sigma}\bar{a}_m^{(3-\sigma)2}b_m B_m^{\sigma 1} + b_m\bar{b}_m A_m^{\sigma 1} \end{array}\right].$$

A necessary condition for the existence of a minimum of (3.14) is the vanishing of the gradient with respect to the simple and cross multipole coefficients $\bar{a}_m^{\sigma 1},\ \bar{a}_m^{(3-\sigma)2}$ and b_m. So, we obtain the following relations:

$$\alpha_m^{\sigma 1}A_m^{\sigma 1}a_m^{\sigma 1} + \bar{\beta}_m^{\sigma 1}\bar{B}_m^{\sigma 1}a_m^{(3-\sigma)2} - (-1)^{\sigma}(\alpha_m^{\sigma 1}\bar{B}_m^{\sigma 1} + A_m^{\sigma 1}\bar{\beta}_m^{\sigma 1})b_m = g_m^{\sigma 1}, \tag{3.15}$$

$$\begin{aligned} &\beta_m^{\sigma 1}B_m^{\sigma 1}a_m^{\sigma 1} + \alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}a_m^{(3-\sigma)2} \\ &- (-1)^{\sigma}(\alpha_m^{(3-\sigma)2}B_m^{\sigma 1} + A_m^{(3-\sigma)2}\beta_m^{\sigma 1})b_m = g_m^{(3-\sigma)2}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} &- (-1)^{\sigma}(B_m^{\sigma 1}\alpha_m^{\sigma 1} + A_m^{\sigma 1}\beta_m^{\sigma 1})a_m^{\sigma 1} \\ &- (-1)^{\sigma}(A_m^{(3-\sigma)2}\bar{\beta}_m^{\sigma 1} + \bar{B}_m^{\sigma 1}\alpha_m^{(3-\sigma)2})a_m^{(3-\sigma)2} \\ &+ (A_m^{(3-\sigma)2}\alpha_m^{\sigma 1} + \bar{B}_m^{\sigma 1}\beta_m^{\sigma 1} + B_m^{\sigma 1}\bar{\beta}_m^{\sigma 1} + A_m^{\sigma 1}\alpha_m^{(3-\sigma)2})b_m \\ &= -(-1)^{\sigma}(h_m^{\sigma 1} + h_m^{(3-\sigma)2}). \end{aligned} \tag{3.17}$$

Exploiting the Schwartz inequality and linear independence of the multipole vectors $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ and $\left\{TF_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ we can show that the determinant of the system (3.15 – 3.17) is greater than zero:

$$\Delta_m^\sigma = \left[\begin{array}{c} A_m^{\sigma 1}\alpha_m^{(3-\sigma)2} + \alpha_m^{\sigma 1} A_m^{(3-\sigma)2} \\ -\left(B_m^{\sigma l}\bar{\beta}_m^{\sigma l} + \beta_m^{\sigma l}\bar{B}_m^{\sigma 1}\right) \end{array}\right]$$

$$+\left(A_m^{\sigma 1}A_m^{(3-\sigma)2} - \left|B_m^{\sigma l}\right|^2\right)\left(\alpha_m^{\sigma 1}\alpha_m^{(3-\sigma)2} - \left|\beta_m^{\sigma l}\right|^2\right) > 0. \tag{3.18}$$

A fact that ensures this system has a unique solution. Its solution is given by the relations (3.2) and (3.3).

Step 2:

In order to prove that these simple and cross multipole coefficients really minimize the quantity given by (3.1), we have to show that the sufficient conditions for a minimum are also satisfied. So, we assume that the integrand in (3.1) is a function of the variables $x_m^{\sigma 1}$, $y_m^{\sigma 1}$, x_m, $x_m^{(3-\sigma)2}$, $y_m^{(3-\sigma)2}$ and y_m , where:

$$\alpha_m^{\sigma 1} = x_m^{\sigma 1} + iy_m^{\sigma 1},\ a_m^{(3-\sigma)2} = x_m^{(3-\sigma)2} + iy_m^{(3-\sigma)2} \text{ and } b_m = x_m + iy_m. \tag{3.19}$$

Then we see that the optimal choice of the simple and cross multipole coefficients defined in (3.2) and (3.3), and which ensure that first derivatives vanish, also make the determinants of the second derivatives greater than zero:

$$\left| Hessien\left[\int\limits_{r_p=A} \left\|T_pG_1\right\|_{L_2(\partial D)}^2 ds_p\right]\right| = \tag{3.20}$$

$$\left(16\alpha_m^{\sigma 1}A_m^{\sigma 1}\alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}\right)\Delta_m^\sigma + 16\left(\mathrm{Re}^4\left(\beta_m^{\sigma 1}B_m^{\sigma 1}\right) + \mathrm{Im}^4\left(\beta_m^{\sigma 1}B_m^{\sigma 1}\right)\right),$$

$$|H_{11}| = |H_{44}| = \left(16\alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}\right)\Delta_m^\sigma > 0, \tag{3.21}$$

$$|H_{22}| = |H_{55}| = \left(8\alpha_m^{(3-\sigma)2}A_m^{(3-\sigma)2}\right)\Delta_m^\sigma > 0, \tag{3.22}$$

$$|H_{33}| = |H_{66}| = \left(16\alpha_m^{\sigma 1}A_m^{\sigma 1}\right)\Delta_m^\sigma > 0. \tag{3.23}$$

So the required sufficient conditions can be satisfied. □

3.2 Circular case

As shown in [4, 5], the lack of orthogonality between the multipole vectors $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ and $\left\{TF_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ does not permit a tractable analytical approach to show that the choice of the simple and cross multipole coefficients given in (3.2) and (3.3) satisfy the conditions (2.5), so ensuring the unique solvability of the modified integral equation given by (2.1). so we treat the problem by considering only the special circular case as an indication of the general case.

Lemma 1. *If the boundary ∂D is a circle of radius , then the simple and cross multipole coefficients of theorem 3.1.1 are given by the relations:*

$$a_m^{11} = a_m^{21} = -\frac{1}{2}\left[\frac{\hat{\alpha}_m^1 \alpha_m^2 - \bar{\beta}_m \hat{\beta}_m}{\Delta_m'}\right], \tag{3.24}$$

$$a_m^{12} = a_m^{22} = -\frac{1}{2}\left[\frac{\alpha_m^1 \hat{\alpha}_m^2 - \beta_m \hat{\varphi}_m}{\Delta_m'}\right], \tag{3.25}$$

$$b_m = \frac{1}{2}\left[\frac{\hat{\alpha}_m^1 \beta_m - \alpha_m^1 \hat{\beta}_m}{\bar{\Delta}_m}\right] = \frac{1}{2}\left[\frac{\bar{\beta}_m \hat{\alpha}_m^2 - \alpha_m^2 \hat{\varphi}_m}{\Delta_m'}\right], \tag{3.26}$$

where

$$a_m^1 = 2\pi a\left[\begin{array}{c} \left|k^2\left(2\mu H_m''(ka) - \lambda H_m(ka)\right)\right|^2 \\ +\left|\frac{2\mu m}{a}\left(kH_m'(ka) - \frac{H_m(ka)}{a}\right)\right|^2 \end{array}\right], \tag{3.27}$$

$$\hat{a}_m^1 = 2\pi a\left[\begin{array}{c} k^4\left(2\mu J_m''(ka) - \lambda J_m(ka)\right)\left(2\mu \bar{H}_m''(ka) - \lambda \bar{H}_m(ka)\right) \\ +\left(\frac{2\mu m}{a}\right)^2\left(k\bar{H}_m'(ka) - \frac{\bar{H}_m(ka)}{a}\right)\left(kJ_m'(ka) - \frac{J_m(ka)}{a}\right) \end{array}\right], \tag{3.28}$$

$$\alpha_m^2 = 2\pi a\left[\begin{array}{c} \left|\mu k^2\left(2\mu H_m''(Ka) + H_m(Ka)\right)\right|^2 \\ +\left|\frac{2\mu m}{a}\left(KH_m'(ka) - \frac{H_m(Ka)}{a}\right)\right| \end{array}\right], \tag{3.29}$$

$$\hat{\alpha}_m^2 = 2\pi a\left[\begin{array}{c} \left(\mu K^2\right)^2\left(2J_m''(Ka) + J_m(Ka)\right)\left(2\mu \bar{H}_m''(ka) + \bar{H}_m(ka)\right) \\ +\left(\frac{2\mu m}{a}\right)^2\left(KJ_m'(Ka) - \frac{J_m(Ka)}{a}\right)\left(K\ \bar{H}_m'(ka) - \frac{\bar{H}_m(Ka)}{a}\right) \end{array}\right], \tag{3.30}$$

$$\beta_m = 2\pi\mu m\left[\begin{array}{c} k^2\left(2\mu H_m''(ka) - \lambda H_m(ka)\right)\left(K\bar{H}_m'(Ka) - \frac{\bar{H}_m(Ka)}{a}\right) \\ +\mu K^2\left(kH_m'(ka) - \frac{H_m(ka)}{a}\right)\left(2\bar{H}_m''(Ka) + \bar{H}_m(Ka)\right) \end{array}\right], \tag{3.31}$$

(3.32)
$$\hat{\beta}_m = 4\pi\mu m \begin{bmatrix} k^2\left(2\mu J_m''(ka) - \lambda J_m(ka)\right)\left(K\bar{H}_m'(Ka) - \dfrac{\bar{H}_m(Ka)}{a}\right) \\ +\mu K^2\left(kJ_m'(ka) - \dfrac{J_m(ka)}{a}\right)\left(2\bar{H}_m''(Ka) + \bar{H}_m(Ka)\right) \end{bmatrix},$$

(3.33)
$$\hat{\varphi}_m = 4\pi\mu m \begin{bmatrix} k^2\left(2\mu \bar{H}_m''(ka) - \lambda \bar{H}_m(ka)\right)\left(KJ_m'(Ka) - \dfrac{J_m(Ka)}{a}\right) \\ +\mu K^2\left(kJ_m'(ka) - \dfrac{\bar{H}_m(ka)}{a}\right)\left(2J_m''(Ka) + J_m(Ka)\right) \end{bmatrix},$$

(3.34)
$$\Delta_m = (2\pi\mu)^2 \begin{bmatrix} \mu k^2K^2\left(2\mu H_m''(ka) - \lambda H_m(ka)\right)\left(2H_m''(Ka) + H_m(Ka)\right) \\ -\left(\dfrac{2\mu m}{a}\right)^2\left(kH_m'(ka) - \dfrac{H_m(ka)}{a}\right)\left(KH_m'(Ka) - \dfrac{H_m(Ka)}{a}\right) \end{bmatrix}.$$

Proof. Using the well-known relations which hold for the inner product of the multipole vectors $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ and $\left\{TF_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ on the circle of radius a we can obtain, after easy calculations [3, 7, 10]:

(3.35)
$$\begin{aligned} g_m^{\sigma 1} &= -\bar{\beta}_m^{\sigma 1}\left\langle T\hat{F}_m^{(3-\sigma)(3-l)},\ TF_m^{\sigma l}\right\rangle_a - \alpha_m^{\sigma 1}\left\langle T\hat{F}_m^{\sigma l},\ TF_m^{\sigma l}\right\rangle_a \\ &= -\left\langle TF_m^{(3-\sigma)(3-l)},\ TF_m^{\sigma l}\right\rangle_a\left\langle T\hat{F}_m^{(3-\sigma)(3-l)},\ TF_m^{\sigma l}\right\rangle_a \\ &\quad - \left\|TF_m^{\sigma l}\right\|_a^2\left\langle T\hat{F}_m^{\sigma l},\ TF_m^{\sigma l}\right\rangle_a. \end{aligned}$$

So

(3.36)
$$g_m^{\sigma 1} = -\left(-(-1)^\sigma \bar{\beta}_m\right)\left(-(-1)^\sigma \hat{\varphi}_m\right) - \alpha_m^1\hat{\alpha}_m^1,$$

(3.37)
$$g_m^{(3-\sigma)2} = -\left(-(-1)^\sigma \beta_m\right)\left(-(-1)^\sigma \hat{\beta}_m\right) - \alpha_m^2\hat{\alpha}_m^2,$$

and

(3.38)
$$\begin{aligned} h_m^{\sigma 1} &= -\bar{\beta}_m^{\sigma 1}\langle T\hat{F}_m^{(3-\sigma)(3-l)}, TF_m^{(3-\sigma)(3-l)}\rangle_a - \alpha_m^{\sigma 1}\langle T\hat{F}_m^{\sigma l}, TF_m^{(3-\sigma)(3-l)}\rangle_a \\ &= -\langle TF_m^{(3-\sigma)(3-l)},\ TF_m^{\sigma l}\rangle_a\langle T\hat{F}_m^{(3-\sigma)(3-l)},\ T_m^{(3-\sigma)(3-l)}F\rangle_a \\ &\quad - \|TF_m^{\sigma l}\|_a^2\langle T\hat{F}_m^{\sigma l},\ TF_m^{(3-\sigma)(3-l)}\rangle_a. \end{aligned}$$

So

(3.39)
$$h_m^{\sigma 1} = -\left(-(-1)^\sigma \bar{\beta}_m\hat{\alpha}_m^2\right) - \alpha_m^1\left(-(-1)^\sigma \hat{\beta}_m\right),$$

(3.40)
$$h_m^{(3-\sigma)2} = -\left(-(-1)^\sigma \beta_m\hat{\alpha}_m^1\right) - \alpha_m^2\left(-(-1)^\sigma \hat{\varphi}_m\right).$$

From (3.38) to (3.40) we obtain (3.24) to (3.26).

Note here, that we have $a_m^{11} = a_m^{21}$ and $a_m^{12} = a_m^{22}$, and we can show [10] that the two ewpressions found for b_m (3.26) which appear to be different, are equal. □

Theorem 2. *If the boundary ∂D is a circle of radius , then the optimal choice of the simple and cross multipole coefficients given by (3.24) to (3.26) yield the exact Green's function for the Neumann problem, i.e.:*

$$G_1^N(p,\ q) = G_{exact}^N(p,\ q)\,. \tag{3.41}$$

Proof. As it was proved in [10], the exact Green's function for the exterior Neumann problem for the circle of radius is given by the relation:

$$\begin{aligned} G_{exact}^N(p,\ q) &= G_0(p,\ q) + \\ &\frac{i}{4\mu K^2}\sum_{m=0}^{+\infty}\left[\begin{array}{l} \left(\dfrac{\hat{\alpha}_m^1\alpha_m^2 - \bar{\beta}_m\hat{\beta}_m}{\Delta_m'}\right)F_m^{11}(p)\otimes F_m^{11}(q) + \\ \left(\dfrac{\hat{\alpha}_m^1\beta_m - \alpha_m^1\hat{\beta}_m}{\Delta_m'}\right)F_m^{11}(p)\otimes F_m^{22}(q) \\ +\left(\dfrac{\alpha_m^1\hat{\alpha}_m^2 - \beta_m\hat{\varphi}_m}{\Delta_m'}\right)F_m^{12}(p)\otimes F_m^{12}(q) + \\ \left(\dfrac{\bar{\beta}_m\hat{\alpha}_m^2 - \alpha_m^2\hat{\varphi}_m}{\Delta_m'}\right)F_m^{12}(p)\otimes F_m^{21}(q) \\ +\left(\dfrac{\hat{\alpha}_m^1\alpha_m^2 - \bar{\beta}_m\hat{\beta}_m}{\Delta_m'}\right)F_m^{21}(p)\otimes F_m^{21}(q) + \\ \left(\dfrac{\hat{\alpha}_m^1\beta_m - \alpha_m^1\hat{\beta}_m}{\Delta_m'}\right)F_m^{21}(p)\otimes F_m^{12}(q) \\ +\left(\dfrac{\alpha_m^1\hat{\alpha}_m^2 - \beta_m\hat{\varphi}_m}{\Delta_m'}\right)F_m^{22}(p)\otimes F_m^{22}(q) \\ +\left(\dfrac{\bar{\beta}_m\hat{\alpha}_m^2 - \alpha_m^2\hat{\varphi}_m}{\Delta_m'}\right)F_m^{22}(p)\otimes F_m^{11}(q) \end{array}\right] \end{aligned} \tag{3.42}$$

.

then from (3.24) to (3.26) we conclude that when ∂D is a circle of radius a, the simple and cross multipole coefficients yield the exact Green's function for the Neumann problem. This result ensures the unique solvability of the modified integral equation defined by (2.1). □

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