Some Drazin invertible elements in Banach algebras and applications to operator equations solutions

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Abstract. In this paper, some conditions of Drazin invertibility of operators has been established as well as some results for operations on Drazin invertible operators leading to the solvability of some oparator equations in Banch algebras.
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1. Introduction

Troughout this paper A denotes a Banach algebra. Recall that an element $a \in A$ is said to be Drazin invertible if there exists $x \in A$ such that

$$ax = xa, \; xax = x \text{ and } a^k = a^{k+1}x \tag{1}$$

for some positive integer k. The element x above is unique if it exists and is denoted by a^D. The least such k is called Drazin index of a, denoted by $\operatorname{ind}(a)$. If $\operatorname{ind}(a) = 1$, then a^D is called group inverse of a and is denoted by $a^{\pm}$. By A^D we mean the set of all Drazin invertible elements of A. For two commutative generalized Drazin invertible elements a, b in a Banach algebra, Kolina gave the expression of $(ab)^D$. Meanwhile, the representation of $(a+b)^D$ was obtained under the condition $ab = ba = 0$ in a Banach algebra. Later, Dordjevic and Wei [10] gave the expression of $(a+b)^D$ under the assumption $ab = 0$ in the context of the Banach algbra of all bounded linear operators on an arbitrary

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complex Banach space. In [2], Castro-Gonzàlez and Kolina obtained a formula for $(a+b)^D$ under the conditions

$$a^\pi b = b, \quad ab^\pi = a, \quad b^\pi aba^\pi = 0, \tag{2}$$

which are weaker than $ab = 0$ in Banach algebra. In [8], Deng and Wei derived necessary and sufficient conditions for the existence of $(P+Q)^D$ under the condition $FQ = QP$, where P, Q are bounded linear operators. Moreover, the expression of $(P+Q)^D$ was given. In [4], Cvetkovic-Illè, Liu and Wei extended the result of [8] to Banach algebra. More results on generalized Drazin inverse can be found in [2, 4, 7, 8, 12, 14].

In [15], Liu, Wu and Yu deduced the explicit expression for the Drazin inverses of product ab and sum $a+b$ under the conditions $a^2b = aba$ and $b^2a = bab$, where a and b are complex matrices. In [18] the corresponding results of [15] have been studied for the pseudo Drazin inverse which is a special case of generalized Drazin inverse [19].

In this paper, we prove some interesting condition of Drazin invertibility of operators, some results for operations on Drazin invertible operators are given, some conditions of solvability of oparator equations and other important results are also given. Our work will extend the above results.

2. Preliminaries

The following lemmas will be useful later.

Lemma 2.1 ([9]). *For $a, b \in A^D$, the following assertions hold.*

1. *Whenever $ab = ba$, we have $a^D b = ba^D$.*
2. *Whenever $ab = ba = 0$, we have $(a+b)^D = a^D + b^D$.*

Lemma 2.2 ([21]). *Let $a, b \in A^D$ such that $ab = ba$. Then $ab \in A^D$ and*

$$(ab)^D = b^D a^D = a^D b^D.$$

Lemma 2.3 ([2]). *Let $a, b \in A$ such that $ab \in A^D$. Then $ba \in A^D$ and*

$$(ba)^D = b\left((ab)^D\right)^2 a.$$

Lemma 2.4 ([5]). *1. Let $a, b \in A$. Then $1 - ab \in A^D$ if and only if $1 - ba \in A^D$.*

2. Let $a, b \in A^D$ and $p^2 = p \in A$, if $ap = pa$ and $bp = pb$, then $ap + b(1-p) \in A^D$ and $(ap + b(1-p))^D = a^D p + b^D (1-p)$.

Proposition 2.5. *Let $a, b, c \in A$ such that $ca = bc$. Then*

$$ca^n = b^n c, \ \forall\, n \in \mathbb{N}.$$

Proof. We proceed by recurrence on n. For $n = 1$, we get $ca = bc$ which is true by hypothesis. So, assume that $ca^n = b^n c$. We have next

$$ca^{n+1} = caa^n = bca^n = bb^n c = b^{n+1} c.$$

The second equality follows from the hypothesis on a, b, c. The third one follows from the recurrence hypothesis.

Proposition 2.6. *Let $a, b, c \in A$ be such that $p^2 = p$, $cp = pc$ and $c(pa) = (bp)c$. Then*

$$c(pa^n) = (b^n p)c, \ \forall n \in \mathbb{N}.$$

Proof. For $n = 0$, the result follows from the hypothesis $cp = pc$. For $n = 1$, it follows from the hypothesis $c(pa) = (bp)c$. So, assume that $c(pa^n) = (b^n p)c$. We have

$$\begin{aligned} c(pa^{n+1}) &= c(paa^n) \\ &= c(pa)a^n \\ &= (b^n p)ca \text{ (This is from the recurrence hypothesis)} \\ &= b^n(pc)a \\ &= b^n(cp)a \text{ (From the hypothesis cp=pc)} \\ &= b^n(bp)c \text{ (From the hypothesis c(pa)=(bp)c)} \\ &= (b^{n+1}p)c. \end{aligned}$$

Definition 2.7. One say that the operator p is idempotent if $p^2 = p$.

Corollary 2.8. *If p is idempotent then $p^D = p$.*

Theorem 2.9. *Let $A, B, C \in B(H)$. Then the equation*

$$AXB = C \tag{1}$$

is consistent if and only if for some $A^{(1)}, B^{(1)}$

$$AA^{(1)}CB^{(1)}B = C \tag{2}$$

in which case the general solution is

$$X = A^{(1)}CB^{(1)} + Y - A^{(1)}AXBB^{(1)} \tag{3}$$

for arbitrary $Y \in B(H)$.

Proof. Let $AA^{(1)}CB^{(1)}B = C$. Then $X_0 = A^{(1)}CB^{(1)}$ is a particular solution of $AXB = C$. Conversely assume that X_0 satisfies $AXB = C$. Then $AA^{(1)}CB^{(1)}B = AA^{(1)}AXBB^{(1)}B = AXB = C$. Further, $A(X - X_0)B = 0$.

3. Main results

In what follows, $a, b \in A$, p and q always mean two arbitrary idempotent elements in the Banach algebra A. We aim to study the Drazin invertibility of the operators $ap(1-q)$, $bq(1-p)$, $ap(1-q)p$ and $bq(1-p)q$.

Theorem 3.1. *Let $a, b \in A^D$ and assume that $ap = pa$, $aq = qa$, $bp = pb$, $bq = qb$ and $pq = qp$. Then*

1. *$ap(1-q) \in A^D$ and $bq(1-p) \in A^D$.*
2. *$ap(1-q) + bq(1-p) \in A^D$ and*

$$(ap(1-q) + bq(1-p))^D = (ap(1-q))^D + (bq(1-p))^D.$$

Proof. We will prove the two assertions in one block. Since $p^2 = p$, we get $p^D = p$. Thus, $p, q \in A^D$. Recall next that $ap = pa$ and $bp = pb$. So, we obtain by Lemma 2.2,

$$(ap)^D = a^D p \text{ and } (bq)^D = b^D q.$$

Next, notice that $aq = qa$, $bp = pb$ and $ap(1-q) = (1-q)ap$. We obtain

$$(ap(1-q))^D = a^D p(1-q) \text{ and } bq(1-p) = (1-p)bq.$$

Consequently,

$$(bq(1-p))^D = b^D q(1-p).$$

Now, observe that

$$ap(1-q)bq(1-p) = bq(1-p)ap(1-q) = 0.$$

We obtain

$$(ap(1-q) + bq(1-p))^D = a^D p(1-q) + b^D q(1-p).$$

Corollary 3.2. *For $a = I = b$ and $pq = qp$, we have*

$$(p(1-q) + q(1-p))^D = p(1-q) + q(1-p).$$

Theorem 3.3. *Let $a, b \in A^D$ and assume that $p^2 = p$, $q^2 = q$ and $pq = qp$. The following assertions hold.*

1. *$ap(1-q)p \in A^D$ and $bq(1-p)q \in A^D$*
2. *$ap(1-q)p + bq(1-p)q \in A^D$ and*

$$(ap(1-q)p + bq(1-p)q)^D = a^D p(1-q)p + b^D q(1-p)q.$$

Proof. The first part is similar to Theorem 3.1. So, let as prove the second. We have

$$ap(1-q)p = ap - apqp = ap - aqp^2 = ap - aqp = ap - apq = ap(1-q) \in A^D.$$

Theorem 3.4. *Let $a, b \in A^D$ and $c \in A$, p idempotent such that $c\,(pa) = (bp)\,c$ and $cp = pc$. Then $c\left(pa^D\right) = \left(b^D p\right) c$.*

Proof. It is straightforward that $\forall\, n \in \mathbb{N}$, we have

$$\begin{aligned} bb^D pc - bb^D bcpa^D &= bb^D pc - bb^D cpaa^D \\ &= bb^D pc\left(1 - aa^D\right) \\ &= \left(bb^D\right)^n pc\left(1 - aa^D\right) \\ &= \left(b^D\right)^n (b)^n pc\left(1 - aa^D\right) \\ &= \left(b^D\right)^n cpa^n\left(1 - aa^D\right), \end{aligned}$$

which yields that

$$\begin{aligned} \|bb^D pc - bb^D bcpa^D\| &= \|\left(b^D\right)^n cpa^n\left(\left(1 - aa^D\right)\right)\| \\ &\leq \|b^D\| \|cp\|^{\frac{1}{n}} \|a^n\left(1 - aa^D\right)\|^{\frac{1}{n}} \longrightarrow 0 \end{aligned}$$

whenever $n \longrightarrow \infty$. Thus,

$$bb^D pc = bb^D bcpa^D,$$

or equivalently,

$$b^D pc = b^D bcpa^D.$$

On the other hand, we have

$$\begin{aligned} cpa^D a - b^D pcaa^D a &= cpa^D a - b^D bpca^D a \\ &= \left(1 - b^D b\right) cpa^D a \\ &= \left(1 - b^D b\right) cp\left(a^D a\right)^n \\ &= \left(1 - b^D b\right) cpa^n\left(a^D\right)^n \\ &= \left(1 - b^D b\right) b^n pc\left(a^D\right)^n. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \|cpa^D a - b^D pcaa^D a\|^{\frac{1}{n}} &= \|\left(1 - b^D b\right) b^n pc\left(a^D\right)^n\|^{\frac{1}{n}} \\ &\leq \|\left(1 - b^D b\right) b^n\|^{\frac{1}{n}} \|pc\|^{\frac{1}{n}} \|a^D\| \longrightarrow 0 \end{aligned}$$

whenever $n \longrightarrow \infty$. So,

$$cpa^D a = b^D pcaa^D a,$$

which means that

$$cpa^D = b^D pcaa^D.$$

Therefore, we deduce that

$$cpa^D = b^D cp.$$

Corollary 3.5. *Let $a, b \in A^D$ and $c \in A$. If $ca = bc$, then $ca^D = b^D c$.*

We obtain the result in [3] as a particular case $p = I$.

Corollary 3.6. *Let $a \in A^D$ and $ca = ac$, then $ca^D = a^D c$.*

We obtain the result in [10] as a particular case $a = b$.

Theorem 3.7. *Let $A, B, C \in B(H)$. The equation*

$$AXB = C \tag{1}$$

is consistent if and only if for some A^D, B^D,

$$AA^D CB^D B = C \tag{2}$$

in which case the general solution is

$$X = A^D CB^D + Y - A^D AYBB^D \tag{3}$$

for arbitrary $Y \in B(H)$.

Proof. If (2) holds then $X = A^D CB^D$ is solution of (1). Conversly, if X is any solution of (1) then

$$C = AXB = AA^D AXBB^D B = AA^D CBB^D.$$

Moreover, it follows from (2) and the definition of A^D and B^D that every X of the from (3) satisfies (1). On the other hand, let X be any solution of (1). Then clearly

$$\begin{aligned} C &= AXB \\ &= AA^D CB^D B + AYB - AA^D AYBB^D B \\ &= AA^D CB^D B + AYB - AYB \\ &= AA^D CB^D B. \end{aligned}$$

Then $X = A^D CB^D$.

Theorem 3.8. *Let $A, B, C \in B(H)$ such that $CA = BC$. Then the equation*

$$AXB = C \tag{4}$$

is consistent if and only if for some A^D, B^D

$$A^D CB = C \tag{5}$$

in which case the general solution is

$$X = A^D CB + Y - A^D AYBB^D \tag{6}$$

for arbitrary $Y \in B(H)$.

Proof. If $CA = BC$ then $CA^D = B^D C$ (see Corrollary 3.2). By Theorem 3.7 we obtain

$$\begin{aligned} AXB = C &\iff AA^D CB^D B = C \\ &\iff AA^D A^D CB = C \\ &\iff A^D CB = C. \end{aligned}$$

Now, whenever $X = A^D CB^D + Y - A^D AYBB^D$ we obtain

$$AXB = AA^D CB^D B + AYB - AA^D AYBB^D B.$$

Consequently,

$$AXB = A^D CB + AYB - AYB.$$

Which means that

$$AXB = A^D CB.$$

Corollary 3.9. *Let $A, C, \in B(H)$ such that $AC = CA$. Then the equation*

$$AXA = C \tag{7}$$

is consistent if and only if for some A^D we have $AA^D CA = C$.

Proof. We have the following implications.

$$AXA = C \implies AA^D CA^D A = C \implies AA^D A^D CA = C \implies AA^D CA = C.$$

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