Edge maximal non-bipartite Hamiltonian graphs without theta graphs of order 7

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Abstract. For a set of graphs $\mathcal{F}$, let $\mathcal{H}(n;\mathcal{F})$ denote the class of non-bipartite Hamiltonian graphs on n vertices that does not contain any graph of $\mathcal{F}$ as a subgraph and $h(n;\mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{H}(n;\mathcal{F})\}$ where $\mathcal{E}(G)$ is the number of edges in G. In this paper, we determine $h(n;\{\theta_4,\theta_5,\theta_7\})$ and we establish an upper bound of $h(n;\theta_7)$ for sufficiently even large n. Our results confirms the conjecture made in [1] for $k = 3$.

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1. Introduction and preliminaries

We consider undirected graphs without loops and multiple edges. Let $V(G)$ be the vertex set and $E(G)$ be the edge set of a graph G. The order of a graph G is the number of vertices of G and is denoted by $\mathcal{V}(G)$. The size of G, denoted by $\mathcal{E}(G)$, is the number of edges of G. A complete k-partite graph is a graph whose vertices can be partitioned into k disjoint sets, such that two vertices are adjacent if and only if they belong to different sets. We often denote by $x_1x_2\ldots x_nx_1$ the cycle C_n having n vertices $x_1, x_2, \ldots, x_n$ and the edges $x_1x_2, x_2x_3, \ldots, x_{n-1}x_n$ and x_nx_1. A theta graph is a cycle C_n with a new edge (a chord) joining two non-adjacent vertices of C_n. The set of all theta graphs of order n will be denoted by θ_n. It is easy to check that the set θ_n contains $\left\lfloor \frac{n}{2} \right\rfloor - 1$ (non-isomorphic) graphs.

If F is a subgraph of G, then $G - F$ is the graph that contains all vertices of G which are not in F and all edges of G connecting two vertices of $G - F$. If P and Q are two subgraphs of G, then $E(P, Q)$ is the set containing all edges of G, which connect a vertex in P and a vertex in Q and $\mathcal{E}(P, Q) = |E(P, Q)|$. An induced subgraph $G[V(Q)]$ of a graph G consists of the vertices in Q and all edges of G connecting two vertices in Q. The join $G = G_1 \vee G_2$ of graphs $G_1$and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph G_1 union G_2 together with all the edges joining $V(G_1)$ and $V(G_2)$.

For a set of graphs S, the Turán number $ex(n, S)$ is defined as the maximum number of edges in a graph of order n having no member of S as a subgraph. If S contains only one graph G, we write simply $ex(n, G)$. The problem was formulated by Turán [14], who showed that $ex(n, K_r) = \lfloor \frac{rn^2}{2(r+1)} \rfloor$, where K_r is the complete graph having r vertices.

We now introduce some additional notation. For a positive integer n and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n; \mathcal{F})$ (and $\mathcal{H}(n; \mathcal{F})$) denote the class of non-bipartite $\mathcal{F}$-free graphs (the subclass of $\mathcal{G}(n; \mathcal{F})$ which consists of all the Hamiltonian members in $\mathcal{G}(n; \mathcal{F})$) on n vertices, and

$$\begin{aligned} f(n; \mathcal{F}) &= \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; \mathcal{F})\}, \\ h(n; \mathcal{F}) &= \max\{\mathcal{E}(G) : G \in \mathcal{H}(n; \mathcal{F})\}. \end{aligned}$$

Hendry and Brandt [10] proved that $h(n; C_5) \leq \frac{(n-3)^2}{4} + 5$ for odd $n \geq 7, n \neq 9$, and $h(9; C_5) = 15$. However, they did not characterize the extremal graphs. Caccetta and Jia [7] characterized the extremal graphs and proved that $f(n; C_5) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $n \geq 9$. Also, they proved $h(n; C_5) \leq \frac{(n-4)^2}{4} + 7$ for even $n \geq 12$. Further, the extremal graphs were characterized. Jia [13] conjectured that $f(n; C_{2k+1}) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $n \geq 4k+2$. Bataineh [1] settled the above conjecture for $n \geq 36k$. Further, he showed that equality holds if and only if $G \in \mathcal{G}^*(n)$ where $\mathcal{G}^*(n)$ is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor (n-2)/2 \rfloor, \lceil (n-2)/2 \rceil}$. Furthermore he proved the following result:

Theorem 1 (Bataineh [1]). *For positive integers $k \geq 3$ and $n > (4k+2)(4k^2 + 10k)$,*

$$h(n; C_{2k+1}) = \begin{cases} \dfrac{(n-2k+1)^2}{4} + 4k - 3, & \text{if } n \text{ is odd} \\ \dfrac{(n-2k)^2}{4} + 4k + 1, & \text{if } n \text{ is even.} \end{cases}$$

For θ_5-graph, Bataineh et al. [2] proved that for $n \geq 5$

$$f(n; \theta_5) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Later on, Bataineh et al. [3], [4] and Jaradat et al. [11] proved the following results

Theorem 2 (Jaradat et al. [11]). *For positive integers n and k, let G be a graph on $n \geq 6k+3$ vertices which contains no θ_{2k+1} as a subgraph, then*

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Theorem 3 (Jaradat et al. [11] and Bataineh et al. [3] and [4]). *For sufficiently large integer n and for $k \geq 3$,*

$$f(n; \theta_{2k+1}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

Caccetta and Jia [7] proved the following results:

Theorem 4 (Caccetta and Jia [7]). *Let $G \in \mathcal{G}(n; C_3, C_5, \ldots, C_{2k+1})$. Then*

$$\mathcal{E}(G) \leq \lfloor \frac{1}{4}(n-2k+1)^2 \rfloor + 2k - 1.$$

Theorem 5 (Caccetta and Jia [7]). *Let $\mathcal{F}_k = \{C_3, C_5, C_7, \ldots, C_{2k+1}\}$. For even $n \geq 4k+4, k \geq 2$, we have*

$$h(n; \mathcal{F}_k) = \frac{(n-4k-4)^2}{4} + 8k - 11.$$

Analogously, In [1], Bataineh proved the following result concerning theta graphs:

Theorem 6 (Bataineh [1]). *Let $\Theta_k = \{\theta_4\} \cup \{\theta_5, \theta_7, \ldots, \theta_{2k+1}\}$, then for $k \geq 5$ and large odd n, we have*

$$h(n; \Theta_k) = \frac{(n-2k+3)^2}{4} + 2k - 3.$$

Jaradat et al. [12] proved the following result.

Theorem 7 (Jaradat et al. [12]). *For sufficiently large odd n, let $H \in \mathcal{H}(n; \theta_7)$ with $\delta(H) \geq 7$. Then*

$$h(n; \theta_7) \leq \frac{(n-3)^2}{4} + 3.$$

Furthermore, the bound is best possible.

Bataineh [1] made the following conjecture

Conjecture 1. *Let $k \geq 3$ be a positive integer. For even $n \geq 4k+4$, $h(n; \theta_{2k+1}) \leq \frac{(n-2k+2)^2}{4} + 2k$.*

In this paper, we investigates the values of $h(n; \mathcal{F})$, for sufficiently large even n where $\mathcal{F} = \{\theta_4, \theta_5, \theta_7\}$ and $\mathcal{F} = \{\theta_7\}$. In fact, we settle the above conjecture for $k = 3$ under a constrain on the minimum degree.

2. Main results

For the sake of completeness, we start this section, by listing the following three results of Jaradat et al. [12] which will be used in the sequel.

Lemma 1 ([12]). *Let $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$ and H contains a cycle C of length 7. If $u \in V(H - C)$, then $\mathcal{E}(u, C) \leq 3$. Also, if $B = \{u \in V(H - C) : \mathcal{E}(u, C) = 3\}$, then $|B| \leq 1$. Further, if $C = x_1x_2x_3x_4x_5x_6x_7x_1$ and $u \in B$, then $N_C(u) = \{x_i, x_{i+1}, x_{i+4}\}$ for some $i = 1, 2, \ldots, 7$ ($x_j = x_{j-7}$ for $j > 7$).*

Lemma 2 ([12]). *Let $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$ such that H contains a cycle C of length 7. If $|B| = 1$ and uv is an edge in the subgraph $H - C - B$, then $\mathcal{E}(uv, C) \leq 3$ where B is as defined in Lemma 1.*

The following remark follows from the fact that if $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$, C is a cycle of length 7 in H and $\mathcal{E}(u, C) = 3$, then $N_C(u) = \{x_i, x_{i+1}, x_{i+4}\}$.

Remark 1 ([12]). Let $H \in \mathcal{H}(n, \{C_3, \theta_4, \theta_5, \theta_7\})$ and H contains a cycle C of length 7. Then $B = \varnothing$ where B is as defined in Lemma 1.

To investigate $h(n; \{\theta_4, \theta_5, \theta_7\})$ and $h(n; \theta_7)$ for even n, we prove the following lemmas.

Lemma 3. *For any $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$, if H contains cycles of lengths 3 and 7, then*

$$\mathcal{E}(H) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5,$$

for sufficiently large even n.

Proof. Let $C_7 = x_1x_2 \dots x_7x_1$ and $C_3 = y_1y_2y_3y_1$ be cycles of length 7 and 3 in H, respectively. Let $A = H[x_1, x_2, \dots, x_7]$ and $R_1 = H - A$. We distinguish two cases:

Case 1. $V(C_3) \subseteq V(R_1)$. Let $R_2 = R_1 - C_3$. By Lemma 2 we have $\mathcal{E}(R_2, A) \leq 2(n-10)$. Notice that if $u \in V(H - C_3)$, then $\mathcal{E}(u, C_3) \leq 1$, otherwise θ_4 is produced as a subgraph of H. Thus, $\mathcal{E}(R_2, C_3) \leq n - 10$. Observe that for $i = 1, 2, \dots, 7$ and $j = 1, 2, 3$, if x_i is adjacent to y_j, then neither x_{i+1} nor x_{i-1} can be adjacent to y_s for some $s = 1, 2, 3$, and $s \neq j$, otherwise θ_5 is produced as a subgraph. Now, if x_{i-1}, x_i and x_{i+1} are all adjacent to the same y_j, then θ_4 is produced as a subgraph, hence, $\mathcal{E}(C_3, A) \leq 4$. By Theorem 2 we have

$$\mathcal{E}(R_2) \leq \left\lfloor \frac{(n-10)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(R_2) + \mathcal{E}(R_2, A) + \mathcal{E}(R_2, C_3) + \mathcal{E}(A) + \mathcal{E}(A, C_3) + \mathcal{E}(C_3) \\
&\leq \left\lfloor \frac{(n-10)^2}{4} \right\rfloor + 2(n-10) + n - 10 + 7 + 4 + 3 \\
&\leq \left\lfloor \frac{n^2 - 8n + 36}{4} \right\rfloor \\
&= \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.
\end{aligned}$$

Case 2. $V(C_3) \nsubseteq V(R_1)$. Then $|V(C_3) \cap V(A)| = 2$ or 1, accordingly, we split this case into two subcases:

Subcase 2.1. $|V(C_3) \cap V(A)| = 2$. Withoutloss of generality assume $x_1, x_2 \in N_A(y_1)$ and let $A_1 = H[y_1, A]$ and $R_3 = H - A_1$, then by Lemma 1, we get $\mathcal{E}(y_1, A) \leq 3$, hence $\mathcal{E}(A_1) \leq 10$. Also, by Theorem 2 we have

$$\mathcal{E}(R_3) \leq \left\lfloor \frac{(n-8)^2}{4} \right\rfloor.$$

Now, we consider the case $\mathcal{E}(y_1, A) = 3$, then $\mathcal{E}(A_1) = 10$. By Lemma 1 $\mathcal{E}(x, A) \leq 2$ for each $x \in V(R_3)$. On the other hand, one can notice that if there is an $x \in V(R_3)$ such that $y_1x \in E(H)$, then $\mathcal{E}(x, A) = 0$ as otherwise a θ_4 or θ_5 or θ_7 is produced as a subgraph of H, which implies that $\mathcal{E}(x, A_1) \leq 1$ and so $\mathcal{E}(R_3, A_1) \leq 2(n-8) - 1$. If $y_1x \notin E(H)$ for each $x \in V(R_3)$, then $\mathcal{E}(x, A_1) = \mathcal{E}(x, A)$, but by Lemma 2 we get $\mathcal{E}(R_3, A_1) = \mathcal{E}(R_3, A) \leq 2(n-8)-1$. Therefore,

$$\mathcal{E}(R_3, A_1) \leq 2(n-8) - 1.$$

Consequently, we have

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R_3)+\mathcal{E}(R_3,A_1)+\mathcal{E}(A_1)\\ &\leq \left\lfloor\frac{(n-8)^2}{4}\right\rfloor+2(n-8)-1+10\\ &= \left\lfloor\frac{n^2-8n+36}{4}\right\rfloor\\ &= \left\lfloor\frac{(n-4)^2}{4}\right\rfloor+5.\end{aligned}$$

We now consider the case $\mathcal{E}(y_1,A)=2$, then $\mathcal{E}(A_1)\leq 9$. Now, for $x\in V(R_3)$ if $xy_1\in E(H)$, then $x_ix\notin E(H)$ for each $i=1,2,3,5,7$ as otherwise a θ_4 or θ_5 or θ_7 is produced as a subgraph of H. Further, y_1 can not be adjacent to both x_4 and x_1 as otherwise $x_4x_3x_2y_1xx_6x_5x_4x$ is a θ_7-graph of H. Thus, $\mathcal{E}(x,A)\leq 1$, which implies that $\mathcal{E}(x,A_1)\leq 2$. Also, if $uv\in E(H-A_1)$ and $y_1u\in E(H)$, then as above $N_{C_7}(u)\subseteq\{x_4\}$ or $N_{C_7}(u)\subseteq\{x_6\}$; and $vx_i\notin E(H)$ for each $i=1,2,4,6$ as otherwise θ_5 or θ_7 is produced as a subgraph of H. Further, v is adjacent to at most one of x_3,x_5 and x_7, to see this, note that: (1) If v is adjacent to both x_3 and x_5, then $vx_5x_4x_3x_2y_1uvx_3$ is a θ_7-graph in H; (2) if v is adjacent to both of x_7 and x_5, then by symmetry we get a θ_7-graph in H; (3) if v is adjacent to both of x_3 and x_7, then $vuy_1x_1x_7x_6x_5vx_5$ is a θ_7-graph in H. In addition, if $vy_1\in E(H)$, then $ux_4,ux_6\notin E(H)$, to see that let $ux_4\in E(H)$, then $ux_4x_3x_2x_1y_1vy_1$ is a θ_7-graph in H. Thus, $\mathcal{E}(uv,A_1)\leq 3$. Therefore, from the above and using Lemma 2, we conclude that

$$\mathcal{E}(R_3,A_1)\leq 2(n-8).$$

And so,

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R_3)+\mathcal{E}(R_3,A_1)+\mathcal{E}(A_1)\\ &\leq \left\lfloor\frac{(n-8)^2}{4}\right\rfloor+2(n-8)+9\\ &= \left\lfloor\frac{n^2-8n+36}{4}\right\rfloor\\ &= \left\lfloor\frac{(n-4)^2}{4}\right\rfloor+5.\end{aligned}$$

Subcase 2.2. $|V(C_3)\cap V(A)|=1$. Without loss of generality assume y_1,y_2 are adjacent to x_1. Let $A_2=H[y_1y_2,A]$ and $R_4=H-A_2$. One can easily see that $\mathcal{E}(y_1y_2,A)=2$, because otherwise θ_4 or θ_5 or θ_7 is produced, hence $\mathcal{E}(A_2)=10$. Now, if $x\in V(R_4)$, then x cannot be adjacent to both y_1 and y_2, as otherwise θ_4 is produced. Moreover, if x is adjacent to either y_1 or y_2, then $N_A(x)\subseteq\{x_3,x_6\}$ as otherwise θ_4 or θ_5 or θ_7 is produced.
Now, let $x,x^*\in R_4$ be adjacent to y_1 or y_2 and assume $N_A(x)=\{x_3,x_6\}$, then x^* is adjacent to at most one of x_3 and x_6. To see this, assume $N_A(x^*)=\{x_3,x_6\}$.

Then, If $xy_1, x^*y_1 \in E(H)$, then $xx_6x_7x_1y_1x^*x_3xy_1$ is θ_7. A similar argument holds if $xy_2, x^*y_2 \in E(H)$. If $xy_1, x^*y_2 \in E(H)$, then $x^*x_3xx_6x_7x_1y_2x^*x_6$ is a θ_7.

Let

$$S = \{x \in R_4 : xy_1 \text{ or } xy_2 \in E(H)\},$$

and

$$S^* = \{x \in R_4 : xy_1, xy_2 \notin E(H)\}.$$

Then from the above argument

$$\mathcal{E}(S, A_2) \leq 2|S| + 1, \tag{1}$$

and by Lemma 2

$$\mathcal{E}(S^*, A_2) \leq 2|S^*| + 1. \tag{2}$$

Hence, combining 1 and 2, we get

$$\begin{aligned}\mathcal{E}(R_4, A_2) &\leq 2|S| + 1 + 2|S^*| + 1 \\ &\leq 2(n-9) + 2.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R_4) + \mathcal{E}(R_4, A_2) + \mathcal{E}(A_2) \\ &\leq \left\lfloor \frac{(n-9)^2}{4} \right\rfloor + 2(n-9) + 2 + 9 \\ &= \left\lfloor \frac{n^2 - 10n + 49}{4} \right\rfloor \\ &< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.\end{aligned}$$

□

Lemma 4. *For any $H \in \mathcal{H}(n, \{C_3, \theta_4, \theta_5, \theta_7\})$, if H contains a cycle of length 5 and a cycle of length 7, then*

$$\mathcal{E}(H) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5,$$

for sufficiently large even n.

Proof. Let $C_5 = y_1y_2y_3y_4y_5y_1$ and $C_7 = x_1x_2x_3 \ldots x_7x_1$ be cycles of length 5 and 7 in H, respectively. As in Lemma 3, we let $R_1 = H - A$ where $A = H[x_1, x_2, \ldots, x_7]$. Now we consider two cases:

Case 1. $V(C_5) \subseteq V(R_1)$. Let $R_5 = R_1 - C_5$. Notice that $A = C_7$ and $H[C_5] = C_5$, otherwise θ_7 or θ_5 is produced as a subgraph, and so $\mathcal{E}(A) = 7$ and

$\mathcal{E}(H[C_5]) = \mathcal{E}(C_5) = 5$. By Lemma 2 we have $\mathcal{E}(R_5, A) \leq 2(n-12)$. Now, if $x \in V(R_5)$, then $\mathcal{E}(x, C_5) \leq 2$, otherwise θ_4 or θ_5 is produced as a subgraph.

Claim 1. Let $xy \in E(R_5)$, then $\mathcal{E}(xy, C_5) \leq 2$.

Proof of the claim. Suppose that $\mathcal{E}(x, C_5) = 2$. Then, by taking into account the symmetry, we have $N_{C_5}(x) = \{y_i, y_{i+2}\}$, otherwise C_3 is produced. Without loss of generality we may assume that $N_{C_5}(x) = \{y_1, y_3\}$, then we have the following possibilities:

1. y is adjacent to y_1. Then the trail $xyy_1y_2y_3xy_1$ is a θ_5-graph.
2. y is adjacent to y_2. Then the trail $xyy_2y_1y_5y_4y_3xy_1$ is a θ_7-graph.
3. y is adjacent to y_3. Then the trail $xyy_3y_2y_1xy_3$ is a θ_5-graph.
4. y is adjacent to y_4. Then the trail $xyy_4y_5y_1y_2y_3xy_1$ is a θ_7-graph.
5. y is adjacent to y_5. Then the trail $xyy_5y_4y_3y_2y_1xy_3$ is a θ_7-graph.

Thus $\mathcal{E}(y, C_5) = 0$, and so $\mathcal{E}(xy, C_5) \leq 2$. This completes the proof of the claim. Since H is a Hamiltonian graph, then there is an edge e in R_5. Thus, by the Claim 1, $\mathcal{E}(e, C_5) \leq 2$, and so $\mathcal{E}(R_5, C_5) \leq 2(n-12) - 2$. Also, by Claim 1, one can see that $\mathcal{E}(C_5, A) \leq 6$. Further, by Theorem 2, we have

$$\mathcal{E}(R_5) \leq \left\lfloor \frac{(n-12)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(R_5) + \mathcal{E}(R_5, A) + \mathcal{E}(R_5, C_5) + \mathcal{E}(A) + \mathcal{E}(A, C_5) + \mathcal{E}(C_5) \\
&\leq \left\lfloor \frac{(n-12)^2}{4} \right\rfloor + 2(n-12) + 2(n-12) - 2 + 7 + 6 + 5 \\
&\leq \left\lfloor \frac{n^2 - 8n + 10}{4} \right\rfloor \\
&\leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor - 1 \\
&< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.
\end{aligned}$$

Case 2. $V(C_5) \nsubseteq V(R_1)$. Then $|V(C_5) \cap V(A)| = 1$ or 2 or 3 or 4. Thus, we split this case into 4 subcases:

Subcase 2.1. $|V(C_5) \cap V(A)| = 1$. Without loss of generality, assume $C_5 = x_1y_1y_2y_3y_4x_1$ is in H, then let $T_1 = H[y_1, y_2, y_3, y_4, A]$ and $D_1 = H - T_1$. From Remark 1, $\mathcal{E}(x, A) \leq 2$ for any $x \in H - A$. Since $\mathcal{E}(H[C_5]) = \mathcal{E}(C_5) = 5, \mathcal{E}(A) = 7$ and $\mathcal{E}(y_j, A) \leq 2$ for $j = 1, 2, 3, 4$, then $\mathcal{E}(T_1) \leq 18$. Also by Theorem 2 we have

$$\mathcal{E}(D_1) \leq \left\lfloor \frac{(n-11)^2}{4} \right\rfloor.$$

Claim 2. For each $x \in V(D_1)$, $\mathcal{E}(x, T_1) \leq 3$.

Proof of the claim. Let $x \in V(D_1)$, then as above both $\mathcal{E}(x, A), \mathcal{E}(x, C_5) \leq 2$. If $\mathcal{E}(x, C_5) \leq 1$, then $\mathcal{E}(x, T_1) \leq 3$. Also, if $xx_1 \in E(H)$, then $\mathcal{E}(x, T_1) \leq 3$ because x_1 is a common vertex of both A and C_5. To this end, we consider the case where $\mathcal{E}(x, C_5) = 2$ and $xx_1 \notin E(H)$. Then, x is either adjacent to both y_1 and y_3 or adjacent to both y_2 and y_4 or to both y_1 and y_4. If $xy_1, xy_4 \in E(H)$, then $C_5^* = xy_1y_2y_3y_4x$ is a cycle of length 5 such that $V(C_5^*) \subseteq V(R_5)$ and so we get Case 1. If $xy_1, xy_3 \in E(H)$, then

1- $xx_2 \notin E(H)$ as otherwise $xy_1y_2y_3y_4x_1x_2xy_3$ is a θ_7.

2- $xx_3 \notin E(H)$ as otherwise $xy_3y_2y_1x_1x_2x_3xy_1$ is a θ_7.

3- $xx_7 \notin E(H)$ as otherwise $xy_1y_2y_3y_4x_1x_7xy_3$ is a θ_7.

4- $xx_6 \notin E(H)$ as otherwise $xy_3y_2y_1x_1x_7x_6xy_1$ is a θ_7.

Thus, $N_{C_7}(x) \subseteq \{x_4, x_5\}$. Also, if $xx_4, xx_5 \in E(H)$, then C_3 is produced. Hence x is adjacent to either x_4 or x_5 but not to both, and so, $\mathcal{E}(x, T_1) \leq 3$. Similarly, by using the symmetry, one can show that if $xy_2, xy_4 \in E(H)$, then $\mathcal{E}(x, T_1) \leq 3$. This completes the proof of the claim.

Therefore, by Claim 2, $\mathcal{E}(D_1, T_1) \leq 3(n - 11)$. Consequently, we have

$$
\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(D_1) + \mathcal{E}(D_1, T_1) + \mathcal{E}(T_1) \\
&\leq \left\lfloor \frac{(n-11)^2}{4} \right\rfloor + 3(n-11) + 18 \\
&= \left\lfloor \frac{n^2 - 10n + 61}{4} \right\rfloor \\
&= \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 9 \\
&< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.
\end{aligned}
$$

Subcase 2.2. $|V(C_5) \cap V(A)| = 2$. Without loss of generality assume that $C_5 = x_1y_1y_2y_3x_2x_1$ is in H. Let $T_2 = H[y_1, y_2, y_3, A]$ and $D_2 = H - T_2$. As above, $\mathcal{E}(x, C_5) \leq 2$ and $\mathcal{E}(x, A) \leq 2$.

Claim 3. For each $x \in V(D_2)$, $\mathcal{E}(x, T_2) \leq 3$.

Proof of the claim. Suppose that $\mathcal{E}(x, T_2) = 4$. Then $\mathcal{E}(x, C_5) = 2$. Note that if $xx_1 \in E(H)$, then x_1 is a common vertex of both C_5 and A and so $\mathcal{E}(x, T_2) \leq 3$, similarly if $xx_2 \in E(H)$, then x_2 is a common vertex of both C_5 and A and so $\mathcal{E}(x, T_2) \leq 3$. Thus, $N_{C_5}(x) = \{y_1, y_3\}$, and

1- $xx_3 \notin E(H)$ as otherwise $xy_3y_2y_1x_1x_2x_3xy_1$ is a θ_7.

2- $xx_4 \notin E(H)$ as otherwise $xy_1y_2y_3x_2x_3x_4xy_3$ is a θ_7.

3- $xx_6 \notin E(H)$ as otherwise $xy_3y_2y_1x_1x_7x_6xy_1$ is a θ_7.

4- $xx_7 \notin E(H)$ as otherwise $xy_1y_2y_3x_2x_1x_7xy_3$ is a θ_7.

Thus, x is adjacent to at most x_5, and so $\mathcal{E}(x, T_2) \leq 3$, as claimed.

Hence, $\mathcal{E}(D_2, T_2) \leq 3(n - 10)$. Recall that for $j = 1, 2, 3, \mathcal{E}(y_j, A) \leq 2$. Observe that y_2 cannot be adjacent to x_1 or x_2, as otherwise C_3 is produced as a subgraph of H. Thus, $N_A(y_2) = \{x_3\}$ or $\{x_4\}$ or $\{x_5\}$ or $\{x_6\}$ or $\{x_7\}$ or $\{x_3, x_7\}$ as otherwise C_3, θ_4, θ_5 or θ_7 is produced as a subgraph of H. If $N_A(y_2) = \{x_3, x_7\}$, then $N_A(y_1) = \{x_1\}$ and $N_A(y_3) = \{x_2\}$ as otherwise C_3, θ_4, θ_5 or θ_7 is produced as a subgraph of H. Thus, $\mathcal{E}(T_2) \leq 14$. By Theorem 2 we have

$$\mathcal{E}(D_2) \leq \left\lfloor \frac{(n-10)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(D_2) + \mathcal{E}(D_2, T_2) + \mathcal{E}(T_2) \\ &\leq \left\lfloor \frac{(n-10)^2}{4} \right\rfloor + 3(n-10) + 14 \\ &= \left\lfloor \frac{n^2 - 8n + 36}{4} \right\rfloor \\ &= \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5. \end{aligned}$$

Subcase 2.3. $|V(C_5) \cap V(A)| = 3$. Without loss of generality, assume that $C_5 = x_1y_1y_2x_3x_2x_1$ is in H, then let $T_3 = H[y_1, y_2, A]$ and $D_3 = H - T_3$. Now, $\mathcal{E}(A) = 7$ and by Lemma 2 $\mathcal{E}(y_1y_2, A) \leq 3$, thus $\mathcal{E}(T_3) \leq 11$. Now, let $x \in V(D_3)$, then x is adjacent to at most one of y_1 and y_2 as otherwise C_3 is produced. Further, by Remark 1, $\mathcal{E}(x, A) \leq 2$. Thus, $\mathcal{E}(x, T_3) \leq 3$. Let $B_1 = \{x \in V(D_3) : \mathcal{E}(x, T_3) = 3\}$.

Claim 4. $|B_1| = 0$.

Proof of the claim. Let $x \in B_1$, then $N_{T_3}(x) = \{y_2, x_2, x_4\}$ or $\{y_2, x_2, x_6\}$ or $\{y_1, x_2, x_5\}$ or $\{y_1, x_2, x_7\}$.

If $N_{T_3}(x) = \{y_2, x_2, x_4\}$, then the trail $x_2x_1y_1y_2x_3x_4xx_2x_3$ is a θ_7-graph. If $N_{T_3}(x) = \{y_2, x_2, x_6\}$, then the trail $xx_6x_7x_1x_2x_3y_2x_2$ is a θ_7-graph. By symmetry we get similar trails if $N_{T_3}(x) = \{y_1, x_2, x_5\}$ or $\{y_1, x_2, x_7\}$. The proof of the claim is complete.

Thus, $\mathcal{E}(x, T_3) \leq 2$ for any $x \in V(D_3)$, which implies that

$$\mathcal{E}(D_3, T_3) \leq 2(n-9).$$

Also, by Theorem 2 we have

$$\mathcal{E}(D_3) \leq \left\lfloor \frac{(n-9)^2}{4} \right\rfloor.$$

Therefore,

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(D_3)+\mathcal{E}(D_3,T_3)+\mathcal{E}(T_3)\\
&\leq \left\lfloor \frac{(n-9)^2}{4} \right\rfloor + 2(n-9)+11\\
&= \left\lfloor \frac{n^2-10n+53}{4} \right\rfloor\\
&= \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 7\\
&< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.
\end{aligned}$$

Subcase 2.4. $|V(C_5)\cap V(A)| = 4$. Without loss of generality, assume that $C_5 = x_1y_1x_4x_3x_2x_1$ is in H, then let $T_4 = H[y_1, A]$ and $D_4 = H - T_4$. By Remark 1 $\mathcal{E}(x, A) \leq 2$ for any $x \in H - A$. Therefore, $\mathcal{E}(y_1, A) = 2$, and so $\mathcal{E}(T_4) = 9$. Now, let $x \in V(D_4)$, if x is not adjacent to y_1, then $\mathcal{E}(x, T_4) \leq 2$; if x is adjacent to y_1, then

1. $xx_1 \notin E(H)$ as otherwise the trail xy_1x_1x is a C_3.
2. $xx_4 \notin E(H)$ as otherwise the trail xy_1x_4x is a C_3.
3. $xx_5 \notin E(H)$ as otherwise the trail $y_1xx_5x_4x_3x_2x_1y_1x_4$ is a θ_7-graph.
4. $xx_7 \notin E(H)$ as otherwise the trail $y_1xx_7x_1x_2x_3x_4y_1x_1$ is a θ_7-graph.

Thus, $N_{C_7}(x) \subseteq \{x_2, x_3, x_6\}$. Now, If x is adjacent to x_2, then it is neither adjacent to x_3 (as otherwise $C_3 = xx_2x_3x$ is produced) nor to x_6 (as otherwise $\theta_7 = xx_6x_5x_4y_1x_1x_2xy_1$ is produced). Similarly if x is adjacent to x_3, then it can not be adjacent to x_6 (as otherwise $\theta_7 = xx_6x_7x_1y_1x_4x_3xy_1$ is produced). Thus, $\mathcal{E}(x, T_4) \leq 2$, and so $\mathcal{E}(D_4, T_4) \leq 2(n-8)$. Also, by Theorem 2 we have

$$\mathcal{E}(D_4) \leq \left\lfloor \frac{(n-8)^2}{4} \right\rfloor.$$

Consequently we have

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(D_4)+\mathcal{E}(D_4,T_4)+\mathcal{E}(T_4)\\
&\leq \left\lfloor \frac{(n-8)^2}{4} \right\rfloor + 2(n-8)+9\\
&= \left\lfloor \frac{n^2-8n+36}{4} \right\rfloor\\
&= \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.
\end{aligned}$$

□

Now, we give the following construction: Let $\mathcal{H}_1$ be the class of graphs that obtained from $\overline{K}_{\frac{n-4}{2}} \vee \overline{K}_{\frac{n-4}{2}}$ by replacing one edge, say $u_1u_2 \in \overline{K}_{\frac{n-4}{2}} \vee \overline{K}_{\frac{n-4}{2}}$, by the path $u_1w_2w_3w_4w_5u_2$ in addition to one of the two edges u_1w_3 and w_2w_4. Note that if $H \in \mathcal{H}_1$, then H is a non-bipartite Hamiltonian graph which has none of $\{\theta_4, \theta_5, \theta_7\}$ as a subgraph of H and $\mathcal{E}(H) = \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5$. Thus, we establish that

$$h(n; \{\theta_4, \theta_5, \theta_7\}) \geq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5 \text{ for even } n. \tag{3}$$

Theorem 8. *Let $H \in \mathcal{H}(n; \{\theta_4, \theta_5, \theta_7\})$, then*

$$h(n; \{\theta_4, \theta_5, \theta_7\}) = \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5,$$

for sufficiently large even n. Furthermore, the bound is best possible.

Proof. By 3, it is suffices to prove the upper bound of $h(n; \{\theta_4, \theta_5, \theta_7\})$. Let $H \in \mathcal{H}(n; \{\theta_4, \theta_5, \theta_7\})$. If H has no cycles of length 7, then by Theorem 1 we have

$$\begin{aligned} \mathcal{E}(H) &\leq \frac{(n-6)^2}{4} + 13 \\ &< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5. \end{aligned}$$

Now, we assume that H has cycles of length 7. If H contains neither cycles of length 3 nor cycles of length 5, then by Theorem 5 we have

$$\mathcal{E}(H) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.$$

To this end, If H contains cycles of length 5, then the results follows from Lemma 4. Finally, if H contains no cycles of length 5 but it contains cycles of length 3, then the results follows from Lemma 3. □

In the following theorem we give an upper bound of $h(n; \theta_7)$ for sufficiently large even n under a constrain of the minimum degree.

Theorem 9. *For sufficiently large even n, let $H \in \mathcal{H}(n; \theta_7)$ with $\delta(H) \geq 22$. Then*

$$h(n; \theta_7) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.$$

Proof. Let $H \in \mathcal{H}(n; \theta_7)$ with $\delta(H) \geq 22$. Suppose that H has θ_5-graph, say $\theta_5 = x_1x_2x_3x_4x_5x_1x_4$. For $i = 1, 2, 3$, let A_i be a set that consist of 6 neighbors of x_i in $H - \theta_5$ selected so that $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $T = H[x_1, x_2, x_3, x_4, x_5, A_1, A_2, A_3]$ and $B = H - T$. Let $u \in V(B)$, if u is adjacent to a vertex in one of the sets A_1, A_2 and A_3, then u cannot be adjacent to any vertex in the other two sets as otherwise H would have a θ_7-graph. Also, if u

is adjacent to a vertex in A_i for some $i = 1, 2, 3$, then u cannot be adjacent to any of x_{i+1} and x_{i-1}, otherwise, H would have a θ_7-graph. Thus, $\mathcal{E}(u, T) \leq 9$, which implies $\mathcal{E}(B, T) \leq 9(n - 23)$. Also, by Theorem 2 we have

$$\mathcal{E}(B) \leq \left\lfloor \frac{(n-23)^2}{4} \right\rfloor \quad \text{and} \quad \mathcal{E}(T) \leq \left\lfloor \frac{(23)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(B) + \mathcal{E}(B, T) + \mathcal{E}(T) \\ &\leq \left\lfloor \frac{(n-23)^2}{4} \right\rfloor + 9(n-23) + \left\lfloor \frac{(23)^2}{4} \right\rfloor \end{aligned}$$

$$\begin{aligned} &\leq \left\lfloor \frac{n^2 - 10n + 230}{4} \right\rfloor \\ &= \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 51 \\ &< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5. \end{aligned}$$

So, we consider that H contains no θ_5-graph. If H contains no θ_4-graph as a subgraph, then by Theorem 8 we have

$$\mathcal{E}(H) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.$$

If H contains θ_4-graph as a subgraph, then let $\theta_4 = x_1x_2x_3x_4x_1x_3$. For $i = 2, 3, 4$, let A_i be a set that consist of 5 neighbors of x_i in H selected so that $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $T = H[\, x_1, x_2, x_3, x_4, A_2, A_3, A_4]$ and $B = H - T$. Also, let $u \in V(B)$. If u is adjacent to a vertex in one of the sets A_2, A_3 and A_4, then u cannot be adjacent to a vertex in the other two sets as otherwise H would have a θ_7-graph. Also, if u is adjacent to a vertex in A_i for some $i = 2, 3, 4$, then u cannot be adjacent to x_{i+1} and x_{i-1}, otherwise H would have a θ_5-graph. Thus, $\mathcal{E}(u, T) \leq 7$. Therefore, $\mathcal{E}(B, T) \leq 7(n - 19)$. By Theorem 2 we have

$$\mathcal{E}(B) \leq \left\lfloor \frac{(n-19)^2}{4} \right\rfloor \quad \text{and} \quad \mathcal{E}(T) \leq \left\lfloor \frac{(19)^2}{4} \right\rfloor.$$

Consequently, we have

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(B)+\mathcal{E}(B,T)+\mathcal{E}(T)\\
&\leq \left\lfloor \frac{(n-19)^2}{4} \right\rfloor + 7(n-19) + \left\lfloor \frac{(19)^2}{4} \right\rfloor\\
&\leq \left\lfloor \frac{n^2-10n+190}{4} \right\rfloor\\
&= \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 41\\
&< \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5.
\end{aligned}$$

□

In the above theorem, we have proved that if G is a θ_7-free graph with n vertices and minimum degree greater than or equal to 22, then $\mathcal{E}(G) \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 5 \leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 6$ which confirm Conjecture 1 in the case $k = 3$. Now consider H is the graph obtained from $\overline{K}_{\frac{n-4}{2}} \vee \overline{K}_{\frac{n-4}{2}}$ by replacing one edge, say $u_1u_2 \in \overline{K}_{\frac{n-4}{2}} \vee \overline{K}_{\frac{n-4}{2}}$, by the path $u_1w_2w_3w_4w_5u_2$ in addition to the two edges u_1w_3 and w_2w_4. Note that H is a non-bipartite Hamiltonian graph which has no θ_7 as a subgraph of H. Furthermore, $\mathcal{E}(H) = \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 6$. Thus, we establish the upper bound of the Conjecture 1 in the case $k = 3$.

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