Some separation axioms using hereditary classes in generalized topological spaces

Rajni Bala
Department of Mathematics
Punjabi University
Patiala- 147002, Punjab
India
rajni.maths@gmail.com

Abstract. we introduce a new class of generalized topological spaces namely T_2 modulo $\mathcal{H}$ and discuss its various properties and characterizations. Also we define $\mathcal{H}$-regular generalized topological spaces and study some properties of the same.

Keywords: generalized topologies, hereditary classes, T_2 modulo $\mathcal{H}$, $\mathcal{H}$-regular spaces, $*$- generalized topology.

1. Introduction and preliminaries

Ideals in topological spaces play an important role in the study of general topology. Many authors like Kurakowski, Hamlett, Jankovic, Noiri, Jha [10, 7, 8, 9, 16] studied various concepts like compactness, separation axioms, continuity in topological spaces via ideals. A topological space (X, τ), together with an ideal $\mathcal{I}$ is called an ideal space and denoted by $(X, \tau, \mathcal{I})$. Kuratowski [10], in his classic text, defined a local function for each subset of X with respect to $\mathcal{I}$ and τ. Vaidyanathaswamy [17, 18] extended the study of ideals and local functions to define the star topology τ^* generated by the ideal $\mathcal{I}$ and the topology τ^*, which is finer than τ. Also the modern research in general topology and real analysis concerns the different variations of compactness, separation axioms and connectedness etc in the various setting of topological spaces. Newcomb [12] in his Ph. D. thesis introduced the concept of compactness modulo an ideal, which is further investigated by Hamlett and Jankovic [7]. Shanthi and Rajesh [15] investigated the separation axioms in topological ordered spaces. In 1994, Hamlett and Jankovic [8] introduced and studied the concept of $\mathcal{I}$- regular spaces. V. Renuka Devi [13], in her Ph. D. thesis, introduced a new class of spaces, called Hausdorff modulo $\mathcal{I}$ (T_2 modulo $\mathcal{I}$) and investigated properties and characterizations for the same. She also studied some properties of $\mathcal{I}$- regular spaces.

On the other hand, generalized open sets are very important tools in topological spaces which is discussed by many authors from time to time. Cśasźar [4] introduced the concept of generalized open sets in generalized topological spaces in 2002. Further in 2008, he [6] defined the concept of hereditary classes instead of ideals in generalized topological spaces and extended the study of hereditary classes to define star generalized topology finer than the given generalized topology. Later many authors studied the various concepts like compactness, connectedness, continuity and separation axioms in the settings of generalized topological spaces. Asaad [2] defined a γ operation on generalized open sets in X and studied its applications. In 2017-2018, Ahmad and Asaad [3, 1] introduced an operation γ on semi generalized open subsets of X and discussed some types of separation axioms, functions and closed spaces with respect to γ.

In this paper we introduce a new class of generalized topological spaces namely Hausdorff modulo $\mathcal{H}$ and discuss its properties and characterizations. Also we define $\mathcal{H}$-regular spaces and investigate them to find their properties. The purpose of this paper is to check whether the properties of Hausdorff modulo $\mathcal{I}$ spaces in ideal topological spaces are satisfied by Hausdorff modulo $\mathcal{H}$ spaces in the settings of generalized topological spaces or not. Some counter examples are also given for the properties which are not satisfied.

We refer to the following concepts:

Definition 1.1 ([4])**.** Let X be a non empty set and $expX$ be the power set of X. A collection $\mu \subset expX$ is called a generalized topology on X if $\emptyset \in \mu$ and μ is closed for arbitrary unions. (X, μ) is called a generalized topological space and the members of μ are called μ-open sets and their complements are called μ-closed sets and $cl_\mu(A)$ is defined as the intersection of all μ-closed sets containing A for each $A \subset X$.

Definition 1.2 ([6])**.** An ideal $\mathcal{I}$ on X is a non empty family of subsets of X satisfying (i) $A \subset B$, $B \in \mathcal{I}$ implies $A \in \mathcal{I}$; (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. If τ is a topology on X and $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space.

Definition 1.3 ([6])**.** A non empty family $\mathcal{H}$ of subsets of X is called a hereditary class if it satisfies only condition (i) of ideals, that is $A \subset B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$. We will call $(X, \mu, \mathcal{H})$ a hereditary space. $\mathcal{H}$ is called codense if $\mu \cap \mathcal{H} = \{\emptyset\}$.

Definition 1.4. [6] The operator $()^* : expX \to expX$, with respect to hereditary class $\mathcal{H}$ and generalized topology μ is defined by $A^* = \{x \in X : U \cap A \notin \mathcal{H}$ for every $U \in \mu, x \in U\}$ for $A \subset X$. This operator defines another operator $c^* : expX \to expX$, by $c^*(A) = A \cup A^*$ for $A \subset X$, which is monotonic, enlarging and idempotent. This operator c^* induces another generalized topology μ^*, called $*$-generalized topology, which is finer than μ. The members of μ^* are called $*$-open sets and their complements are called $*$-closed sets.

Definition 1.5. Let $f : (X, \mu) \to (Y, \nu)$ be a mapping between two generalized topological spaces. Then f is called (μ, ν)-continuous [4] if $f^{-1}(V)$ is μ-open set in X for each ν-open set V in Y. f is called (μ, ν)-open mapping [11] if $f(U)$ is

ν-open set in Y for each μ-open set U in X. f is called (μ,ν)-homeomorphism if it is (μ,ν)-continuous, (μ,ν)-open bijection.

Definition 1.6 ([14])**.** A generalized topological space (X,μ) is called μ-Hausdorff if for each pair of distinct points x and y in X, there exist two disjoint μ-open sets U and V such that $x \in U$ and $y \in V$.

2. Results

Definition 2.1. A hereditary space $(X,\mu,\mathcal{H})$ is said to be *Hausdorff mod* $\mathcal{H}$ or T_2 *mod* $\mathcal{H}$ if for each pair of distinct points x and y in X, there exists μ-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathcal{H}$.

Theorem 2.2. *Every μ-Hausdorff space is T_2 mod $\mathcal{H}$.*

Proof. Proof is obvious, since $\emptyset \in \mathcal{H}$.

The converse of Theorem 2.2 need not be true in general, shown as in Example 2.3. Even if $\mathcal{H}$ is codense, then also the converse need not be true, shown in Example 2.5. Although the converse is true in case of ideal topological spaces for codense ideal, that is, if $(X,\tau,\mathcal{I})$ is ideal topological space, which is T_2 *mod* $\mathcal{I}$, where $\mathcal{I}$ is codense, then $(X,\tau,\mathcal{I})$ is Hausdorff. In particular, if we take $\mathcal{H} = \{\emptyset\}$, then μ-Hausdorffness and Hausdorffness mod $\mathcal{H}$ are equivalent.

Example 2.3. Let $X = \{a,b,c\}$, $\mu = \{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},X\}$ and $\mathcal{H} = \{\emptyset,\{b\}\}$. Then X is T_2 *mod* $\mathcal{H}$, but not μ-Hausdorff.

Theorem 2.4. *A space (X,μ) is μ-Hausdorff if and only if (X,μ) is Hausdorff mod $\mathcal{H}$, where $\mathcal{H} = \{\emptyset\}$.*

Example 2.5. Let $X = \{a,b,c\}$, $\mu = \{\emptyset,\{b\},\{a,b\},\{a,c\},X\}$ and $\mathcal{H} = \{\emptyset,\{a\}\}$. Then $\mathcal{H}$ is codense and X is T_2 *mod* $\mathcal{H}$, but not μ-Hausdorff.

Theorem 2.6. *If a hereditary space $(X,\mu,\mathcal{H})$ is Hausdorff mod $\mathcal{H}$ and $\mathcal{H} \subset \mathcal{K}$, then $(X,\mu,\mathcal{K})$ is Hausdorff mod $\mathcal{K}$.*

The following theorem gives characterizations of T_2 *mod* $\mathcal{H}$ spaces.

Theorem 2.7. *Let $(X,\mu,\mathcal{H})$ be a hereditary space. Then the following are equivalent:*

1. *$(X,\mu,\mathcal{H})$ is T_2 mod $\mathcal{H}$.*
2. *For each $x,y \in X$ such that $y \neq x$, there exists a μ-open set U containing x such that $y \notin U^*$.*

3. *For each* $x \in X$, $\cap\{U_x^* : U_x \in \mu, x \in U_x\}$ *is either* $\emptyset$ *or* $\{x\}$.

Proof. $1 \Rightarrow 2$. Let $x, y \in X$ such that $y \neq x$, then by hypothesis, there exist μ-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathcal{H}$. Therefore $(U \cap V)^* = \emptyset$ and so $U^* \cap V = \emptyset$. Therefore $y \notin U^*$, which proves 2.

$2 \Rightarrow 3$. For each $x \in X$, and each $y \neq x$, there exists a μ-open set U containing x such that $y \notin U^*$. Therefore $y \notin \cap\{U_x^* : U_x \in \mu, x \in U_x\}$ and so $\cap\{U_x^* : U_x \in \mu, x \in U_x\}$ is either $\emptyset$ or $\{x\}$, which proves 3.

$3 \Rightarrow 1$. Let x and y be any two distinct points in X. Then by hypothesis, $y \notin \cap\{U_x^* : U_x \in \mu, x \in U_x\}$, which implies that $y \notin U_x^*$ for some $U_x \in \mu, x \in U_x$. Therefore there exists a μ-open neighbourhood V_y of y such that $U_x \cap V_y \in \mathcal{H}$ and therefore $(X, \mu, \mathcal{H})$ is T_2 mod $\mathcal{H}$, which proves 1.

It is proved that for an ideal space $(X, \tau, \mathcal{I})$, (X, τ) is T_2 mod $\mathcal{I}$ if and only if (X, τ^*) is T_2 mod $\mathcal{I}$. But if we take hereditary space $(X, \mu, \mathcal{H})$ instead of ideal space, where (X, μ^*) is T_2 mod $\mathcal{H}$ then (X, μ) need not be T_2 mod $\mathcal{H}$ as shown in the following example:

Example 2.8. Let $X = \{a, b\}$, $\mu = \{\emptyset, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. Then the corresponding star generalized topology is $\mu^* = \{\emptyset, X, \{a\}, \{b\}\}$, which is T_2 *mod* $\mathcal{H}$, but μ is not T_2 *mod* $\mathcal{H}$.

Theorem 2.9. *Let* X *be any set.* $A \subset X$ *and* $\mathcal{H}$ *be a hereditary class on* X*. Then* $\mathcal{H}_A = \{A \cap H : H \in \mathcal{H}\} = \{H \in \mathcal{H} : H \subset A\}$ *is a hereditary class on* A.

Theorem 2.10. *Let* $(X, \mu, \mathcal{H})$ *be* T_2 *mod* $\mathcal{H}$ *space and* $A \subset X$*. Then* $(A, \mu_A, \mathcal{H}_A)$ *is* T_2 *mod* $\mathcal{H}_A$*, where* μ_A *is subspace generalized topology on* A.

Proof. Let x and y be two distinct points in A, then there exists μ-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathcal{H}$. Then $x \in U \cap A \in \mu_A$, $y \in V \cap A \in \mu_A$ and $(U \cap A) \cap (V \cap A) = (U \cap V) \cap A \in \mathcal{H}_A$. Therefore $(A, \mu_A, \mathcal{H}_A)$ is T_2 *mod* $\mathcal{H}_A$.

Theorem 2.11. *Let* $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ *be a mapping, where* $(X, \mu, \mathcal{H})$ *is a hereditary space and* (Y, ν) *is a generalized topological space. Then* $f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}$ *is a hereditary class on* Y*. If* f *is an injection mapping and* $\mathcal{K}$ *is a hereditary class on* Y*, then* $f^{-1}(\mathcal{K}) = \{f^{-1}(K) : K \in \mathcal{K}\}$ *is a hereditary class on* X.

The following theorem shows that the property of being T_2 *mod* $\mathcal{H}$ is preserved under open bijection mapping in generalized topological space:

Theorem 2.12. *Let* $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ *be an* (μ, ν)*-open bijection and* $(X, \mu, \mathcal{H})$ *be* T_2 *mod* $\mathcal{H}$*. Then* (Y, ν) *is* T_2 *mod* $f(\mathcal{H})$.

Proof. Let y_1 and y_2 be two distinct points in Y. Since f is bijection, there exists distinct points x_1 and x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then there exists μ-open sets U and V such that $x_1 \in U$, $x_2 \in V$ and $U \cap V \in \mathcal{H}$. Then $f(U)\cap f(V) = f(U\cap V) \in f(\mathcal{H})$. Since f is (μ, ν)-open mapping, $f(U)$ and $f(V)$ are ν-open sets in Y such that $y_1 = f(x_1) \in f(U)$ and $y_2 = f(x_2) \in f(V)$. Therefore (Y, ν) is T_2 *mod* $f(\mathcal{H})$.

Theorem 2.13. *Let $f : (X, \mu) \to (Y, \nu, \mathcal{K})$ be a (μ, ν)-continuous injection mapping and $(Y, \nu, \mathcal{K})$ be T_2 mod $\mathcal{K}$. Then (X, μ) is T_2 mod $f^{-1}(\mathcal{K})$.*

Proof. Let x_1 and x_2 be two distinct points in X. Since $(Y, \nu, \mathcal{K})$ is T_2 *mod* $\mathcal{K}$, there exist ν-open sets U and V such that $f(x_1) \in U$, $f(x_2) \in V$ and $U \cap V \in \mathcal{K}$. Since f is injection, $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) \in f^{-1}(\mathcal{K})$. Since f is (μ, ν)-continuous mapping, $f^{-1}(U)$ and $f^{-1}(V)$ are μ-open sets in X such that $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Therefore (X, μ) is T_2 *mod* $f^{-1}(\mathcal{K})$.

Definition 2.14. Let (X, μ) be a generalized topological space and $\mathcal{A}$ be a collection of subsets of X. Then the smallest hereditary class on X containing $\mathcal{A}$, denoted by $\mathcal{H}(\mathcal{A})$, is called the hereditary class generated by $\mathcal{A}$.

It can be easily seen that $\mathcal{H}(\mathcal{A}) = \{H : H \subset A \in \mathcal{A}\}$.

Theorem 2.15. *Let $(X_\alpha, \mu_\alpha, \mathcal{H}_\alpha)$ be a collection of T_2 mod $\mathcal{H}_\alpha$ hereditary spaces for each $\alpha \in \Lambda$, where Λ is indexing set. If $P_\alpha : \Pi X_\alpha \to X_\alpha$ is the projection mapping for each α, $\mathcal{A} = \{P_\alpha^{-1}(H_\alpha) : H_\alpha \in \mathcal{H}_\alpha, \alpha \in \Lambda\}$, $\mathcal{H}(\mathcal{A})$ is hereditary class generated by $\mathcal{A}$ and $\mathcal{H}$ is hereditary class finer than $\mathcal{H}(\mathcal{A})$ on ΠX_α, then ΠX_α with product generalized topology is T_2 mod $\mathcal{H}$.*

Proof. Let x and y be two distinct points in ΠX_α, then there exists some $\alpha \in \Lambda$ such that $x_\alpha \neq y_\alpha$. Since each $(X_\alpha, \mu_\alpha, \mathcal{H}_\alpha)$ is T_2 mod $\mathcal{H}_\alpha$, there exist μ_α-open sets U_α and V_α such that $x_\alpha \in U_\alpha$, $y_\alpha \in V_\alpha$ and $U_\alpha \cap V_\alpha \in \mathcal{H}_\alpha$. Then $P_\alpha^{-1}(U_\alpha) \cap P_\alpha^{-1}(V_\alpha) = P_\alpha^{-1}(U_\alpha \cap V_\alpha) \in \mathcal{H}(\mathcal{A}) \subset \mathcal{H}$. Since P_α is projection mapping, $P_\alpha^{-1}(U_\alpha)$ and $P_\alpha^{-1}(V_\alpha)$ are generalized open sets in ΠX_α containing x and y respectively. Therefore ΠX_α with product generalized topology is T_2 mod $\mathcal{H}$.

Theorem 2.16. *Let (X_α, μ_α) be a collection of generalized topological spaces for each $\alpha \in \Lambda$, where Λ is indexing set. Let $\mathcal{H}$ be hereditary class on ΠX_α. If ΠX_α is T_2 mod $\mathcal{H}$, then (X_α, μ_α) is T_2 mod $P_\alpha(\mathcal{H})$ for each $\alpha \in \Lambda$, where P_α is the projection mapping for each α.*

Proof. Each X_α is (μ, ν)-homeomorphic to a subspace Y of the space ΠX_α, so each $P_\alpha|Y : Y \to X_\alpha$ is a (μ, ν)-homeomorphism. Since ΠX_α is T_2 mod $\mathcal{H}$, Y is T_2 mod $\mathcal{H}_Y$ and so X_α is T_2 mod $(P_\alpha|Y)(\mathcal{H}_Y)$. Since $(P_\alpha|Y)(\mathcal{H}_Y) \subset (P_\alpha)(\mathcal{H}_Y) \subset P_\alpha(\mathcal{H})$, therefore each (X_α, μ_α) is T_2 mod $P_\alpha(\mathcal{H})$.

Hamlett and Jankovic introduced and studied $\mathcal{I}$-regular topological spaces. Here we introduce $\mathcal{H}$-regularity for generalized topological spaces.

Definition 2.17. A hereditary space $(X, \mu, \mathcal{H})$ is said to be $\mathcal{H}$-regular if for each μ-closed set F and $x \notin F$, there exists disjoint μ-open sets U and V such that $x \in U$ and $F - V \in \mathcal{H}$.

Theorem 2.18. *Let $(X, \mu, \mathcal{H})$ be a hereditary space. Then the following are equivalent:*

1. *X is $\mathcal{H}$-regular.*
2. *For each $x \in X$ and μ-open set U containing x, there exists a μ-open set V containing x such that $cl_\mu(V) - U \in \mathcal{H}$.*
3. *For each $x \in X$ and μ-open set U containing x, there exists a μ-closed set F containing x such that $F - U \in \mathcal{H}$.*
4. *For each $x \in X$ and μ-closed set A not containing x, there exists a μ-open set V containing x such that $cl_\mu(V) \cap A \in \mathcal{H}$.*

Proof. $1 \Rightarrow 2$. Let $x \in X$ and U be a μ-open set containing x. Then $X - U$ is a μ-closed set not containing x. Since X is $\mathcal{H}$-regular, there exist disjoint μ-open sets V and W such that $x \in V$ and $(X-U)-W \in \mathcal{H}$. If $(X-U)-W = H \in \mathcal{H}$, then $(X - U) \subset W \cup H$. Since V and W are disjoint, $V \subset X - W$ implies $cl_\mu(V) \subset X - W$ and so $cl_\mu(V) - U \subset (X - W) - U = H \in \mathcal{H}$. Therefore $cl_\mu(V) - U \in \mathcal{H}$, this proves 2.

$2 \Rightarrow 4$. Let A be a μ-closed set in X such that $x \notin A$. Then by hypothesis, there exists a μ-open set V containing x such that $cl_\mu(V) - (X - A) \in \mathcal{H}$. Therefore $cl_\mu(V) \cap A \in \mathcal{H}$, which proves 4.

$4 \Rightarrow 1$. Let A be a μ-closed set in X such that $x \notin A$. Then by hypothesis, there is a μ-open set V containing x such that $cl_\mu(V) \cap A \in \mathcal{H}$. Then $A - (X - cl_\mu(V)) \in \mathcal{H}$. V and $X - cl_\mu(V)$ are disjoint μ-open sets such that $x \in V$ and so $A - (X - cl_\mu(V)) \in \mathcal{H}$. Hence X is $\mathcal{H}$-regular, which proves 1.

The equivalence of 2 and 3 is obvious.

Corollary 2.19. *Let $(X, \mu, \mathcal{H})$ be a hereditary space, where $\mathcal{H}$ be codense. Then the following are equivalent:*

1. *X is $\mathcal{H}$-regular.*
2. *For each $x \in X$ and μ-open set U containing x, there exists a μ-open set V containing x such that $V^* - U \in \mathcal{H}$.*
3. *For each $x \in X$ and μ-closed set A not containing x, there exists a μ-open set V containing x such that $V^* \cap A \in \mathcal{H}$.*

Proof. $\mathcal{H}$ is codense if and only if $cl(V) = V^*$ for every μ-open set V.

The following theorems show that $\mathcal{H}$-regularity is hereditary property as well as it is preserved under (μ,ν)-homeomorphism:

Theorem 2.20. *Let $(X,\mu,\mathcal{H})$ be a $\mathcal{H}$-regular hereditary space and $A \subset X$. Then $(A,\mu_A,\mathcal{H}_A)$ is $\mathcal{H}_A$-regular.*

Proof. Let $F \subset A$ be μ_A-closed in A and $x \in A$ such that $x \notin F$. Then $F = A \cap K$ where K is μ-closed in X and $x \notin K$. Since $(X,\mu,\mathcal{H})$ is $\mathcal{H}$-regular, there exists disjoint μ-open sets U and V such that $x \in U$ and $K - V \in \mathcal{H}$. So $A \cap U$ and $A \cap V$ are μ_A-open sets in A such that $x \in A \cap U$ and $(A \cap U) \cap (A \cap V) = (U \cap V) \cap A = \emptyset$. If $K - V = H \in \mathcal{H}$, then $K \subset H \cup V$ and so $F = A \cap K \subset A \cap (H \cup V) = (A \cap H) \cup (A \cap V)$ implies $F - (A \cap V) \subset A \cap H \in \mathcal{H}_A$. Therefore $F - (A \cap V) \in \mathcal{H}_A$. Hence $(A,\mu_A,\mathcal{H}_A)$ is $\mathcal{H}_A$-regular.

Theorem 2.21. *Let $(X,\mu,\mathcal{H})$ be a $\mathcal{H}$-regular hereditary space and let $f : (X,\mu,\mathcal{H}) \to (Y,\nu,f(\mathcal{H}))$ be a (μ,ν)-homeomorphism. Then $(Y,\nu,f(\mathcal{H}))$ is $f(\mathcal{H})$-regular.*

Proof. Let F be ν-closed in Y and $y \in Y$ such that $y \notin F$. Let $x = f^{-1}(y)$. Since f is (μ,ν)-continuous, $f^{-1}(F)$ is μ-closed in X not containing x. Since $(X,\mu,\mathcal{H})$ is $\mathcal{H}$-regular, there exist disjoint μ-open sets U and V in X such that $x \in U$ and $f^{-1}(F) - V \in \mathcal{H}$. Let $f^{-1}(F) - V = H \in \mathcal{H}$, so $f^{-1}(F) \subset H \cup V \Rightarrow f(f^{-1}(F)) \subset f(H \cup V) \Rightarrow F \subset f(H) \cup f(V) \Rightarrow F - f(V) \subset f(H) \in f(\mathcal{H}) \Rightarrow F - f(V) \in f(\mathcal{H})$. Therefore $f(U)$ and $f(V)$ are disjoint ν-open sets in Y such that $y \in f(U)$ and $F - f(V) \in f(\mathcal{H})$. Hence $(Y,\nu,f(\mathcal{H}))$ is $f(\mathcal{H})$-regular.

Theorem 2.22. *Let (X_α,μ_α) be a collection of generalized topological spaces for each $\alpha \in \Lambda$, where Λ is indexing set. Let $\mathcal{H}$ be hereditary class on ΠX_α. If ΠX_α is $\mathcal{H}$-regular, then (X_α,μ_α) is $P_\alpha(\mathcal{H})$-regular for each $\alpha \in \Lambda$, where P_α is the projection mapping for each α.*

Theorem 2.23. *Let $(X,\mu,\mathcal{H})$ be a $\mathcal{H}$-regular hereditary space and let x and y be two distinct points in X. Then either $cl_\mu(\{x\}) = cl_\mu(\{y\})$ or $cl_\mu(\{x\}) \cap cl_\mu(\{y\}) \in \mathcal{H}$.*

Proof. If $x \in cl_\mu(\{y\})$ and $y \in cl_\mu(\{x\})$. Then $cl_\mu(\{x\}) \subset cl_\mu(cl_\mu(\{y\})) = cl_\mu(\{y\}) \subset cl_\mu(cl_\mu(\{x\})) = cl_\mu(\{x\})$ and so $cl_\mu(\{x\}) = cl_\mu(\{y\})$. If $y \notin cl_\mu(\{x\})$. Since $(X,\mu,\mathcal{H})$ is $\mathcal{H}$-regular, there exists a μ-open set V containing y such that $cl_\mu(V) \cap cl_\mu(\{x\}) \in \mathcal{H}$, therefore $cl_\mu(\{x\}) \cap cl_\mu(\{y\}) \subset cl_\mu(\{x\}) \cap cl_\mu(V) \in \mathcal{H}$. Hence $cl_\mu(\{x\}) \cap cl_\mu(\{y\}) \in \mathcal{H}$. This completes the proof.

Theorem 2.24. *Let $(X,\mu,\mathcal{H})$ be a hereditary space. If each point of X has a μ-closed neighbourhood A which is $\mathcal{H}_A$-regular, then $(X,\mu,\mathcal{H})$ is $\mathcal{H}$-regular.*

Proof. Let $x \in X$ and U be a μ-open set containing x. Then there exists a μ-closed neighbourhood A of x, which is $\mathcal{H}_A$-regular. $A \cap U$ is μ_A-open set in A containing x. Therefore there is a μ_A-closed set F containing x in A such that $F - (A \cap U) \in \mathcal{H}_A$. Since F is μ_A-closed in A and A is μ-closed in X, F is μ-closed in X containing x and $F - U \subset F - (A \cap U) \in \mathcal{H}_A \subset \mathcal{H}$. Thus F is μ-closed set containing x in X such that $F - U \in \mathcal{H}$. Hence $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular.

Theorem 2.25. *Let $(X, \mu, \mathcal{H})$ be a hereditary space, which is $\mathcal{H}$-regular. Then for every nonempty set A and a μ-closed set F in X such that $F \cap A = \emptyset$, there exist disjoint μ-open sets U and V such that $A \cap U \neq \emptyset$ and $F - V \in \mathcal{H}$.*

Proof. Let $(X, \mu, \mathcal{H})$ be $\mathcal{H}$-regular. Let A be a nonempty set and F be a μ-closed set in X such that $F \cap A = \emptyset$. Then for each $x \in A$, there exist disjoint μ-open sets U and V such that $x \in U$ and $F - V \in \mathcal{H}$. Also $A \cap U \neq \emptyset$.

It is proved that the star topology with respect to a given ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular if the given ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular. That is if $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular then $(X, \tau^*, \mathcal{I})$ is $\mathcal{I}$-regular. But in case of generalized topological spaces, this result need not be true. That is if the hereditary space $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular then $(X, \mu^*, \mathcal{H})$ need not be $\mathcal{H}$-regular as shown in the following example:

Example 2.26. Let $X = \{a, b, c\}$, $\mu = \{\emptyset, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. Then the corresponding star generalized topology is $\mu^* = \{\emptyset, \{a, c\}, \{b, c\}, X\}$. Then $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular, but $(X, \mu^*, \mathcal{H})$ is not $\mathcal{H}$-regular, since $\{a\}$ is a μ^*-closed set not containing c and there do not exist disjoint μ^*-open sets U and V such that $c \in U$ and $\{a\} - V \in \mathcal{H}$.

Although there exist spaces where this result is true, shown in the following example:

Example 2.27. Let $X = \{a, b\}$, $\mu = \{\emptyset, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. Then the corresponding star generalized topology is $\mu^* = \{\emptyset, \{a\}, \{b\}, X\}$. Then $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular, as well as $(X, \mu^*, \mathcal{H})$ is $\mathcal{H}$-regular.

The following example shows that if the star generalized topology is $\mathcal{H}$-regular, then the given generalized topology need not be $\mathcal{H}$-regular. But if the hereditary class is codense, then this will hold. Therefore it can be concluded that $\mathcal{H}$-regularity of generalized topology and $\mathcal{H}$-regularity of star generalized topology do not depend on each other.

Example 2.28. Let $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{H} = \{\emptyset, \{b\}\}$. Then the corresponding star generalized topology is $\mu^* = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then $\{a, c\}$ is a μ-closed not containing b and there do not exist disjoint μ-open sets U and V such that $b \in U$ and $\{a, c\} - V \in \mathcal{H}$. Therefore $(X, \mu, \mathcal{H})$ is not $\mathcal{H}$-regular, but $(X, \mu^*, \mathcal{H})$ is $\mathcal{H}$-regular.

Theorem 2.29. *Let $(X, \mu, \mathcal{H})$ be a hereditary space such that $(X, \mu^*, \mathcal{H})$ is $\mathcal{H}$-regular and $\mathcal{H}$ is codense. Then $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular.*

Proof. Let A be μ-closed set in X such that $x \notin A$. Since $\mu \subset \mu^*$, A is μ^*-closed, therefore by Corollary 2.19, there exists a μ^*-open set V containing x such that $V^* \cap A \in \mathcal{H}$. Since V is μ^*-open set containing x, there exist $U \in \mu$ and $H \in \mathcal{H}$ such that $x \in U - H \subset V$. Therefore $U^* \subset V^*$ implies $U^* \cap A \subset V^* \cap A \in \mathcal{H}$ and so $U^* \cap A \in \mathcal{H}$. Hence $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular.

The hereditary space in Example 2.28 is a T_2 mod $\mathcal{H}$ space which is not $\mathcal{H}$-regular. The following example shows that the $\mathcal{H}$-regular space need not be T_2 mod $\mathcal{H}$. Hence T_2 mod $\mathcal{H}$ and $\mathcal{H}$-regular are independent concepts:

Example 2.30. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then $(X, \mu, \mathcal{H})$ is $\mathcal{H}$-regular, which is not T_2 mod $\mathcal{H}$.

Acknowledgement

I am thankful to Dr. Navpreet Singh Noorie for his kind guidance and help.

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Accepted: 20.03.2019