Rough set theory applied to UP-algebras

Moin A. Ansari*
Ali N.A. Koam
Azeem Haider
Department of Mathematics
College of Science
Post Box 2097, New Campus
Jazan University
Jazan, KSA
maansari@jazanu.edu.sa
akoum@jazanu.edu.sa
aahaider@jazanu.edu.sa

Abstract. In this paper, the concept of roughness in UP-algebras is introduced. We study the lower and upper approximations of UP-subalgebras and UP-ideals and prove that the lower/upper approximation of UP-subalgebra (resp., UP-ideals) is a UP-subalgebra (resp., UP-ideals). A connection between rough sets and UP-Algebras with their weak and strong ideals have also been taken under consideration and some related results have been shown.

Keywords: UP-algebra, UP-ideal, congruence, lower and upper approximations, definable.

1. Introduction

The notion of rough sets and its approximations (lower/upper) spaces were introduced in early 1980s by Pawlak in his papers [26], [27] and [28] to deal with uncertain knowledge in information system, artificial intelligence and cognitive sciences in fields such as machine learning, knowledge aquisition, decision analysis etc. As an R and D (research and developement) a connection between rough set theory with algebraic systems (structures) came into existence. As a result many authors introduced lot of concepts. Kuroki [12] introduced roughness in semigroups and its ideals which can be considered to be concept of roughness in classical algebras. Then Biswas and Nanda [18] introduced the notion of rough groups and rough subgroups. Xiao and Zhang [15] studied rough prime ideals and rough fuzzy prime ideals in semigroups. The basic logical algebras have been established and investigated widely by many authors. Li. and Yin [7] defined ϑ-lower and T-upper fuzzy rough approximation operators on a semigroup whereas Qi. and Liu. [16] studied the cocepts of rough appoximations in Boolean Algebras. Davvaz [3] introduced the concept of roughness in rings.

$*$. Corresponding author

Hu and Li. [14] gone through BCH Algebras. Prabpayak and Leerawat [19] introduced KU-ideals which can be consider to be an interesting idea in logical algebras, further they studied homomorphisms of KU-Algebras [6] and investigated some related results and properties of KU-algebras. Yaqoob et al. [13] introduced cubic KU-ideals which further became an interesting direction for study of diffrent types of classical properties, logical properties, fuzzificational (intuitionistic) properties, Neutrosophic properties in modern modern algebra. Ameri et al. [17] , studied rough set theory applied to hyper BCK-Algebras, where Dudek et al. [21] studied rough set theory applied to hyper BCI-Algebras. Further Jun et al. [23] gone through the concepts of roughness in BCC-algebras.

Fuzzy sets was introduced by Zadeh [25] in 1965. From then till now many authors have considered and studied fuzziness in different branches of sciences technologies and engineering. Fuzzyness [10], Neutrosophic [4] and in different types of Logical Algebras [22], [20], [11] are some recent trend and interest of study for numerous researchers and authors. Including numerous authors Moin and Ali [11] have studied roughness in KU-algebras recently. Ahn [2] et. al have studied rough fuzzy ideals in BCK/BCI algebras.

Both logical and classical algebras have been the basic building tools in the study of different types and directions of applied algebras more precisely in computer applications related to artificial intelligence which simulate a human being in dealing with certainty and uncetainty in information with the help of logical techniques. Tasks related to these concepts can easily be solved by these techniques. Some imoprtant types of basic logical algebras are BCI/BCK algebras, BL-algebras [8] and many more. Torkzadeh and Ghorbani [9] studied rough filters in B-Algebras. Different types of these algebras are perfect BL-algebras and local BL-algebras, SBL-algebras etc. which have been studied by many authors. These concepts are an interesting part for roughness and softness with or without their hyper structures.

We have applied roughness concept in UP-algebras which is introduced recently by Impan [1]. Some classical results based on rough set theory is applied to UP-algebras and related results have been studied based on this concept. By means of lower and upper approximations we have shown properties of rough ideals of a UP-algebra. It is shown that a strong UP-ideal with respect to its upper and lower approximation of a UP-algebra is again a strong ideal.

2. Preliminaries

In this section we shall define some basic concepts including UP-algebras, UP-subalgebras, UP-ideals with examples based on them.

Definition 1 ([6]). *An algebra* $(X, *, 0)$ *of type* $(2, 0)$ *with a single binary operation* $*$ *that satisfies the following identities: for any* $x, y, z \in X$,

$(ku1) : (x * y) * [(y * z) * (x * z)] = 0,$

$(ku2) : x * 0 = 0,$

$(ku3): 0 * x = x$,
$(ku4): x * y = 0 = y * x$ *implies* $x = y$.

In a KU-algebra X a binary relation $' \leq'$ can be considered by: $x \leq y$ if and only if $y * x = 0$.

Proposition 1 ([6])**.** $(X, *, 0)$ *is a KU-algebra if and only if it satisfies:*
$(ku5): (y * z) * (x * z) \leq (x * y)$,
$(ku6): 0 \leq x$,
$(ku7): x \leq y,\ y \leq x$ *implies* $x = y$,
$(ku8): x \leq y$ *if and only if* $y * x = 0$.

Proposition 2. *In a KU-algebra, the following identities are true [20]*
(1) $z * z = 0$,
(2) $z * (x * z) = 0$,
(3) $x \leq y$ *imply* $y * z \leq x * z$,
(4) $z * (y * x) = y * (z * x)$, *for all* $x, y, z \in X$,
(5) $y * [(y * x) * x] = 0$.

Definition 2 ([1])**.** *By a UP-algebra we mean an algebra* $(A, *, 0)$ *of type* $(2, 0)$ *with a single binary operation* $*$ *that satisfies the following identities: for any* $x, y, z \in X$,
$(UP-1): (y * z) * [(x * y) * (x * z)] = 0$,
$(UP-2): 0 * x = x$,
$(UP-3): x * 0 = 0$,
$(UP-4): x * y = 0 = y * x$ *implies* $x = y$.

Example 1 ([1])**.** Let X be a universal set. Define a binary operation $*$ on the power set of X by putting $A * B = B \cap A' = A' \cap B = B - A$ for all $A, B \in P(X)$. Then $(P(X); *, \emptyset)$ is a UP-algebra which is the power UP-algebra of type 1.

Example 2 ([1])**.** Let X be a universal set. Define a binary operation $*$ on the power set of X by putting $A * B = B \cup A' = A' \cup B\ \forall\ A, B \in P(X)$. Then $(P(X); *, X)$ is a UP-algebra which is a power UP-algebra of type 2.

Example 3. Let $A = \{0, a, b, c\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

It is easy to see that $A = \{0, a, b, c\}$ is UP-algebra.

Example 4. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	0
b	0	b	0	0	0
c	0	b	b	0	0
d	0	b	b	d	0

Here $A = \{0, a, b, c, d\}$ is UP-algebra.

Example 5. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Here $A = \{0, a, b, c, d\}$ is UP-algebra.

Example 6. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	0
b	0	a	0	c	0
c	0	a	0	0	0
d	0	a	b	c	0

Here $A = \{0, a, b, c, d\}$ is UP-algebra.

Proposition 3. *In a UP-algebras A the following properties hold for any $x, y, z \in A$:*

(1) $x * x = 0$,
(2) $x * y = 0$ *and* $y * z = 0 \Rightarrow x * z = 0$,
(3) $x * y = 0 \Rightarrow (z * x) * (z * y) = 0$,
(4) $x * y = 0 \Rightarrow (y * z) * (x * z) = 0$,
(5) $x * (y * x) = 0$,
(6) $(y * x) * x = 0 \iff x = y * x$, *and*
(7) $x * (y * y) = 0$

Proposition 4. *Let $A = (A, *, 0)$ be UP-algeras, then define a binary relation $\leq$ on A as follows: for all $x, y, z \in A$*

$x \leq y \Leftrightarrow x * y = 0$. *Based on this binary relation we have that in an UP-algebra A, the following properties are true for any $x, y, z \in A$:*

(1) $x \leq x$,
(2) $x \leq y$ *and* $y \leq x \Rightarrow x = y$,
(3) $x \leq y$ *and* $y \leq z \Rightarrow x \leq z$,
(4) $x \leq y \Rightarrow z * x \leq z * y$,
(5) $x \leq y \Rightarrow y * z \leq x * z$,
(6) $x \leq y * x$, *and*
(7) $x \leq y * y$.

Definition 3. *Let $A = (A, *, 0)$ be a UP-algebra. Then a subset S of A is called UP-subalgebras of A if the constant 0 of A is in S and $(S, *, 0)$ itself form a UP-algebra. Clearly, A and $\{0\}$ are UP-algebras of A.*

Definition 4. *Let A be a UP-algebra. Then a subset B of A is called a UP-ideal of A if it satisfies:*
(i) *The constant 0 of A is in B and*
(ii) *for ant $x, y, z \in A$, $x * (y * z) \in B$ and $y \in B \Rightarrow x * z \in B$.*
Clearly, A and $\{0\}$ are UP-ideals of A.

Example 7. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	0	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

We find here that $A = (A; *, 0)$ is a UP-algebras. Further $\{0, a, b\}$ and $\{0, a, c\}$ are UP-ideals of A.

Definition 5. *Let S be a nonempty subset of a UP-algebra A and $0 \in S$. Then,*
(1) *S is called a weak UP-ideal of A if $y * x \in S$ and $y \in S \Rightarrow x \in S$, for all $x, y \in A$;*
(2) *S is called a strong UP-ideal of A if $(y * x) \cap S \neq \emptyset$ and $y \in S \Rightarrow x \in S$, for all $x, y \in A$.*

3. Rough approximations in UP-algebras

Let V be a set and E an equivalence relation on V and let $P(V)$ denote the power set of V. For all $a \in V$, let $[a]_E$ denote the equivalence class of a with respect to E. Define the functions $E_-, E^- : P(V) \to P(V)$ as follows: $\forall\ S \in P(V)$,

$$E_-(S) = \{a \in V : [a]_E \subseteq S\}$$

and

$$E^-(S) = \{x \in V : [a]_E \cap S \neq \emptyset\}.$$

The pair (V, E) is called an approximation space. Let S be a subset of V. Then S is said to be definable if $E_{-}(S) = E^{-}(S)$ and rough otherwise. $E_{-}(S)$ is called the lower approximation of S while $E^{-}(S)$ is called the upper approximation.

Throughout this section A will represent a UP-algebra. Let I be a UP-ideal of A. Define a relation Θ on A by $(a, b) \in \Theta$ if and only if $a * b \in I$ and $b * a \in I$. Then Θ is an equivalence relation on A related to a UP-ideal I of A. Moreover satisfies $(a, b) \in \Theta$ and $(u, v) \in \Theta$ imply $(a * u, b * v) \in \Theta$.

Hence Θ is a congruence relation on A. Let I_a denote the equivalence class of a with respect to the equivalence relation Θ related to a UP-ideal I of A, and A/I denote the collection of all equivalence classes, that is, $A/I = \{I_a : a \in A\}$. Then $I_0 = I$. If $I_a * I_b$ is defined as the class containing $a * b$, that is, $I_a * I_b = I_{a*b}$, then $(A/I, *, I_0)$ is a UP-algebra. Let Θ be an equivalence relation on A related to a UP-ideal I of A. For any nonempty subset S of A, the lower and upper approximation of S are denoted by $\underline{\Theta}(I, S)$ and $\overline{\Theta}(I, S)$ respectively, that is,

$$\underline{\Theta}(I, S) = \{a \in A : I_a \subseteq S\}$$

and

$$\overline{\Theta}(I, S) = \{a \in a : I_a \cap S \neq \emptyset\}.$$

If $I = S$, then $\underline{\Theta}(I, S)$ and $\overline{\Theta}(I, S)$ are denoted by $\underline{\Theta}(I)$ and $\overline{\Theta}(I)$, respectively.

Definition 6 ([28])**.** Given an approximation space (U, Θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough set in (U, Θ) if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$.

Definition 7 ([28])**.** Let (U, Θ) be an approximation space and X be a nonempty subset of U.

(i) If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called definable.
(ii) If $\underline{Apr}(X) = \emptyset$, then X is called empty interior.
(iii) If $\overline{Apr}(X) = U$, then X is called empty exterior.

Example 8. Let $A = \{0, a, b, c, d\}$ be a set in which $*$ is defined by the following cayley table

$*$	0	a	b	c	d
0	0	0	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

We find here that $A = (A; *, 0)$ is a UP-algebras. Further $\{0, a, b\}$ and $\{0, a, c\}$ are UP-ideals of A.

So $I = \{0, a\}$ is a UP-ideal of A ($I \lhd A$) and let Θ be an equivalence relation on A related to I. Then $I_0 = K_1 = K$, $K_2 = \{2\}$, $K_3 = \{3\}$, and $K_4 = \{4\}$.

Hence

$$\begin{aligned}\underline{\Theta}(K,\{0,a\}) &= \{0,a\} \lhd A\\ \underline{\Theta}(K,\{0,b\}) &= \{b\}\\ \underline{\Theta}(K,\{0,c\}) &= \{c\}\\ \underline{\Theta}(K,\{0,a,b,c\}) &= \{0,a,b,c\} \lhd A\end{aligned}$$

and

$$\begin{aligned}\overline{\Theta}(K,\{0,a\}) &= \{0,a\} \lhd A\\ \overline{\Theta}(K,\{0\}) &= \{0,a\}\\ \overline{\Theta}(K,\{b\}) &= \{0,b\}\\ \overline{\Theta}(K,\{a,b,c\}) &= \{0,a,b,c\} \lhd A\\ \overline{\Theta}(K,\{0,b,c\}) &= \{0,a,b,c\} \lhd A\\ \overline{\Theta}(K,\{a,b,c,d\}) &= \{0,a,b,c,d\} \lhd A.\end{aligned}$$

In above example 8, we know that there exists a non-UP-ideal S of A such that their lower and upper approximation are UP-ideals of A. Also we choose some non-UP-ideals S of A such that their lower and upper approximation are UP-ideals of A.

Proposition 5. *Let Θ and Ξ be equivalence relations on A related to UP-ideals I and J of A, respectively. If S and T are nonempty subsets of A. Then*

(1) $\underline{\Theta}(I,S) \subseteq S \subseteq \overline{\Theta}(I,S)$;
(2) $\underline{\Theta}(I,\emptyset) = \emptyset = \overline{\Theta}(I,\emptyset)$
(3) $\overline{\Theta}(I,S\cup T) = \overline{\Theta}(I,S) \cup \overline{\Theta}(I,T)$;
(4) $\underline{\Theta}(I,S\cap T) = \underline{\Theta}(I,S) \cap \underline{\Theta}(I,T)$;
(5) $S \subseteq T$ *implies* $\underline{\Theta}(I,S) \subseteq \underline{\Theta}(I,T)$ *and* $\overline{\Theta}(I,S) \subseteq \overline{\Theta}(I,T)$;
(6) $\underline{\Theta}(I,S) \cup \underline{\Theta}(I,T) \subseteq \underline{\Theta}(I,S\cup T)$;
(7) $\overline{\Theta}(I,S\cap T) \subseteq \overline{\Theta}(I,S) \cap \overline{\Theta}(I,T)$;
(8) $\Theta \subseteq \Xi$ *and* $I \subseteq J$ *implies* $\underline{\Xi}(J,S) \subseteq \overline{\Theta}(I,S)$ *and* $\overline{\Theta}(I,S) \subseteq \overline{\Xi}(J,S)$.

Proof. (1) If $x \in \underline{\Theta}(I,S)$, then $x \in I_x \subseteq S$. Hence $\underline{\Theta}(I,S) \subseteq$S. Next, if $x \in S$, then, since $x \in I_x$, we have $I_x \cap S \neq \phi$, and so $x \in \overline{\Theta}(I,S)$. Thus $S \subseteq \overline{\Theta}(I,S)$.

(2) is straightforward.

(3) Note that

$$\begin{aligned}x \in \overline{\Theta}(I,S\cup T) &\iff I_x \cap (S\cup T) \neq \phi\\ &\iff (I_x\cap S)\cup(I_x\cap T) \neq \phi\\ &\iff I_x\cap S \neq \phi \ \ or\ I_x \cap T \neq \phi\\ &\iff x \in \overline{\Theta}(I,S) \ or\ a \in \overline{\Theta}(I,T)\\ &\iff x \in \overline{\Theta}(I,S)\cup\overline{\Theta}(I,T).\end{aligned}$$

Thus

$$\overline{\Theta}(I,S\cup T) = \overline{\Theta}(I,S)\cup\overline{\Theta}(I,T).$$

(4) Note that

$$\begin{aligned} x \in \underline{\Theta}(I, S \cap T) &\iff I_x \subseteq S \cap T \\ &\iff I_x \subseteq S \ \ \textit{and} \ \ I_x \subseteq T \\ &\iff x \in \underline{\Theta}(I, S) \textit{ and } x \in \underline{\Theta}(I, T) \\ &\iff x \in \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T). \end{aligned}$$

Thus

$$\underline{\Theta}(I, S \cap T) = \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T).$$

(5) Since $S \subseteq T$ if and only if $S \cap T = S$, by (3) we have

$$\underline{\Theta}(I, S) = \underline{\Theta}(I, S \cap T) = \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T).$$

This implies that $\underline{\Theta}(I, S) \subseteq \underline{\Theta}(I, T)$. Note also that $S \subseteq T$ if and only if $S \cup T = T$, by (2) we have

$$\overline{\Theta}(I, T) = \overline{\Theta}(I, S \cup T) = \overline{\Theta}(I, S) \cup \overline{\Theta}(I, T).$$

This implies that $\overline{\Theta}(I, S) \subseteq \overline{\Theta}(I, T)$.

(6) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, by (4) we have

$$\underline{\Theta}(I, S) \subseteq \underline{\Theta}(I, S \cup T) \quad \text{ and } \quad \underline{\Theta}(I, T) \subseteq \underline{\Theta}(I, S \cup T).$$

This implies $\underline{\Theta}(I, S) \cup \underline{\Theta}(I, T) \subseteq \underline{\Theta}(I, S \cup T)$.

(7) Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, by (4) we have

$$\overline{\Theta}(I, S \cap T) \subseteq \overline{\Theta}(I, S) \quad \text{ and } \quad \overline{\Theta}(I, S \cap T) \subseteq \overline{\Theta}(I, T).$$

This implies $\overline{\Theta}(I, S \cap T) \subseteq \overline{\Theta}(I, S) \cap \overline{\Theta}(I, T)$.

(8) Since $\Theta \subseteq \Xi$. If $x \in \underline{\Xi}(J, S)$, then $J_x \subseteq S$. But $\Theta \subseteq \Xi$, then $I_x \subseteq J_x \subseteq S$, that is, $I_x \subseteq S$. Thus $x \in \underline{\Theta}(I, S)$. Hence

$$\underline{\Xi}(J, S) \subseteq \underline{\Theta}(I, S).$$

Now let x be any element of $\overline{\Theta}(S)$. So $I_x \cap S \neq \phi$, then there exists $y \in I_y \cap S$ such that $y \in I_y$ and $y \in S$. Hence $(y, x) \in \Theta$, that is $y * x \in I$. Since $I \subseteq J$, it follows that $y * x \in J$ and $x * y \in J$ so that $(y, x) \in \Xi$, that is, $y \in J_x$. Therefore $y \in J_x \cap S$, which means that $x \in \Xi(J, S)$. This completes the proof. □

Proposition 6. *Let Θ be an equivalence relation on A related to a UP-ideal I of A. If S is a nonempty subset of A. Then*

(1) $\underline{\Theta}(I, \underline{\Theta}(I, S)) = \underline{\Theta}(I, S)$;
(2) $\overline{\Theta}(I, \overline{\Theta}(I, S)) = \overline{\Theta}(I, S)$;
(3) $\overline{\Theta}(I, \underline{\Theta}(I, S)) = \underline{\Theta}(I, S)$;
(4) $\underline{\Theta}(I, \overline{\Theta}(I, S)) = \overline{\Theta}(I, S)$;
(5) $\underline{\Theta}(I, S) = (\overline{\Theta}(I, S^c))^c$;
(6) $\overline{\Theta}(I, S) = (\underline{\Theta}(I, S^c))^c$;
(7) $\underline{\Theta}(I, I_x) = X = \overline{\Theta}(I, I_x)$, *for all $x \in A$.*

Proof. The proof is straightforward. □

Proposition 7. *Let Θ be an equivalence relation on X related to a UP-ideal I of A. If S is a nonempty subset of A. Then*

(1) $\overline{\Theta}(I,S) * \overline{\Theta}(I,T) \subseteq \overline{\Theta}(I,S*T)$;

(2) If Θ is congruence relation, then $\underline{\Theta}(I,S) * \underline{\Theta}(I,T) \subseteq \underline{\Theta}(I,S*T)$.

Proof. (1) Let c be any element of $\overline{\Theta}(I,S) * \overline{\Theta}(I,T)$. Then $c = p * q$ with $p \in \overline{\Theta}(I,S)$ and $q \in \overline{\Theta}(I,T)$. Thus there exist elements $x, y \in S$ such that

$$x \in I_p \cap S \quad \text{and} \quad y \in I_q \cap T.$$

Thus $x \in I_p$, $y \in I_q$, $x \in S$, and $y \in T$. Since Θ is a congruence on S, it follows that

$$x * y \in I_p * I_q \in I_{p*q}.$$

On the other hand, since $x * y \in S * T$. We have $x * y \in I_{p*q} \cap S * T$, and so $c = p * q \in \overline{\Theta}(I, S * T)$. Thus we have

$$\overline{\Theta}(I,S) * \overline{\Theta}(I,T) \subseteq \overline{\Theta}(I,S*T).$$

(2) Assume that Θ is complete, let c be any element of $\underline{\Theta}(I,S) * \underline{\Theta}(I,T)$. Then $c = p * q$ with $p \in \underline{\Theta}(I,S)$ and $q \in \underline{\Theta}(I,T)$. It follows that $I_p \subseteq S$ and $I_q \subseteq T$. Since Θ is a congruence relation on S, we have

$$I_{p*q} = I_p * I_q \subseteq S * T.$$

So $c = p * q \in \underline{\Theta}(I, S * T)$. Thus

$$\underline{\Theta}(I,S) * \underline{\Theta}(I,T) \subseteq \underline{\Theta}(I,S*T).$$

This completes the proof. □

Proposition 8. *Let Θ and Ξ be equivalence relations on A related to UP-ideals I and J of A, respectively. If S and T are nonempty subsets of A. Then*

(1) $\overline{\Theta \cap \Xi}(I \cap J, S) \subseteq \overline{\Theta}(I,S) \cap \overline{\Xi}(J,S)$;

(2) $\underline{\Theta \cap \Xi}(I \cap J, S) \supseteq \underline{\Theta}(I,S) \cap \underline{\Xi}(J,S)$.

Proof. (1) Note that $\Theta \cap \Xi$ is also a congruence relation on S. Let $c \in \overline{\Theta \cap \Xi}(I \cap J, S)$, then $[I \cap J]_c \cap S \neq \phi$. Then there exists an element $x \in [I \cap J]_c \cap S$. Since $(x, c) \in \Theta \cap \Xi$, we have

$$(x,c) \in \Theta \text{ and } (x,c) \in \Xi.$$

Thus we have $x \in I_c$ and $x \in J_c$. Since $x \in S$, we have $x \in I_c$, $x \in S$ and $x \in J_c$, $x \in S$. This implies that

$$x \in I_c \cap S \quad \text{and} \quad x \in J_c \cap S$$

$$I_c \cap S \neq \phi \quad \text{and} \quad J_c \cap S \neq \phi.$$

So $c \in \overline{\Theta}(I,S)$ and $c \in \overline{\Xi}(J,S)$, hence $c \in \overline{\Theta}(I,S) \cap \overline{\Xi}(J,S)$. Thus we obtain

$$\overline{\Theta \cap \Xi}(I \cap J, S) \subseteq \overline{\Theta}(I,S) \cap \overline{\Xi}(J,S).$$

(2) Since $\Theta \cap \Xi \subseteq \Theta$ and $\Theta \cap \Xi \subseteq \Xi$, which implies that

$$\underline{\Theta}(I,S) \subseteq \underline{\Theta \cap \Xi}(I \cap J, S) \quad \text{and} \quad \underline{\Xi}(J,S) \subseteq \underline{\Theta \cap \Xi}(I \cap J, S)$$

$$\Longrightarrow \underline{\Theta}(I,S) \cap \underline{\Xi}(J,S) \subseteq \underline{\Theta \cap \Xi}(I \cap J, S).$$

This completes the proof. □

Theorem 1. *Let (A, Θ) be an approximation space. Then*
(1) *for every $S \subseteq X$, $\underline{\Theta}(I,S)$ and $\overline{\Theta}(I,S)$ are definable sets,*
(2) *for every $x \in X$, I_x is definable set.*

Proof. (1) By Proposition 6 part (1) and (3), we have

$$\underline{\Theta}(I, \underline{\Theta}(I,S)) = \underline{\Theta}(I,S) = \overline{\Theta}(I, \underline{\Theta}(I,S)).$$

Hence $\underline{\Theta}(I,S)$ is definable. On the other hand by Proposition 6 (2) and (4), we have

$$\overline{\Theta}(I, \overline{\Theta}(I,S)) = \overline{\Theta}(I,S) = \underline{\Theta}(I, \overline{\Theta}(I,S)).$$

Therefore $\overline{\Theta}(I,S)$ is a definable set.

(2) By Proposition 6 (7) the proof is clear. □

Definition 8. *A nonempty subset S of A is called an upper (resp. a lower) rough UP-subalgebra of A if the upper (resp. nonempty lower) approximation of S is a UP-subalgebra of A. If S is both an upper and a lower rough UP-subalgebra of A, we say that S is a rough UP-subalgebra of A.*

Theorem 2. *Let Θ be an congruence relation on A related to a UP-ideal I of A. If S is a UP-subalgebra of I, then*
(1) *$\overline{\Theta}(I,S)$ is a UP-subalgebra of A.*
(2) *$\underline{\Theta}(I,S)$ is a UP-subalgebra of A.*

Proof. (1) Let $x, y \in \overline{\Theta}(I,S)$. Then

$$I_x \cap S \neq \emptyset \text{ and } I_y \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in I_x$ and $b \in I_y$. It follows that $(a, x) \in \Theta$ and $(b, y) \in \Theta$. Since Θ is a congruence relation on A, we have $(a * b, x * y) \in \Theta$. Hence $a * b \in I_{x*y}$. Since S is a UP-subalgebra of A, we get $a * b \in S$, and therefore $a * b \in I_{x*y} \cap S$, that is, $I_{x*y} \cap S \neq \emptyset$. This shows that $x * y \in \overline{\Theta}(I,S)$, and consequently $\overline{\Theta}(I,S)$ is a UP-subalgebra of A.

(2) Let $x, y \in \underline{\Theta}(I,S)$. Then $I_x \subseteq S$ and $I_y \subseteq S$. Since S is a UP-subalgebra of A, it follows that

$$I_{x*y} = I_x * I_y \subseteq S$$

so that $x * y \in \underline{\Theta}(I,S)$. Hence $\underline{\Theta}(I,S)$ is a UP-subalgebra of A. □

The following example shows that the converse of Theorem 2(1) may not be true.

Example 9. Let $A = \{0, a, b, c, d\}$ be a UP-algebra with the Cayley's table as follows:

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	b	d
b	0	0	0	a	d
c	0	0	0	0	d
d	0	a	a	a	0

Let $I = \{0, a, b\}$ be a UP-ideal of A ($I \lhd A$) and let Θ be an equivalence relation on A related to I. Then $I_0 = I_a = I_b = I$, $I_c = \{c\}$, and $I_d = \{d\}$. Note that $S = \{a, c\}$ is not a UP-subalgebra of A, but $\overline{\Theta}(I, S) = \{0, a, b, c\}$ is UP-subalgebra of A.

Definition 9. *A nonempty subset S of A is called an upper (resp. a lower) rough UP-ideal of A if the upper (resp. nonempty lower) approximation of S is a UP-ideal of A. If S is both an upper and a lower rough UP-ideal of A, we say that S is a rough UP-ideal of A.*

Theorem 3. *Let Θ be a congruence relation on A related to a UP-ideal I of A. If S is a UP-ideal of A containing I, then*

(1) *$\overline{\Theta}(I, S)$ is a UP-ideal of A.*

(2) *$\underline{\Theta}(I, S)$ is a UP-ideal of A.*

Proof. (1) Let S be a UP-ideal of A containing I. Obviously $0 \in \overline{\Theta}(I, S)$. Let $x, y, z \in A$ be such that $y \in \overline{\Theta}(I, S)$ and $x * (y * z) \in \overline{\Theta}(I, S)$. Then

$$I_y \cap S \neq \emptyset \text{ and } I_{x*(y*z)} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in I_y$ and $b \in I_{x*(y*z)}$. Hence $(a, y) \in \Theta$ and $(b, (x*(y*z))) \in \Theta$, which implies $y*a \in A \subseteq S$ and $(x*(y*z))*b \in I \subseteq S$. Since $a, b \in S$ and S is a UP-ideal, we get

$$y \in S \text{ and } x * (y * z) \in S,$$

it follows from Definition 5 (2) that $x * z \in S$. Note that $x * z \in I_{x*z}$, thus $x*z \in I_{x*z} \cap S$, that is, $I_{x*z} \cap S \neq \emptyset$. Hence $x*z \in \overline{\Theta}(I, S)$ and therefore $\overline{\Theta}(I, S)$ is a UP-ideal of A.

(2) Let S be a UP-ideal of A containing I. Let $x \in I_0$. Then $x \in I \subseteq S$, and so $I_0 \subseteq S$. Hence $0 \in \underline{\Theta}(I, S)$. Let $x, y, z \in X$ be such that $y \in \underline{\Theta}(I, S)$ and $x * (y * z) \in \underline{\Theta}(I, S)$. Then

$$I_y \in S \text{ and } I_x * (I_y * I_z) = I_{x*(y*z)} \subseteq S.$$

Let $w \in I_{x*z} = I_x * I_z$. Then $w = I_x * I_z$ for some $a \in I_x$ and $c \in I_z$. From $a \in I_x$ and $c \in I_z$, we have $(a, x) \in \Theta$ and $(c, z) \in \Theta$. Taking $b \in I_y$ then we get $(b, y) \in \Theta$. Since Θ is a congruence relation, we get

$$(a * (b * c), x * (y * z)) \in \Theta \text{ and so } a * (b * c) \in I_{x*(y*z)} \subseteq S.$$

Since S is a UP-ideal of A, it follows from Definition 5 (2) that $w \in a * c \in S$, so that $I_{x*z} \subseteq S$. Hence $x * z \in \underline{\Theta}(I, S)$ and therefore $\underline{\Theta}(I, S)$ is a UP-ideal of A. □

Theorem 4. *Let Θ be an congruence relation on A related to a UP-ideal I of A. If S is a weak UP-ideal of A containing I, then*

(1) *$\overline{\Theta}(I, S)$ is a weak UP-ideal of A.*

(2) *$\underline{\Theta}(I, S)$ is a weak UP-ideal of A.*

Proof. (1) Let S be a weak UP-ideal of A containing I. Obviously $0 \in \overline{\Theta}(I, S)$. Let $x, y \in A$ be such that $y \in \overline{\Theta}(I, S)$ and $y * x \in \overline{\Theta}(I, S)$. Then

$$I_y \cap S \neq \emptyset \text{ and } I_{y*x} \cap S \neq \emptyset,$$

and so there exist $a, b \in S$ such that $a \in I_y$ and $b \in I_{y*x}$. Hence $(a, y) \in \Theta$ and $(b, (y * x)) \in \Theta$, which implies

$$y * a \in I \subseteq S \text{ and } (y * x) * b \in I \subseteq S.$$

Since $a, b \in S$ and S is a weak UP-ideal, we get $y \in S$ and $y * x \in S$, it follows from definition 5 (2) that $x \in S$. Note that $x \in I_x$, thus $x \in I_x \cap S$, that is, $I_x \cap S \neq \emptyset$. Hence $x \in \overline{\Theta}(I, S)$ and therefore $\overline{\Theta}(I, S)$ is a weak UP-ideal of A.

(2) Let S be a weak UP-ideal of A containing I. Let $x \in I_0$. Then $x \in I \subseteq S$, and so $I_0 \subseteq S$. Hence $0 \in \underline{\Theta}(I, S)$. Let $x, y \in A$ be such that $y \in \underline{\Theta}(I, S)$ and $y * x \in \underline{\Theta}(I, S)$. Then

$$I_y \in S \text{ and } I_y * I_x = I_{y*x} \subseteq S.$$

Let $w \in I_x$. Then $w = I_x$ for some $a \in I_x$. From $a \in I_x$, we have $(a, x) \in \Theta$. Taking $b \in I_y$ then we get $(b, y) \in \Theta$. Since Θ is a congruence relation, we get

$$(b * a, y * x) \in \Theta \text{ and } b * a \in I_{y*x} \subseteq S.$$

Since S is a weak UP-ideal of A, it follows from definition 5 (2) that $w = a \in S$, so that $I_x \subseteq S$. Hence $x \in \underline{\Theta}(I, S)$ and therefore $\underline{\Theta}(I, S)$ is a weak UP-ideal of A. □

Theorem 5. *Let Θ be an congruence relation on A related to a UP-ideal I of A. If S is a strong UP-ideal of A containing I, then*

(1) *$\overline{\Theta}(I, S)$ is a strong UP-ideal of A.*

(2) *$\underline{\Theta}(I, S)$ is a strong UP-ideal of A.*

Proof. (1) Let $x, y \in X$ be such that

$$(y * x) \cap \overline{\Theta}(I, S) \neq \emptyset \text{ and } y \in \overline{\Theta}(I, S).$$

Then $I_y \cap S \neq \emptyset$ and so there exist $z \in A$ such that $z = y * x$ and $z \in \overline{\Theta}(I, S)$. Hence $I_z \cap S \neq \emptyset$ and so there exist $c, d \in X$ such that

$$c \in I_z \cap S \text{ and } d \in I_y \cap S.$$

Hence $c\Theta z$ and $d\Theta y$ where $c, d \in S$. Thus we $z * c \in I \subseteq S$ and $y * d \in I \subseteq S$. Since S is a strong UP-ideal and $c, d \in S$, we have $z \in S$ and $y \in S$. Thus we have proved that $(y*x) \cap I \neq \emptyset$ and $y \in I$. Since S is a strong UP-ideal, we have $x \in S$ and so $I_x \cap S \neq \emptyset$ which means that $\overline{\Theta}(I, S)$ is a strong UP-ideal of S.

(2) Let $x, y \in X$ be such that

$$(y * x) \cap \underline{\Theta}(I, S) \neq \emptyset \text{ and } y \in \underline{\Theta}(I, S).$$

Let $a \in I_x$ and $b \in I_y$. Then $a\Theta x$ and $b\Theta y$. Since Θ is a congruence relation on A, $b * a\Theta y * x$. Since $(y * x) \cap \underline{\Theta}(I, S) \neq \emptyset$, then there exist $t \in A$ such that $t \in y * x$ and $t \in \underline{\Theta}(I, S)$. Now, $t \in b * a\Theta y * x$ implies that there exist $z \in b * a$ such that $z\Theta t$ and so $I_t = I_z \subseteq S$. Hence $z \in S$ and so $(b * a) \cap S \neq \emptyset$. On the other hand, we have $b \in I_y \subseteq S$. Since S is a strong UP-ideal of A, then we have $a \in S$ which implies $I_x \subseteq S$ that means $x \in \underline{\Theta}(I, S)$. Therefore, $\underline{\Theta}(I, S)$ is a strong UP-ideal of S. □

4. Conclusion

We have studied the connection between rough sets and UP-algebras. We have presented definitions and examples of the lower and upper approximations of a UP-algebra and UP-subalgebras with respect to UP-ideals. In the future further study is possible in the direction of roughness with different types of ideals in UP-algebras.

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Accepted: 7.07.2018