# On asymptotically lacunary statistical equivalent (Wijsman sense) set sequences via ideal and modulus function

**Wadei Al-Omeri**
*Department of Mathematics*
*Al-Balqa Applied University*
*Salt, 19117*
*Jordan*
*wadeimoon1@hotmail.com*

**Abstract.** For any non-trivial ideal $\mathcal{I} \subseteq \wp(\mathbb{N})$, and any non-empty closed subset $A_k \subseteq X$, where $(X, \rho)$ is a metric space. Let $f$ be a modulus function, The objective of this paper is to introduce and study the new notation by using a modulus function, $\mathcal{I}(W_L)$, $\mathcal{I}(fW)$, $\mathcal{I}[WN]^L_\theta$, $\mathcal{I}(Wf^L)$, $\mathcal{I}(^W_\theta f)$, $\mathcal{I}(WS^L)$ and $\mathcal{I}(WS^L_\theta)$. Which are natural combinations of the definitions for asymptotically lacunary equivalent ($Wijsman$ *sense*). In addition, some relations among these new notions are also obtained.

## 1. Introduction and preliminaries

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [7] (see, also [26]). as follows: if $K$ is a subset of natural numbers $\mathbb{N}$, $\mathrm{K}_n$ will denote the set $\{\kappa \in \mathrm{K} : \kappa \leqslant n\}$ and $|\mathrm{K}_n|$ will denote the cardinality of $\mathrm{K}_n$. Natural density of K [9] is given by

$$\delta(\mathrm{K}) := \lim_n \frac{1}{n}|\mathrm{K}_n|$$

, if $\bar{\delta}(\mathrm{K}) = \underline{\delta}(\mathrm{K})$ then we say that the natural density of K exists and it is denoted simply by $\delta(\mathrm{K})$. A sequence $(x_n)$ of real numbers is said to be statistically convergent to $L$ if for arbitrary $\varepsilon > 0$, the set $\mathrm{K}(\varepsilon) = n \in N : |x_n - L| \geq \varepsilon$ has natural density zero. The concept of statistical convergence plays an important role in the summability theory and functional analysis. The relationship between the summability theory and statistical convergence has been introduced by Schoenberg [26]. In [3], Borwein introduced and studied strongly summable functions. The concept of Wijsman statistical convergence is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades in 2012. Similar to this concept, the concept of Wijsman lacunary statistical convergence was presented by Ulusu and Nuray in 2012. Kostyrko et al. [11] generalized statistical convergence with the help of an admissible ideal $\mathcal{I}$ of subsets of $\mathbb{N}$, the set of positive integers and called it $\mathcal{I}$-convergence. Quite recently, Savaş et al. [25] unified the notions of statistical convergence and $\mathcal{I}$-convergence to introduce new concepts of $\mathcal{I}$-statistical convergence.

In order to compare the rate of growth of two sequences, Marouf [16] defined asymptotically equivalent sequences of real numbers and studied its relations with certain matrix-transformed sequences. In 2003, Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In 2006, many authors have shown their interest to solve different problems arising in this area. In this work, we define asymptotically equivalent sequences using lacunary sequences, ideals, Wijsman sense and a modulus function and obtain some revelent connections between these notions.

## 2. Preliminaries

**Definition 2.1** ([16]). *Two nonnegative sequences $x = (x_n)$ and $y = (y_n)$ are said to be asymptotically equivalent to multiple $l$ provided that*

$$\lim_{\kappa \to \infty} \frac{x_n}{y_n} = L,$$

*(denoted by $x \sim y$) and simply asymptotically equivalent if L=1.*

**Definition 2.2** ([21]). *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$.*

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ \kappa \leq n : \left| \frac{x_n}{y_n} - L \right| \geq \varepsilon \right\} \right| = 0.$$

*(denoted by $x \sim^{S_L} y$)*

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $K_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Let, $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. Using lacunary sequences, Fridy et al. [8] defined $S_\theta$-convergence, a generalized statistical convergence as follows.

**Definition 2.3** ([22]). *Let $\theta = k_r$ be a lacunary sequence, the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$.*

$$\lim_{r} \frac{1}{h_r} \left| \left\{ \kappa \in I_r : \left| \frac{x_n}{y_n} - L \right| \geq \varepsilon \right\} \right| = 0.$$

*(denoted by $x \sim^{S_\theta^L} y$) and simply asymptotically lacunary statistically equivalent if L=1.*

**Definition 2.4** ([22]). *Let $\theta = k_r$ be a lacunary sequence, the two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are strongly asymptotically lacunary equivalent of multiple L provided that.*

$$\lim_{r} \frac{1}{h_r} \sum_{\kappa \in I_r} \left| \frac{x_n}{y_n} - L \right| = 0.$$

*(denoted by* $x \sim^{N_\theta^L} y$*) and strongly simply asymptotically lacunary equivalent if* $L = 1$.

Let $(X, \sigma)$ be a metric space. For any point $x \in X$ and any nonempty subset $A$ of $X$, we define the distance from $x$ to $A$ by

$$d(x, A) = \inf_{a \in A} \sigma(x, A).$$

**Definition 2.5** ([1])**.** *Let* $(X, \sigma)$ *be a metric space. For any nonempty closed subset* $A$, $A_k \subseteq X$, *we say that the sequence* $\{A_k\}$ *is Wijsman convergent to* $A$ *if*

$$\lim_{k \to \infty} d(x, A_k) = d(x, A),$$

*for each* $x \in X$ *In this case we write* $W - \lim A_k = A$.

**Definition 2.6** ([18])**.** *Let* $(X, \sigma)$ *a metric space. For any nonempty closed subset* $A$, $A_k \subseteq X$, *we say that the sequence* $\{A_k\}$ *is Wijsman statistically convergent to* $A$ *if* $\{d(x, A_k)\}$ *is statistically convergent to* $d(x, A)$*; that is, for* $\varepsilon > 0$ *and for each* $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} |\{\kappa \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

*In this case we write* $st - \lim_W A_k = A$ *or* $A_k \to A(Ws)$.

Also the concept of bounded sequence for sequences of sets was given by Nuray and Rhoades.

**Definition 2.7** ([18])**.** *Let* $(X, \sigma)$ *a metric space. For any nonempty closed subset,* $A_k \subseteq X$, *we say that the sequence* $\{A_k\}$ *is bounded if*

$$\sup_k d(x, A_k) < \infty,$$

*for each* $x \in X$. *In this case we write* $\{A_k\} \in L_\infty$.

**Definition 2.8** ([27])**.** *Let* $(X, \sigma)$ *a metric space and let* $\theta = \{K_r\}$ *be a lacunary sequence. For any nonempty closed subset* $A$, $A_k \subseteq X$, *we say that the sequence* $\{A_k\}$ *is Wijsman statistically convergent to* $A$ *if* $\{d(x, A_k)\}$ *is statistically convergent to* $d(x, A)$*; that is, for* $\varepsilon > 0$ *and for each* $x \in X$,

$$\lim_{h_r \to \infty} \frac{1}{h_r} |\{\kappa \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

*In this case we write* $S_\theta - \lim_W = A$ *or* $A_k \to A(WS_\theta)$.

**Definition 2.9** ([10, 28])**.** *family* $I \in \wp(X)$ *is said to be an ideal in* $X$ *if*

1. $\emptyset \in \mathcal{I}$;

2. $A \in \mathcal{I}$ *and* $B \subset A$ *implies* $B \in \mathcal{I}$;

3. $A \in \mathcal{I}$ *and* $B \in \mathcal{I}$ *implies* $A \cup B \in \mathcal{I}$.

**Definition 2.10.** *A non-empty family* $\mathcal{F} \in \wp(X)$ *is said to be a filter in* $X$ *if*

1. $\emptyset \notin \mathcal{F}$;

2. $A \in \mathcal{F}$ *and* $A \subset B$ *implies* $B \in \mathcal{F}$;

3. $A \in \mathcal{F}$ *and* $B \in \mathcal{F}$ *implies* $A \cap B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called admissible if it contains all the singleton sets. Moreover, if $\mathcal{I}$ is a non-trivial ideal on $X$, then $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is a filter on $X$ and conversely. The filter $\mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

Using ideals, Kostyrko et al. [11] defined $\mathcal{I}$-convergence, a stronger convergence in a metric space, whereas Dass et al. [6] unified this idea with statistical convergence for real sequences.

**Definition 2.11** ([11])**.** *Let* $\mathcal{I} \in \wp(x)$ *be a non-trivial ideal in* $\mathbb{N}$ *and* $(X, \sigma)$ *be a metric space. A sequence* $x \in (x_k)$ *in* $X$ *is said to be* $\mathcal{I}$*-convergence to* $\xi$ *if for each* $\epsilon > 0$, *the set*

$$A(\epsilon) = \left\{ K \in \mathbb{N} : \rho(x_k, \xi) \geq \epsilon \right\} \in \mathcal{I}.$$

**Definition 2.12** ([25])**.** *Let* $\mathcal{I} \in \wp(x)$ *be a non-trivial ideal in* $\mathbb{N}$ *and* $(X, \sigma)$ *be a metric space. A sequence* $x \in (x_k)$ *in* $X$ *is said to be* $\mathcal{I}$*-statistically convergence to* $\xi$ *provided that for each* $\epsilon > 0$ *and every* $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{\kappa \leq n : |x_k - \xi| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

*In this case we write* $\mathcal{I}$-$S - \lim_{k\to\infty} x_k = \xi$ *or* $x_k \to \xi(\mathcal{I} - S)$

**Definition 2.13** ([13])**.** *Let* $\mathcal{I} \in \wp(x)$ *be a non-trivial ideal in* $\mathbb{N}$ *the two non-negative sequences* $x = (x_k)$ *and* $y = (y_k)$ *are said to be asymptotically* $\mathcal{I}$-$[C, 1]$-*equivalent of multiple* $L$ *provided that for each* $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{\kappa=1}^{n} \left| (\frac{x_n}{y_n}) - L \right| \geq \delta \right\} \in \mathcal{I}$$

(*denoted by* $x \sim^{[C,1]^{L_\mathcal{I}}} y$) *and simply asymptotically* $\mathcal{I}$-$[C, 1]$-*equivalent if* $L = 1$.

**Definition 2.14** ([12])**.** *Let* $\mathcal{I} \in \wp(x)$ *be a non-trivial ideal in* $\mathbb{N}$ *and let* $\theta = \{K_r\}$ *be a lacunary sequence. The non-negative sequences* $x = (x_k)$ *and* $y =$

$(y_k)$ *are said to be asymptotically lacunary statistical equivalent multiple* $L$ *with respect to the ideal* $\mathcal{I}$ *provided that for each* $\epsilon > 0$ *and* $\delta \geq 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_n}{y_n} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}$$

(*denoted by* $x \sim^{\mathcal{I}(S_\theta)} y$) *and simply asymptotically lacunary statistical equivalent with respect to the ideal if* $L = 1$.

**Definition 2.15** ([13]). *Let* $\mathcal{I} \in \wp(x)$ *be a non-trivial ideal in* $\mathbb{N}$. *The non-negative sequences* $x = (x_k)$ *and* $y = (y_k)$ *are said to be asymptotically* $\mathcal{I}$-*statistical equivalent multiple* $L$ *provided that for each* $\epsilon > 0$ *and* $\delta \geq 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|(\frac{x_n}{y_n}) - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}$$

(*denoted by* $x \sim^{S^{L(\mathcal{I})}} y$) *and simply asymptotically* $\mathcal{I}$-*statistical equivalent if* $L = 1$.

Nakano [17] introduced the notion of a modulus function in 1953 as follows. By a modulus function, we mean a function $f$ from $[0, \infty)$ to $[0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$;
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$;
3. $f$ is increasing;
4. $f$ is continuous from the right at 0;

It follows that $f$ must be continuous on $[0, 1)$. A modulus may be bounded or unbounded. Many authors, including Connor [5], Kolk [14], Maddox [15], Ö ztürk et al. [20], Pehlivan et al. [23, 24] and many others used a modulus $f$ to construct some sequence spaces. Recently, Bilgin [4] used modulus function to define some notions of asymptotically equivalent sequences and studied some of their connections.

**Definition 2.16** ([19]). *Let* $(X, \sigma)$ *be a metric space and let* $\theta = \{K_r\}$ *be a lacunary sequence. For any nonempty closed subset* $A_k$, $B_k \subseteq X$, *such that* $d(x, A_k) > 0$ *and* $d(x, B_k) > 0$ *each* $x \in X$. *We say that the sequence* $\{A_k\}$, $\{B_k\}$ *are asymptotically equivalent* ($Wijsman$ *sense*) *if for each* $x \in X$,

$$\lim_{\kappa \to \infty} \frac{d(x, A_k)}{d(x, B_k)} = L,$$

(*denoted by* $A_k \sim B_k$).

**Definition 2.17** ([19]). *Let $(X, \sigma)$ be a metric space. For any nonempty closed subset $A_k$, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are asymptotically statistically equivalent $(Wijsman\ sense)$ of multiple $L$ if for each $x \in X$,*

$$\lim_n \frac{1}{n}\left|\left\{\kappa \leq n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\}\right| = 0$$

*(denoted by $\{A_k\} \sim^{WS_L} \{B_k\}$) and simply asymptotically statistical equivalent $(Wijsman\ sense)$ if $L$ = 1.*

**Definition 2.18** ([19]). *Let $(X, \sigma)$ be a metric space and let $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset $A_k$, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are asymptotically lacunary equivalent $(Wijsman\ sense)$ of multiple $L$ if for each $x \in X$,*

$$\lim_r \frac{1}{h_r} \sum_{\kappa \in I_r} \left|\frac{d(x, A_k)}{d(x, A_k)}\right| = L$$

*(denoted by $\{A_k\} \sim^{WN_\theta^L} \{B_k\}$) and simply asymptotically lacunary equivalent $(Wijsman\ sense)$ if $L$ = 1.*

**Definition 2.19** ([19]). *Let $(X, \sigma)$ a metric space and let $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset $A_k$, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly asymptotically lacunary equivalent $(Wijsman\ sense)$ of multiple $L$ if for each $x \in X$,*

$$\lim_r \frac{1}{h_r} \sum_{\kappa \in I_r} \left|\frac{d(x, A_k)}{d(x, A_k)} - L\right| = 0$$

*(denoted by $\{A_k\} \sim^{[WN]_\theta^L} \{B_k\}$) and simply strongly asymptotically lacunary equivalent $(Wijsman\ sense)$ if $L = 1$.*

**Definition 2.20** ([19]). *Let $(X, \sigma)$ a metric space and let $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset $A_k$, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are asymptotically lacunarily statistical equivalent $(Wijsman\ sense)$ of multiple $L$ if for each $x \in X$,*

$$\lim_r \frac{1}{h_r}\left|\left\{\kappa \in I_r : \left|\frac{d(x, A_k)}{d(x, A_k)} - L\right| \geq \varepsilon\right\}\right| = 0$$

*(denoted by $\{A_k\} \sim^{WS_\theta^L} \{B_k\}$) and simply asymptotically lacunarily statistical equivalent $(Wijsman\ sense)$ if $L = 1$.*

## 3. Results and discussion

We now consider our main results. We begin with the following definitions.

**Definition 3.1.** *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, $(X, \sigma)$ a metric space. For any nonempty closed subset $A_k$, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly asymptotically equivalent $(Wijsman\ sense)$ of multiple $L$ with respect to the ideal $\mathcal{I}$ provided that for each $\delta \geq 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \delta \right\} \in \mathcal{I}$$

*(denoted by $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\}$) and simply strongly asymptotically equivalent with respect to the ideal $(Wijsman\ sense)$, if $L = 1$.*

**Definition 3.2.** *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$ and $(X, \sigma)$ a metric space. The nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent $(Wijsman\ sense)$ multiple $L$ with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$ and $\delta \geq 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

*(denoted by $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$) and simply asymptotically statistical equivalent $(Wijsman\ sense)$with respect to the ideal $\mathcal{I}$, if $L = 1$.*

**Definition 3.3.** *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, $(X, \sigma)$ a metric space and $\theta = \{K_r\}$ be a lacunary sequence. The nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent $(Wijsman\ sense)$ multiple $L$ with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$ and $\delta \geq 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

*(denoted by $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$ and simply asymptotically lacunary statistical equivalent $(Wijsman\ sense)$with respect to the ideal $\mathcal{I}$, if $L = 1$.*

**Definition 3.4.** *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, $(X, \sigma)$ a metric space and $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset $A_k$, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly asymptotically lacunary statistically equivalent $(Wijsman\ sense)$ of multiple $L$ with respect to the ideal $\mathcal{I}$ provided that for each $\delta \geq 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \in \mathcal{I}$$

*(denoted by $\{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$) and simply strongly asymptotically lacunary statistically equivalent $(Wijsman\ sense)$ with respect to the ideal $\mathcal{I}$, if $L = 1$.*

**Definition 3.5.** *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, $(X, \sigma)$ a metric space and $f$ be a modulus function. For any nonempty closed subset $A_k$, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are $f$-asymptotically equivalent ($Wijsman$ $sense$) of multiple $L$ with respect to the ideal $\mathcal{I}$ provided that for each $\varepsilon \geq 0$,*

$$\left\{k \in \mathbb{N} : f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}$$

*(denoted by $\{A_k\} \sim^{\mathcal{I}(f_W)} \{B_k\}$ and simply $f$-asymptotically equivalent (Wijsman sense) with respect to the ideal $\mathcal{I}$, if $L = 1$.*

**Definition 3.6.** *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, $(X, \sigma)$ a metric space and $f$ be a modulus function and $\theta = \{K_r\}$ be a lacunary sequence. For any nonempty closed subset $A_k$, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly $f$-asymptotically lacunary equivalent ($Wijsman$ $sense$) of multiple $L$ with respect to the ideal $\mathcal{I}$ provided that for each $\varepsilon \geq 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}$$

*(denoted by $\{A_k\} \sim^{\mathcal{I}(^{W}_{\theta} f)} \{B_k\}$ and simply strongly $f$-asymptotically lacunary equivalent ($Wijsman$ $sense$) with respect to the ideal $\mathcal{I}$, if $L = 1$.*

**Definition 3.7.** *Let $\mathcal{I} \in \wp(x)$ be a non-trivial ideal in $\mathbb{N}$, $(X, \sigma)$ a metric space. For any nonempty closed subset $A_k$, $B_k \subseteq X$, such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ each $x \in X$. We say that the sequence $\{A_k\}$, $\{B_k\}$ are strongly $f$-asymptotically equivalent ($Wijsman$ $sense$) of multiple $L$ with respect to the ideal $\mathcal{I}$ provided that for each $\delta \geq 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \geq \epsilon\right\} \in \mathcal{I}$$

*(denoted by $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$) and simply strongly $f$-asymptotically equivalent with respect to the ideal ($Wijsman$ $sense$), if $L = 1$.*

**Theorem 3.8** ([2, 24])**.** *Let $f$ be a modulus function and let $0 < \delta < 1$. Then for $y \neq 0$ and each $\left(\frac{x}{y}\right) > \delta$, we have $f\left(\frac{x}{y}\right) \leq \frac{2f(1)}{\delta}\left(\frac{x}{y}\right)$.*

**Theorem 3.9.** *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $(X, \rho)$ be a metric space and $f$ be a modulus function. Then,*

1. *If $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\}$ then $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$ and*
2. *$\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$ then $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\} \Leftrightarrow \{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$.*

**Proof.** (1) Let $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\}$ and $\epsilon \geq 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. We can write

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = \frac{1}{n}\sum_{k=1} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) + \frac{1}{n}\sum_{k=2} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right),$$

where the first summation runs over $\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \leq \delta$, and the complement of first a summation is $\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| > \delta$. Moreover, using the definition of the modulus function f , we have

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) < \epsilon + \left(\frac{2f(1)}{\delta}\right)\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|.$$

Thus, for any $\gamma > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \gamma\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{(\gamma-\epsilon)\delta}{2f(1)}\right\}.$$

Since $\{A_k\} \sim^{\mathcal{I}(W_L)} \{B_k\}$, it follows the final set, and hence, the first set in above expression belongs to $\mathcal{I}$. This prove that $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$.

(2) If $\lim_{k\to\infty} \frac{f(k)}{k} = \beta > 0$, then we have $f(k) \geq k\beta$ for all $k > 0$. Know suppose that $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$. Since

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{1}{n}\sum_{k=1}^{n} \beta\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = \beta\left(\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)$$

it follows that for each $\epsilon > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \beta\epsilon\right\}.$$

Since $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$ $\epsilon > 0$ it follows that the later set belongs to $\mathcal{I}$, from (1) and (2), the theorem is proved. □

**Theorem 3.10.** *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $(X,\rho)$ be a metric space and $f$ be a modulus function. Then,*

1. *If $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$ then $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$ and*
2. *If $f$ is bounded, then $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\} \Leftrightarrow \{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$.*

**Proof.** (1) Suppose $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$, and let $\epsilon \geq 0$ be given, then we can write

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|}}^{} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)$$

$$\geq \frac{f(\varepsilon)}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon\right\}\right|.$$

Consequently, for $\gamma > 0$, we have

$$\left\{n \in \mathbb{N} : \left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon\right\}\right| \geq \frac{\gamma}{f(\epsilon)}\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \gamma\right\}.$$

Since $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$, it follows by Definition 3.7 that the

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \gamma\right\} \in \mathcal{I},$$

and therefore $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$.

(2) Suppose $f$ is bounded and $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$. Since $f$ is bounded, there exists a real number $M$ such that $\sup f(t) \leq M$. Moreover, for $\epsilon > 0$, we can write

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) = \frac{1}{n}\Bigg[ \sum_{\substack{k=1 \\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon}}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)$$

$$+ \sum_{\substack{k=1 \\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|<\epsilon}}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\Bigg]$$

$$\leq \frac{M}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{\epsilon}{2}\right\}\right| + f(\frac{\epsilon}{2}).$$

Thus if we denote the sets

$$B(\epsilon) = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right\}$$

and

$$A(\epsilon) = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{\epsilon}{2}\right\}\right| \geq \frac{f(\epsilon)}{2M}\right\},$$

then we have the inclusion $B(\epsilon) \subset A(\epsilon)$. Since $\{A_k\} \sim^{\mathcal{I}(WS^L)} \{B_k\}$ it follows that $A(\epsilon) \in \mathcal{I}$, and therefore $B(\epsilon) \in \mathcal{I}$, by applying the operators $\epsilon \to 0$ This shows that $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$. $\square$

**Theorem 3.11.** *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $(X, \rho)$ be a metric space, $f$ be a modulus function and $\theta = \{K_r\}$ be a lacunary sequence. If $\liminf_r q_r > 1$, then $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\} \Rightarrow \{A_k\} \sim^{\mathcal{I}(^W_\theta f)} \{B_k\}$.*

**Proof.** Suppose $\liminf_r q_r > 1$, then there exist $\delta > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta$. This implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$. Let $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$. For a sufficiently large $r$, we obtain the following:

$$\begin{aligned}
\frac{1}{k_r}\sum_{k=1}^{k_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) &\geq \frac{1}{k_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\\
&= \left(\frac{h_r}{k_r}\right)\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\\
&\geq \left(\frac{\delta}{1+\delta}\right)\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right).
\end{aligned}$$

Which gives for $\epsilon > 0$

$$\begin{aligned}
&\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \epsilon\right\}\\
\subseteq &\left\{r \in \mathbb{N} : \frac{1}{k_r}\sum_{k=1}^{k_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{\epsilon\delta}{1+\delta}\right\}.
\end{aligned}$$

Since $\{A_k\} \sim^{\mathcal{I}(Wf^L)} \{B_k\}$, its followes that the set

$$\left\{r \in \mathbb{N} : \frac{1}{k_r}\sum_{k=1}^{k_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \frac{\epsilon\delta}{1+\delta}\right\} \in \mathcal{I},$$

and hence the set

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \epsilon\right\} \in \mathcal{I}.$$

This shows that $\{A_k\} \sim^{\mathcal{I}(^W_\theta f)} \{B_k\}$. □

**Theorem 3.12.** *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $(X, \rho)$ be a metric space and $f$ be a modulus function. Then,*

1. *If $\{A_k\} \sim^{\mathcal{I}[WN]^L_\theta} \{B_k\}$ then $\{A_k\} \sim^{\mathcal{I}(^W_\theta f)} \{B_k\}$ and*
2. *$\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$ then $\{A_k\} \sim^{\mathcal{I}[WN]^L_\theta} \{B_k\} \Leftrightarrow \{A_k\} \sim^{\mathcal{I}(^W_\theta f)} \{B_k\})$.*

**Proof.** The proof is similar to the proof of Theorem 3.9 □

**Theorem 3.13.** *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $\theta = \{K_r\}$ be a lacunary sequence, $(X, \rho)$ be a metric space and $f$ be a modulus function. Then,*

1. *If* $\{A_k\} \sim^{\mathcal{I}(^{W}_{\theta} f)} \{B_k\}$ *then* $\{A_k\} \sim^{\mathcal{I}(WS^L_\theta)} \{B_k\}$ *and*

2. *If* $f$ *is bounded, then* $\{A_k\} \sim^{\mathcal{I}(^{W}_{\theta} f)} \{B_k\} \Leftrightarrow \{A_k\} \sim^{\mathcal{I}(WS^L_\theta)} \{B_k\}$.

**Proof.** (1) Suppose $\{A_k\} \sim^{\mathcal{I}(^{W}_{\theta} f)\{B_k\}}$, and let $\epsilon \geq 0$ be given, then

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) &\geq \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon}} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\\ &\geq \frac{f(\varepsilon)}{h_r}\left|\left\{k\in I_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon\right\}\right|.\end{aligned}$$

For $\eta > 0$, if we denote sets

$$\begin{aligned}&\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\left\{k\in I_r\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \eta\right\}\\ &\subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \eta f(\epsilon)\right\}.\end{aligned}$$

Since $\{A_k\} \sim^{\mathcal{I}[WN]^L_\theta} \{B_k\}$, so

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) \geq \eta f(\epsilon)\right\} \in \mathcal{I}.$$

By the definition of ideal

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\left\{k\in I_r\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \eta\right\} \in \mathcal{I}$$

and therefore $\{A_k\} \sim^{\mathcal{I}(WS^L_\theta)} \{B_k\}$. (2) Suppose $f$ is bounded and $\{A_k\} \sim^{\mathcal{I}(WS^L_\theta)} \{B_k\}$. Since $f$ is bounded, there exists a real number $M > 0$ such that $|f(t)| \leq M$ for all $t \geq 0$, by using this fact

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right) &= \frac{1}{h_r}\Bigg[\sum_{\substack{k\in I_r \\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\epsilon}} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\\ &+ \sum_{\substack{k\in I_r \\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|<\epsilon}} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\Bigg]\\ &\leq \frac{M}{h_r}\left|\left\{k\in I_r : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{\epsilon}{2}\right\}\right| + f(\frac{\epsilon}{2}).\end{aligned}$$

Thus if we denote the sets

$$B(\epsilon) = \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\right)\right\}$$

and

$$A(\epsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \frac{\epsilon}{2} \right\} \right| \geq \frac{f(\epsilon)}{2M} \right\},$$

then we have the inclusion $B(\epsilon) \subset A(\epsilon)$. Since $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$ it follows that $A(\epsilon) \in \mathcal{I}$, and therefore $B(\epsilon) \in \mathcal{I}$, by applying the operators $\epsilon \to 0$ This shows that $\{A_k\} \sim^{\mathcal{I}(^W_\theta f)} \{B_k\}$. □

**Theorem 3.14.** *Let $\mathcal{I} \subset \wp(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $(X, \rho)$ be a metric space, $\theta = \{K_r\}$ be a lacunary sequence and let $A_k$, $B_k \subseteq X$ be a non-empty closed subset of $X$.*

1. *If $\{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$ then $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$ and $\{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$ is proper.*
2. *If $\{A_k\}, \{B_k\} \in \mathcal{L}_\infty$ such that $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$, then $\{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$.*
3. *$\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\} \cap \ell_\infty = \{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\} \cap \ell_\infty$.*

*Here $\ell_\infty$ denotes the class of bounded sequences.*

**Proof.** (1) Suppose $A_k$, $B_k \subseteq X$ be a non-empty closed subset of $X$ such that $\{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$ . We show that $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$. Let $\epsilon > 0$. Since

$$\begin{aligned}\sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| &\geq \sum_{\substack{k=1 \\ \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \epsilon}} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \\ &\geq \varepsilon . \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \epsilon \right\} \right|,\end{aligned}$$

given $\delta > 0$

$$\frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \epsilon \right\} \right| \textit{implies that } \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \epsilon\delta.$$

$$\begin{aligned}&\left\{ n \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \epsilon \right\} \right| \geq \delta \right\} \\ \subseteq &\left\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \epsilon\delta \right\}.\end{aligned}$$

Since $\{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$, it follows by the definition that the set in the right side belong to $\mathcal{I}$ which immediately implies

$$\left\{ n \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \epsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}.$$

This shows that $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$.

We next give an example to show that the inclusion $\{A_k\} \sim^{\mathcal{I}[WN]_\theta^L} \{B_k\} \subset \{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$ is proper. Consider the sequences $A_k$, $B_k$ defined by

$$A_k = \begin{cases} \{k\},, & \text{if } k_{r-1} < k < k_{r-1} + \left[\sqrt{h_r}\right] r = 1, 2, ... \\ \{0\}, & \text{otherwise} \end{cases}$$

$\{B_k\} = 0$ for all $k$. Then clearly $\{A_k\} \notin \ell_\infty$. We have for every $\epsilon > 0$ and for each $x \in X$,

$$\frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \epsilon\right\}\right| = \frac{\left[\sqrt{h_r}\right]}{h_r} \to 0 \,\text{as}\, r \to \infty.$$

That is $\{A_k\} \sim^{\mathcal{I}(WS_\theta^1)} \{B_k\}$. On the other hand,

$$\frac{1}{h_r}\sum_{k \in I_r}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \nrightarrow 0 \,\text{as}\, r \to \infty.$$

Hence $\{A_k\} \nsim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$. □

(2) *Suppose* $A_k$, $B_k \in \ell_\infty$ *such that* $\{A_k\} \sim^{\mathcal{I}(WS_\theta^L)} \{B_k\}$. *Then we can find a number* $M$ *such that*

$$\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \leq M,$$

*for all* $k \in \mathbb{N}$. *Given* $\epsilon > 0$, *We have*

$$\begin{aligned} \frac{1}{h_r}\sum_{k \in I_r}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \epsilon}} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \\ &+ \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| < \epsilon}} f\left(\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right|\right) \\ &\leq \frac{M}{h_r}\left|\left\{k \in I_r : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \frac{\epsilon}{2}\right\}\right| + \frac{\epsilon}{2}. \end{aligned}$$

*This shows that* $\{A_k\} \nsim^{\mathcal{I}[WN]_\theta^L} \{B_k\}$.

(3) *This immediately follows from* (1) *and* (2).

## 4. Conclusions

We observe that if the modulus function $f$ satisfies $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then the notions $\mathcal{I}(W_L)$ and $\mathcal{I}[WN]_\theta^L$ respectively coincide with the notions $\mathcal{I}(Wf^L)$ and $\mathcal{I}(^W_\theta f)$. However, if f is bounded, the notions $\mathcal{I}(Wf^L)$ and $\mathcal{I}(^W_\theta f)$ coincides respectively with the notions $\mathcal{I}(WS^L)$ and $\mathcal{I}(WS_\theta^L)$.

**References**

[1] M. Baronti and P. L. Papini, *Convergence of sequences of sets*, in Methods of Functional Analysis in Approximation Theory, Birkhäuser, Basel, Switzerland, 76 (1986), 135-155.

[2] M. Baśarir, S. Altundağ, *On asymptotically equivalent difference sequences with respect to a modulus function*, Ricerche Mat., 60 (2011), 299-311.

[3] D. Borwein, *Linear functionals with strong Cesaro summability*, Journal London Math.Soc., 40 (1965), 628-634.

[4] T. Bilgin, *$f$-Asymptotically lacunary equivalent sequences*, Acta Univ. Apulensis., 28 (2011), 271-278.

[5] J. S. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., 32 (1989), 194-198.

[6] P. Das, E. Savas and S. Ghosal, *On generalizations of certain summability methods using ideals*, Appl. Math. Lett., 24 (2011), 1509-1514

[7] H. Fast, *Sur la convergence ststistique*, Colloq. Math., 2 (1951) 241-244.

[8] J. A. Fridy and C. Orhan, *Lacunary statistical convergent*, Pacific J. Math., 160 (1993), 43-51.

[9] H. Halberstem, K. F. Roth, *Sequence*, Springer-Verlag, New York, 1993.

[10] K. Kuratowski, *Topology*, Vol. I. NewYork: Academic Press, 1966.

[11] P. Kostyrko, T. Salat and W.Wilczynski, *$\mathcal{I}$-convergence*, Real Anal. Exchange, 26 (2) (2000/2001), 669-686.

[12] V. Kumar and A. Sharma, *Asymptotically lacunary equivalent sequences defined by ideals and modulus function*, Mathematical Sciences., 6 (2012), 1-5.

[13] V. K. Kaushik, A. Sharma, *On asymptotically generalized statistical equivalent sequences via ideals*, Tamkang Journal of Mathematics., 43 (2012), 469-478.

[14] E. Kolk, *On strong boundedness and summability with respect to a sequence of moduli*, TartuUl. Toimetised., 960 (1993), 41-50.

[15] I.J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Camb. Phil. Soc., 100 (1986), 161-166.

[16] M. Marouf, *Asymptotic equivalence and summability,* Internat. J. Math. Math. Sci., 16 (1993), 755-762.

[17] H. Nakano, *Concave modulars*, J. Math. Soc. Japan., 5 (1953), 29-49.

[18] F. Nuray and B. E. Rhoades, *Statistical convergence of sequences of sets*, Fasciculi Mathematici., 49 (2012), 87-99.

[19] F. Nuray, *On asymptotically lacunary statistical equivalent set sequences*, Journal of Mathematics, 2013.

[20] E. Öztürk, T. Bilgin, *Strongly summable sequence spaces defined by a modulus*, Indian J. Pure Appl. Math., 25 (1994), 621-625.

[21] R. F. Patterson, *On asymptotically statistical equivalent sequences*, Demonstratio Mathematica., 36 (2003), 149-153.

[22] R. F. Patterson and E. Savaş, *On asymptotically lacunary statistically equivalent sequences*, Thai Journal of Mathematics, 4 (2006), 267-272.

[23] S. Pehlivan, B. Fisher, *On some sequence spaces*, Indian J. Pure Appl. Math., 25 (1994), 1067-1071.

[24] S. Pehlivan, B. Fisher, *Some sequence spaces defined by a modulus*, Math. Slovaca., 45 (1995), 275-280.

[25] E. Savaş, P. Das, *A generalized statistical convergence via ideals,* Appl.Math. Lett., 24 (2011), 826-830.

[26] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, 66 (1959), 361-375.

[27] U. Ulusu and F. Nuray, *Lacunary statistical convergence of sequence of sets*, Progress in Applied Mathematics., 4 (2012), 99-109.

[28] R. Vaidyanathaswamy, *Set topology*, Chelsea Publishing Company, 1960.

Accepted: 18.04.2018