On the Diophantine equation $X^4 + hY^3 = Z^4 + hW^3$

S.D. Alavi
Department of Mathematics
Urmia University
Urmia 57561-51818
Iran
dalir_alavi@yahoo.com

A.S. Janfada*
Department of Mathematics
Urmia University
Urmia 57561-51818
Iran
asjanfada@gmail.com
a.sjanfada@urmia.ac.ir

A. Abbaspour
Department of Mathematics
Urmia University
Urmia 57561-51818
Iran
ali3507reza@gmail.com

Abstract. First we prove that, under a certain condition, the Diophantine equation $X^4 + hY^3 = Z^4 + hW^3$, where h is a rational number, has infinitely many nontrivial integral solutions. Using this result we deduce that, under a condition, for $k \geq 2$, the Diophantine equation $X^4+Y_1^3+Y_2^3+\cdots+Y_k^3 = Z^4+W_1^3+W_2^3+\cdots+W_k^3$ has infinitely many nontrivial integral solutions. Then, we conjecture, by some evidences, that the above conditions may be removed.

Keywords: diophantine equation, elliptic curve

1. Introduction

One of the ancient challenging problems is to find the rational solutions as well as the number of solutions of the symmetric Diophantine equations. A symmetric Diophantine equation in n variables is of the form

$$f(x_1, x_2, \ldots, x_n) = f(z_1, z_2, \ldots, z_n),$$

where f is a function with integer coefficients [1]. Actually, most of the studies are done on degrees 3 to 6 since the higher degrees are much more complicated.

*. Corresponding author

The more attractive case is the equation

$$AX^m + BY^n = AZ^m + BW^n, \ A, B \in \mathbb{Z}. \tag{1}$$

Most people concentrate on the case $m = n$ in lower degrees. For example, see [2] for the equation $X^4+hY^4 = Z^4+hW^4$, $h \in \mathbb{Q}$, and [7] for the equation $AX^3+BY^3 = AZ^3 + BW^3$, $A, B \in \mathbb{Z}$. For more details on the symmetric Diophantine equations (1) in low degrees we cite the comprehensive classic references [3, 4].

The equation (1) is also a special case of the more general Diophantine equation

$$\sum_{i=1}^{m} a_i {X_i}^k = \sum_{j=1}^{n} b_j {Z_j}^l,$$

where $a_i, b_j \in \mathbb{Z}$.

Consider the elliptic curve

$$E : y^2 = x^3 + ax^2 + bx + c$$

where $a, b, c \in \mathbb{Q}$, and assume $b \neq 0$. Then the Mordell-Weil group $E(\mathbb{Q})$ of rational points on E is a finitely generated and hence satisfies

$$E(\mathbb{Q}) \equiv \mathbb{Z}^r \times E(\mathbb{Q})_{\text{tors}},$$

where r is called the rank of E and $E(\mathbb{Q})_{\text{tors}}$ the torsion subgroup of $E(\mathbb{Q})$. For more details we cite the classical references [8, 9, 10]. It is clear that an elliptic curve with positive rank has infinitely many rational points. For this reason, one of the methods to show that a Diophantine equation has infinitely many rational solutions is to transfer it by a birational map to an elliptic curve with positive rank.

By a trivial solution of (1) we mean $X = Z$ and $Y = W$ and we consider all solutions are nontrivial. In this article we are interested on the the solutions of the equation (1) in the case $m = 4, n = 3$ and prove the following main theorem by the elliptic curve method mentioned above. The second and third authors have experienced this method in [5] to solve the equation $X^6+6Y^3 = Z^6\pm 6W^3$.

Theorem 1.1. *Given a nonzero rational number h, if the elliptic curve*

$$y^2 = x^3 - 9x^2 - 8h^2x$$

has positive rank, then the Diophantine equation

$$X^4 + hY^3 = Z^4 + hW^3, \tag{2}$$

has infinitely many integral solutions.

In fact, putting $h = B/A$ we get exactly the equation (1). Using Theorem 1.1 we solve an extensive Diophantine equation.

Corollary 1.2. *In Theorem 1.1 if, in addition,* $h = \sum_{i=1}^{k} h_i^3$ *for some positive integer* k*, then the Diophantine equation*

$$X^4 + \sum_{i=0}^{k} Y_i^3 = Z^4 + \sum_{i=0}^{k} W_i^3, \tag{3}$$

has infinitely many integral solutions.

Appealing to Remark 2.1, as an evidence, we conjecture that the condition of Theorem 1.1 may be removed.

Conjecture 1.3. *For any rational number* h*, the Diophantine equation*

$$X^4 + hY^3 = Z^4 + hW^3$$

has infinitely many integral solutions.

Now, the next conjecture is clear.

Conjecture 1.4. *The Diophantine equation*

$$X^4 + \sum_{i=0}^{k} Y_i^3 = Z^4 + \sum_{i=0}^{k} W_i^3$$

has infinitely many integral solutions.

2. Proofs

Proof of Theorem 1.1. Let the nonzero rational number h be such that the elliptic curve $y^2 = x^3 - 9x^2 - 8h^2x$ has positive rank. Put

$$X = u - h,\ Y = u + v,\ Z = u + h,\ W = u - 2v.$$

Then, with some straightforward calculations, the equation (2) turns to

$$8u^3 + 8uh^2 - 9u^2v + 9uv^2 - 9v^3 = 0.$$

Next, taking $u - v = 1$ we get

$$9v^2 = u^3 - 9u^2 - 8h^2u.$$

Finally, putting $x = u,\ y = 3v$, we obtain the elliptic curve

$$E_h : y^2 = x^3 - 9x^2 - 8h^2x. \tag{4}$$

Now, let the rational number h be such that the elliptic curve E_h has positive rank. Take a non-torsion point $(x, y) = (\frac{r}{s^2}, \frac{t}{s^3})$ on E_h, where $r, s, t \in \mathbb{Z}$. Let (X_0, Y_0, Z_0, W_0) be the corresponding rational solution of (2). Then

$$\begin{aligned} X_0 &= u - h, \\ Y_0 &= u + v, \\ Z_0 &= u + h, \\ W_0 &= u - 2v. \end{aligned}$$

h	1	2	6	12	13	15	18	19	24	25	26	27	32	36	41	46	47	49
t	2	2	2	2	3	2	3	4	2	2	2	2	9	5	2	2	2	5
R	1	1	1	1	1	2	1	1	1	1	1	1	1 or 2	1	1	1	1	2

Table 1: E_h has null rank and $R = \text{rank}(E_{ht^3}) > 0$

We have $u = \dfrac{r}{s^2}$ and $v = \dfrac{t}{3s^3}$. From the fact $u - v = 1$ we get $t = 3rs - 3s^3$.

Clearly, if (X_0, Y_0, Z_0, W_0) is a rational solution of (2), then

$$(\mu^3 X_0, \mu^4 Y_0, \mu^3 Z_0, \mu^4 W_0)$$

is also a rational solution, for any $\mu \in \mathbb{Q}$. Let $h = c/d$, where $c, d \in \mathbb{Z}$, $d \neq 0$, and take $\mu = 3sd$. Since $X_0 = u - h = \dfrac{r}{s^2} - h$, we have

$$\mu^3 X_0^3 = 27d^3 rs - 27d^3 s^3 h.$$

Adding $\pm 27s^3d^3$ to the right hand side and using the relation $t = 3rs - 3s^3$ turns the above equation to

$$\mu^3 X_0^3 = 9d^3 t + 27s^3 d^3 - 27d^2 s^3 c.$$

The same calculations with $\mu^3 Y_0, \mu^4 Z_0, \mu^3 W_0$ yields the following integral solution.

$$\begin{aligned} X &= \mu^3 X_0^3 = 9d^3 t + 27s^3 d^3 - 27d^2 s^3 c, \\ Y &= \mu^4 Y_0^3 = 81rs^2 d^4 + 27sd^4 t, \\ Z &= \mu^3 Z_0^3 = 9d^3 t + 27s^3 d^3 + 27d^2 s^3 c, \\ W &= \mu^4 W_0^3 = 81rs^2 d^4 - 54sd^4 t. \end{aligned}$$

Therefore, we have infinitely many integral solutions □

Remark 2.1. If the rank of E_h is zero, then we may find an appropriate t such that replacing h in (2) with ht^3 yields an elliptic curve with positive rank. Table 1 shows all such h up to 50. Note that for those h not in the table, E_h has positive rank. In fact, Table 1 guarantees the truth of Conjecture 1.3 for $1 \leq h \leq 50$.

Proof of Corollary 1.2. In Theorem 1.1 take $h = \sum_{i=1}^k h_i^3$. Then we have

$$X^4 + \sum_{i=0}^k h_i^3 Y^3 = Z^4 + \sum_{i=0}^k h_i^3 W^3.$$

Putting $Y_i = h_i Y$ and $W_i = h_i W$ now gives the result. □

Tables 2 and 3 shows some h in the cases $k = 2$ and $k = 3$, respectively. All rank computations in Tables 1, 2 and 3 are done using MWRANK software.

h	9	16	28	35	65	91	126	133	189	243
h_1, h_2	1, 2	2, 2	1, 3	2, 3	1, 4	3, 4	1, 5	2, 5	4, 5	3, 6
R	1	1	1	1	2	2	2	1	1	1

Table 2: $R = \text{rank}(E_h)$ for $h = h_1^3 + h_2^3$

h	10	17	29	43	55	73	80	129	134	160
h_1, h_2, h_3	1, 1, 2	2, 1, 2	1, 1, 3	2, 2, 3	1, 3, 3	1, 2, 4	2, 2, 4	1, 4, 4	1, 2, 5	1, 3, 5
R	1	2	1	1	2	1	1	1	1	1

Table 3: $R = \text{rank}(E_h)$ for $h = h_1^3 + h_2^3 + h_3^3$

3. A subfamily of E_h with positive rank

In this section, we attempt to construct a subfamily of E_h with positive rank. This guarantees the existence of positive-rank elliptic curves E_h in Theorem 1.1. Following the method of MacLeod [6], if in the elliptic curve (4) we impose $x = h^2$, then we will have a point of infinite order if

$$h^2 - 17 = t^2. \tag{5}$$

We try to parameterize (5). To do this, we use tangent-chord method. By a simple search we find the solution $h = 9, t = 8$. The line $t = 8 + k(h - 9)$ meets (5) at one further point giving

$$h = \frac{9k^2 - 16k + 9}{k^2 - 1}, \ k \neq \pm 1$$

Substituting h in (4) and defining

$$x = \frac{u}{(k^2 - 1)^4}, \ y = \frac{v}{(k^2 - 1)^6}$$

gives the subfamily of elliptic curves

$$v^2 = u^3 - 9(k^2 - 1)^4 u^2 - 8(k^2 - 1)^6 (9k^2 - 16k + 9)u$$

with rank at least 1.

Acknowledgements

The authors thank the referee for the constructive criticism and valuable suggestions to improve the paper.

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Accepted: 15.11.2017