On hyperconnected spaces via m-structures

Hanan Al-Saadi
Umm Al-Qura University
Faculty of Applied Sciences
Department of Mathematics
P.O. Box 11155
Makkah 21955
Saudi Arabia
hasa112@hotmail.com

Ahmad Al-Omari*
Al al-Bayt University
Faculty of Sciences
Department of Mathematics
P.O. Box 130095, Mafraq 25113
Jordan
omarimutah1@yahoo.com

Takashi Noiri
2949-1 Shiokita-cho
Hinagu, Yatsushiro-shi
Kumamoto-ken, 869-5142
Japan
t.noiri@nifty.com

Abstract. In this paper, we introduce and investigate the notion of m-hyperconnectedness in a topological space (X, τ) with a minimal structure m_X on X. Several characterizations and preservation theorems of m-hyperconnectedness are obtained.

Keywords: m-structure, m-hyperconnected, semi-m_X-open, semi-m_X-interior, somewhere dense.

1. Introduction

A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a minimal structure [11] if $\phi \in m_X$ and $X \in m_X$. In [2], the present authors introduced and investigated the notion of m^*-connected spaces, m-separated sets and m-connected sets in a topological space (X, τ) with a minimal structure m_X. In this paper, we introduced the notion of m-hyperconnectedness in a topological space (X, τ) with a minimal structure m_X. We obtain several characterizations and preservation theorems of m-hyperconnectedness. And also, we investigate the

*. Corresponding author

relationship between m-hyperconnectedness and hyperconnectedness. Recently papers [3, 4, 12] have introduced some new classes of sets via m-structures.

2. Minimal structures

Definition 2.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X [11] if $\emptyset \in m_X$ and $X \in m_X$. Each member of m_X is said to be *m_X-open* and the complement of an m_X-open set is said to be *m_X-closed*.

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be

1. α-open [10] if $A \subset Int(Cl(Int(A)))$,
2. semi-open [7] if $A \subset Cl(Int(A))$,
3. preopen [9] if $A \subset Int(Cl(A))$,
4. b-open [6] if $A \subset Int(Cl(A)) \cup Cl(Int(A))$,
5. β-open [1] or semi-preopen [5] if $A \subset Cl(Int(Cl(A)))$.

The family of all α-open (resp. semi-open, preopen, b-open, semi-preopen) sets in (X, τ) is denoted by $\alpha(X)$ (resp. SO(X), PO(X), BO(X), SPO(X)).

Definition 2.3. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [8] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 2.4. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), BO(X), SPO(X)), then we have
(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\mathrm{bCl}(A)$, $\mathrm{spCl}(A)$),
(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\mathrm{bInt}(A)$, $\mathrm{spInt}(A)$).

Lemma 2.5 (Maki et al. [8])**.** *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,
(2) *If* $(X \setminus A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Lemma 2.6 (Popa and Noiri [11])**.** *Let X be a nonempty set with an m-structure m_X and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 2.7. An m-structure m_X on a nonempty set X is said to have

1. *property* $\mathcal{B}$ [8] if the union of any family of sets belonging to m_X belongs to m_X.
2. *property* $\mathcal{I}$ if the intersection of any finite family of sets belonging to m_X belongs to m_X.

Remark 2.8. Let (X, τ) be a topological space. Then the families $\alpha(X)$, SO(X), PO(X), BO(X) and SPO(X) are m-structures on X with property $\mathcal{B}$.

Lemma 2.9 (Popa and Noiri [11])**.** *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*
(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,
(2) A *is* m_X*-closed if and only if* $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is* m_X*-closed.*

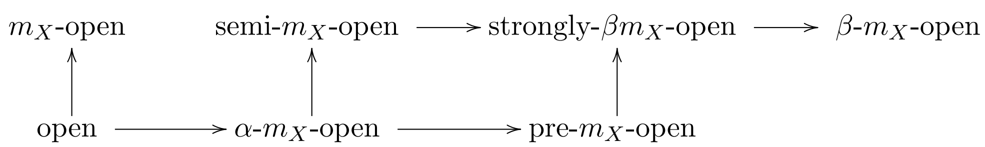
A topological space (X, τ) with an m-structure m_X on X is called a mixed space and is denoted by (X, τ, m_X).

Definition 2.10. A subset A of a mixed space (X, τ, m_X) is said to be:

1. m_X-dense if $mCl(A) = X$.
2. m_X-nowhere dense if $Int(mCl(A)) = \phi$.
3. α-m_X-open if $A \subseteq Int(mCl(Int(A)))$.
4. semi-m_X-open if $A \subseteq mCl(Int(A))$.
5. pre-m_X-open if $A \subseteq Int(mCl(A))$.
6. β-m_X-open if $A \subseteq Cl(Int(mCl(A)))$.
7. semi-m_X^*-open if $A \subseteq Cl(mInt(A))$.
8. strongly-βm_X-open if $A \subseteq mCl(Int(mCl(A)))$.

Lemma 2.11. *If $\tau \subseteq m_X$, then every semi-m_X-open set is semi-m_X^*-open.*

If $\tau \subseteq m_X$, the following diagram holds:



Lemma 2.12. *Let A be a subset of a mixed space (X, τ, m_X). Then the following properties hold:*

1. *A is semi-m_X-open if and only if there exists $B \in \tau$ such that $B \subseteq A \subseteq mCl(B)$.*
2. *If there exists $B \in m_X$ such that $B \subseteq A \subseteq Cl(B)$, then A is semi-m^*_X-open.*
3. *A is semi-m^*_X-open if and only if $Cl(A) = Cl(mInt(A))$.*

3. m-hyperconnected spaces

Definition 3.1. A mixed space. (X, τ, m_X) is said to be

1. m-hyperconnected (resp. hyperconnected [13]) if A is m_X-dense (resp. dense) for every nonempty open set A of X,
2. $m^\star$-connected [2] if X cannot be written as the disjoint union of a nonempty m_X-open set and a nonempty open set.

Example 3.2. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $m_X = \{\phi, X, \{a\}, \{a, c\}, \{a, b\}\}$. Then the mixed space (X, τ, m_X) is m-hyperconnected.*

If $\tau \subseteq m_X$, the following diagram holds:

$$\begin{array}{ccc} (X,\tau,m_X) \text{ is } m\text{-hyperconnected} & \longrightarrow & (X,\tau) \text{ is hyperconnected} \\ \downarrow & & \downarrow \\ (X,\tau,m_X) \text{ is } m^*\text{-connected} & \longrightarrow & (X,\tau) \text{ is connected} \end{array}$$

Theorem 3.3. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. The following properties are equivalent:*

1. *X is m-hyperconnected;*
2. *A is m_X-dense or m_X-nowhere dense for every subset A of X;*
3. *$A \cap B \neq \phi$ for every nonempty open subset A and every nonempty m_X-open subset B of X;*
4. *$A \cap B \neq \phi$ for every nonempty semi-m_X-open subset A and every nonempty m_X-open subset B of X.*

Proof. (1) $\Rightarrow$ (2): Let X be m-hyperconnected and $A \subseteq X$. Suppose that A is not m_X-nowhere dense. Then $Cl[X - mCl(A)] = X - Int(mCl(A)) \neq X$. Since $Int(mCl(A)) \neq \phi$, by (1) $mCl(Int(mCl(A))) = X$. Since $mCl(Int(mCl(A))) = X \subseteq mCl(A)$, then $mCl(A) = X$. Thus A is m_X-dense.

(2) $\Rightarrow$ (3): Suppose that $A \cap B = \phi$ for some nonempty sets $A \in \tau$ and $B \in m_X$. Then $mCl(A) \cap B = \phi$ and A is not m_X-dense. Moreover, since $A \in \tau$, $\phi \neq A \subseteq Int(mCl(A))$ and A is not m_X-nowhere dense.
(3) $\Rightarrow$ (4): Suppose that $A \cap B = \phi$ for some nonempty semi-m_X-open set A and some nonempty m_X-open set B of X. Since A is nonempty, $Int(A)$ is nonempty and by (3) $\phi \neq Int(A) \cap B \subseteq A \cap B$. This is a contradiction.
(4) $\Rightarrow$ (1): Suppose that X is not m-hyperconnected. Then there exists a nonempty open set V such that $mCl(V) \neq X$. Therefore, there exists a point $x \notin mCl(V)$ and by Lemma 2.6 $U \cap V = \phi$ for some $U \in m_X$ containing x. Since V is open, V is semi-m_X-open. This is contrary to (4). □

Lemma 3.4. *Let (X, τ, m_X) be an m-hyperconnected mixed space. Then every open mixed subspace $(Y, \tau_Y, m_X(Y))$ of (X, τ, m_X) is m-hyperconnected.*

Proof. Let $Q = Y \cap O$ be a nonempty open set in Y, then $mCl_Y(Q) = Y \cap mCl(Y \cap O) = Y \cap X = Y$ and hence Q is m-dense in Y. □

Definition 3.5. Let (X, τ, m_X) be a mixed space and $A \subseteq X$. The semi-m_X-closure (resp. semi-m_X^*-closure, pre-m_X-closure, strongly-βm_X-closure) of A, denoted by S-$mCl(A)$ (resp. S^*-$mCl(A)$, P-$mCl(A)$, $s\beta$-$mCl(A)$), is defined by the intersection of all semi-m_X-closed (resp. semi-m_X^*-closed, pre-m_X-closed, strongly-βm_X-closed) sets of X containing A.

Lemma 3.6. *Let (X, τ, m_X) be a mixed space. The following properties hold for a subset $A \subseteq X$.*

1. S-$mCl(A) = A \cup mInt(Cl(A))$;
2. S^*-$mCl(A) = A \cup Int(mCl(A))$.

Theorem 3.7. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. The following properties are equivalent:*

1. *X is m-hyperconnected;*
2. *H is m_X-dense for every nonempty strongly-βm_X-open subset $H \subseteq X$;*
3. *S^*-$mCl(H) = X$ for every nonempty strongly-βm_X-open subset $H \subseteq X$.*

Proof. (1) $\Rightarrow$ (2): Let (X, τ, m_X) be an m-hyperconnected space. Let H be any nonempty strongly-βm_X-open subset of X. We have $Int(mCl(H)) \neq \phi$. Thus $X = mCl(Int(mCl(H))) = mCl(H)$.
(2) $\Rightarrow$ (3): Let H be any nonempty strongly-βm-open subset of X. Thus, by Lemma 3.6 S^*-$mCl(H) = H \cup Int(mCl(H)) = H \cup Int(X) = X$.
(3) $\Rightarrow$ (1): Let G be any open set. Then G is strongly-βm_X-open and by (3) we have $X = S^*$-$mCl(G) = G \cup Int(mCl(G)) \subset mCl(G)$. Therefore, X is m-hyperconnected. □

Lemma 3.8. *If a mixed space (X, τ, m_X) is m-hyperconnected, then every set having nonempty interior is semi-m_X-open.*

Proof. Let A be a nonempty set containing a nonempty open set U. Since X is m-hyperconnected and U is m-dense, we have $mCl(U) = X$ and so $U \subseteq A \subseteq mCl(U) = X$. Hence, by Lemma 2.12, A is semi-m_X-open. □

Theorem 3.9. *Let (X, τ, m_X) be a mixed space such that the intersection of an open set and an m_X-open set is m_X-open. Then X is m-hyperconnected if and only if the union of a non dense set and a non m_X-dense set is non m_X-dense.*

Proof. Suppose that X is m-hyperconnected. Let U be non dense and V be non m_X-dense in X. If one is empty, then there is nothing to prove. If not, there exist a nonempty open set G and an m_X-open H such that $U \cap G = \phi = V \cap H$. Now, $(G \cap H) \cap (U \cup V) = (G \cap H \cap U) \cup (G \cap H \cap V) = \phi$. Since X is m-hyperconnected, $G \cap H \neq \phi$. By assumption, $G \cap H \in m_X$ and so $U \cup V$ is non m_X-dense. Conversely, suppose that X is not m-hyperconnected. Then there exist a nonempty open set U and a nonempty m_X-open set V such that $U \cap V = \phi$. Then $(X - U) \cup (X - V) = X$. If $X - U$ is dense, then $Cl(X - U) = X$ which implies that $U = \phi$ and if $X - V$ is m_X-dense, then $mCl(X - V) = X$ which implies that $V = \phi$. This is a contradiction. Therefore, $X - U$ is non dense and $X - V$ is non m_X-dense. By assumption, $(X - U) \cup (X - V) = X$ is non m_X-dense and so X is non m_X-dense, which is a contradiction. Therefore, X is m-hyperconnected. □

Theorem 3.10. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. If A is m-hyperconnected set of X and $A \subseteq B \subseteq mCl(A)$, then B is m-hyperconnected.*

Proof. Let V be any nonempty open set of the subspace B. Then $V = U \cap B$ for some open set U of X. Since V is nonempty and $B \subseteq mCl(A)$, $U \cap A$ is a nonempty open set of A and hence $mCl_A(U \cap A) = A$. Moreover, we have $A = mCl(U \cap A) \cap A \subseteq mCl(U \cap B) \cap B = mCl_B(V)$ and hence $B = mCl(A) \cap B = mCl_B(A) \subseteq mCl_B(V)$. This shows that B is an m-hyperconnected set of X. □

The semi-m_X-interior of a subset A of a mixed space (X, τ, m_X), denoted by $S\text{-}mInt(A)$, is defined by the union of all semi-m_X-open sets of X contained in A.

Definition 3.11. A function $f : (X, \tau, m_X) \to (Y, \sigma)$ is said to

1. semi-m-continuous if for every open set A of Y, $f^{-1}(A)$ is semi-m_X-open.
2. almost semi-m-continuous if for every nonempty regular open set A of Y, $f^{-1}(A) \neq \phi$ implies $S\text{-}mInt[f^{-1}(A)] \neq \phi$.

Definition 3.12. A function $f : (X, \tau) \to (Y, \sigma, m_Y)$ is said to almost semi-m-open if for every nonempty regular open set B of X, $S\text{-}mInt[f(B)] \neq \phi$

Lemma 3.13. *Every semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$ is almost semi-m-continuous.*

Proof. Let $f : (X, \tau, m_X) \to (Y, \sigma)$ be a semi-m-continuous function. Suppose that G is a nonempty regular open subset of (Y, σ, m_Y) such that $f^{-1}(G) \neq \phi$. This implies that $f^{-1}(G)$ is a nonempty semi-m_X-open set in (X, τ, m_X).

Thus, $f^{-1}(G) \subseteq mCl(Int(f^{-1}(G)))$ and hence $\phi \neq f^{-1}(G) \subseteq f^{-1}(G) \cap mCl(Int(f^{-1}(G))) = S\text{-}mInt(f^{-1}(G)$. Consequently, $f : (X, \tau, m_X) \to (Y, \sigma)$ is almost semi-m-continuous.

□

Theorem 3.14. *Let (X, τ, m_X) be a mixed space and $\tau \subseteq m_X$. Then, the following properties are equivalent:*

1. *(X, τ, m_X) is m-hyperconnected;*
2. *Every almost semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$, where (Y, σ) is a Hausdorff space, is constant;*
3. *Every semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$, where (Y, σ) is a Hausdorff space, is constant;*
4. *Every semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$, where (Y, σ) is the two point discrete space, is constant.*

Proof. (1) ⇒ (2): Let X be an m-hyperconnected space. Suppose that there exist a Hausdorff space Y and an almost semi-m-continuous function $f : (X, \tau, m_X) \to (Y, \sigma)$ such that f is not constant. There exist two distinct points x and y in X such that $f(x) \neq f(y)$. Since Y is Hausdorff, then there exist two open sets A and B in X such that $f(x) \in A$, $f(y) \in B$ and $A \cap B = \phi$. Take $G = Int(Cl(A))$ and $H = Int(Cl(B))$. This implies that G and H are nonempty regular open and $G \cap H = \phi$. Since f is almost semi-m-continuous, $S\text{-}mInt[f^{-1}(G)] \neq \phi$ and $S\text{-}mInt[f^{-1}(H)] \neq \phi$. Since $S\text{-}mInt[f^{-1}(G)]$ is nonempty semi-m_X-open, $Int(S\text{-}mInt[f^{-1}(G)]) \neq \phi$. We have $Int(S\text{-}mInt[f^{-1}(G)]) \cap S\text{-}mInt[f^{-1}(H)] \subseteq S\text{-}mInt[f^{-1}(G)] \cap S\text{-}mInt[f^{-1}(H)] \subseteq f^{-1}[G \cap H] = \phi$. Since $Int(S\text{-}mInt[f^{-1}(G)])$ is m_X-open and $S\text{-}mInt[f^{-1}(H)]$ is semi-m_X-open, by Theorem 3.3 (4) X is not m-hyperconnected. This is a contradiction.

(2) ⇒ (3): It follows from Lemma 3.13 and (2).

(3) ⇒ (4): It follows from (3).

(4) ⇒ (1): Suppose (X, τ, m_X) is not m-hyperconnected. Then there exists a nonempty open set U such that $mCl(U) \neq X$. Let $Y = \{a, b\}$, $\sigma = \{\phi, \{a\}, \{b\}, Y\}$ and define $f : (X, \tau, m_X) \to (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} a, & x \in mCl(U); \\ b, & x \notin mCl(U). \end{cases}$$

Then f is not a constant function. Since $mCl(U) = mCl(Int(U)) \subseteq mCl(Int(mCl(U)))$, $mCl(U)$ is semi-m_X-open. Hence σ is the two point discrete topology on $\{a, b\}$ and f is a semi-m-continuous function. Moreover, f is not constant. This is contrary to the hypothesis. Therefore, (X, τ, m_X) is an m-hyperconnected space. □

Theorem 3.15. *Let (X, τ, m_X) be an m-hyperconnected mixed space and $\tau \subseteq m_X$. If $f : (X, \tau, m_X) \to (Y, \sigma)$ is an almost semi-m-continuous surjection, then Y is hyperconnected.*

Proof. Suppose Y is not hyperconnected. Then there exist disjoint nonempty open sets $A \subseteq Y$ and $B \subseteq Y$. Take $G = Int(Cl(A))$ and $H = Int(Cl(B))$. Then G and H are nonempty regular open sets and $G \cap H = \phi$.

We have $Int(S\text{-}mInt[f^{-1}(G)]) \cap S\text{-}mInt[f^{-1}(H)] \subseteq S\text{-}mInt[f^{-1}(G)] \cap S\text{-}mInt[f^{-1}(H)] \subseteq f^{-1}(G) \cap f^{-1}(H) = \phi$. Since f is almost semi-m-continuous, $S\text{-}mInt[f^{-1}(G)] \neq \phi$ and $S\text{-}mInt[f^{-1}(H)] \neq \phi$.

Hence $Int(S\text{-}mInt[f^{-1}(G)])$ is not empty. By Theorem 3.3, X is not m-hyperconnected. This is a contradiction. □

Theorem 3.16. *Let (Y, σ, m_Y) be an m-hyperconnected mixed space and $\sigma \subseteq m_Y$. If $f : (X, \tau) \to (Y, \sigma, m_Y)$ is an almost semi-m-open injection, then X is hyperconnected.*

Proof. Let A and B be nonempty open sets of X. Take $G = Int(Cl(A))$ and $H = Int(Cl(B))$. Then G and H are nonempty regular open sets. Since f is almost semi-m-open, $S\text{-}mInt[f(G)] \neq \phi$ and $S\text{-}mInt[f(H)] \neq \phi$. Hence $Int(S\text{-}mInt[f(G)]) \neq \phi$. Since Y is an m-hyperconnected space, then by Theorem 3.3, $\phi \neq Int(S\text{-}mInt[f(G)]) \cap S\text{-}mInt[f(H)] \subseteq f(G) \cap f(H)$. Since f is injective, then $G \cap H \neq \phi$. Then $A \cap B \neq \phi$ and hence X is hyperconnected. □

4. m_X-somewhere dense sets

Definition 4.1. A subset A of a mixed space (X, τ, m_X) is said to be m_X-somewhere dense if $Int(mCl(A)) \neq \phi$. In other words, a subset A of a mixed space (X, τ, m_X) is said to be m_X-somewhere dense if there exists a nonempty open set G such that $G \subseteq mCl(A)$. If A is not m_X-somewhere dense in X, then it is m_X-nowhere dense in X.

Theorem 4.2. *Let (X, τ, m_X) be a mixed space, $A \cap B \in m_X$ for any $A \in \tau$ and $B \in m_X$ and m_X have property $\mathcal{B}$. Then, for a subset N of X, the following properties are equivalent:*

1. *N is m_X-nowhere dense in X,*
2. *$X - mCl(N)$ is dense in X,*
3. *For each nonempty open set U in X, there exists a nonempty m_X-open set V in X such that $V \subseteq U$ and $V \cap N = \phi$.*

Proof. (1) $\Rightarrow$ (2): Let N be m_X-nowhere dense in X. Then $Int(mCl(N)) = \phi$ and $Cl(X - mCl(N)) = X$. Hence $X - mCl(N)$ is dense in X.
(2) $\Rightarrow$ (3): By letting $V = U \cap (X - mCl(N))$, we have the desired result.
(3) $\Rightarrow$ (1): Suppose that $Int(mCl(N)) \neq \phi$. Then, for any nonempty m_X-open set V such that $V \subseteq Int(mCl(N))$, $V \cap N \neq \phi$. Because, if $V \cap N = \phi$, then $V \cap mCl(N) = \phi$ and $V \cap Int(mCl(N)) = \phi$. Therefore, $V = \phi$. This is a contradiction. Hence $V \cap N \neq \phi$. This shows that (3) $\Rightarrow$ (1). □

Theorem 4.3. *Let (X, τ, m_X) be a mixed space. Then, every nonempty strongly-βm_X-open set of X is m_X-somewhere dense.*

Proof. Suppose that A is a nonempty strongly-βm_X-open set. Then $A \subseteq mCl(Int(mCl(A))) \subseteq mCl(mCl(A)) \subseteq mCl(A)$. Therefore the set $Int(mCl(A))$ is nonempty open and $Int(mCl(A)) \subseteq mCl(A)$. Thus A is m_X-somewhere dense. □

Theorem 4.4. *Let (X, τ, m_X) be a mixed space. The union of an arbitrary family of nonempty m_X-somewhere dense subsets of (X, τ, m_X) is m_X-somewhere dense.*

Proof. Assume that $\{A_\alpha : \alpha \in \Delta\}$ is a family of m_X-somewhere dense sets. Then, for each $\alpha \in \Delta$ there exists a nonempty open set G_α such that $G_\alpha \subseteq mCl(A_\alpha) \subseteq mCl(\cup_{\alpha \in \Delta} A_\alpha)$. Hence $\cup_{\alpha \in \Delta} A_\alpha$ is an m_X-somewhere dense set. □

Theorem 4.5. *Let (X, τ, m_X) a hyperconnected mixed space and $A \cap B \in m_X$ for any $A \in \tau$ and $B \in m_X$. If M is open and N is m_X-somewhere dense, then $M \cap N$ is m_X-somewhere dense.*

Proof. Suppose that N is an m_X-somewhere dense subset of (X, τ, m_X). Then there is a nonempty open set G which is contained in $mCl(N)$. Therefore $M \cap G \subseteq M \cap mCl(N) \subseteq mCl(M \cap N)$. Since (X, τ, m_X) is hyperconnected, then $M \cap G \neq \phi$, Hence $M \cap N$ is m_X-somewhere dense. □

A function $f : (X, m_X) \to (Y, m_Y)$ is said to be M-continuous [11] if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subseteq V$.

Theorem 4.6. *If a function $f : (X, \tau, m_X) \to (Y, \sigma, m_Y)$ is open and M-continuous, then the image of each m_X-somewhere dense set is m_X-somewhere dense.*

Proof. Let A be an m_X-somewhere dense subset of (X, τ, m_X). Then there is a nonempty open set G such that $G \subseteq mCl(A)$. Now, $f(G) \subseteq f(mCl(A))$. Since f is open and M-continuous, then $f(G)$ is open and $f(mCl(A)) \subseteq mCl(f(A))$ (Theorem 3.1 of [11]). Therefore $f(A)$ is m_X-somewhere dense. □

Lemma 4.7. *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{I}$. If G is m_X-open in X, then $mCl(A) \cap G \subseteq mCl(A \cap G)$ for every $A \subseteq X$.*

Proposition 4.8. *Let (X, τ, m_X) be an m-hyperconnected mixed space such that m_X has property $\mathcal{I}$. If Y is an m_X-open set in X, then Y is m-hyperconnected.*

Proof. Let U be any nonempty open set of Y. Then there exists an open set V of X such that $U = V \cap Y$. By Lemma 4.7, we have $mCl(V) \cap Y \subseteq mCl(V \cap Y) \cap Y = mCl(U) \cap Y = mCl_Y(U)$. Since X is m-hyperconnected, we obtain that $mCl(V) = X$ and hence $Y = mCl_Y(U)$ then Y is m-hyperconnected. □

Theorem 4.9. *Let (X, τ, m_X) be a mixed space. Then $X - A$ is m_X-somewhere dense if and only if there exists a proper closed set F such that $mInt(A) \subseteq F$.*

Proof. Suppose $X - A$ is m_X-somewhere dense, then there exists a nonempty open set G such that $G \subseteq mCl(X - A)$. Thus $mInt(A) = X - mCl(X - A) \subseteq X - G$, take $F = X - G$ which is a proper closed set and hence $mInt(A) \subseteq F$. Conversely, suppose that there exists a proper closed set F such that $mInt(A) \subseteq F$. Then $X - F \subseteq X - mInt(A) = mCl(X - A)$ and $X - F$ is a nonempty open set. Therefore, $X - A$ is m_X-somewhere dense. □

Theorem 4.10. *Let (X, τ, m_X) be a mixed space and $m_X \subseteq \tau$. Then A or $X - A$ is m_X-somewhere dense.*

Proof. Suppose that A is not m_X-somewhere dense, then $Int(mCl(A)) = \phi$, so we have $Cl(A) \subseteq mCl(A) \neq X$. Then $X - Cl(A)$ is a nonempty open subset of $X - A$ and hence $X - Cl(A) \subseteq mCl(X - A)$. Therefore, $mInt(A) \subseteq Cl(A)$ and, by Theorem 4.9, $X - A$ is m_X-somewhere dense. □

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