An algorithm to compute the number of Rosenberg hypergroups of order less than 7

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Abstract. Using some special type of Boolean matrices, named very good matrices, we enumerate the non-isomorphic Rosenberg hypergroups of order less than 7. Moreover, the regular and reversible Rosenberg hypergroups are identified. Also the algorithm behind the method is presented.

Keywords: Rosenberg hypergroup, binary relation, regular reversible hypergroup, Boolean matrix.

1. Introduction

One of the most investigated hypergroups associated with binary relations is that introduced by Rosenberg [20] in 1998. It represents object of study of numerous papers written in order to compute the number of finite such hyperstructures. This line of research has been inspired by Migliorato's paper [19], where the structure of all non-isomorphic hypergroups of order 3 and of total regular abelian hypergroups have been determined. Later on Bayon and Lygeros [1, 2, 3] computed the number of finite abelian hypergroups and H_v-groups.

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Turning back to Rosenberg hypergroups, several algorithms written in $C\#$, Mathematica or MS Visual Basic have been presented by Spartalis and Mamaloukas [21], Massouros and Tsitouras [17, 18], Cristea et al. [9] and Jafarpour et al. [15], where particular types of such hypergroups have been computed. More exactly, in [9] the authors determined the number of Rosenberg hypergroups that are (general) mutually associative or complementary hypergroups. The paper [15], like the current one, deals with regular reversible Rosenberg hypergroups. Both papers use an algorithm based on representation of binary relations by Boolean matrices. All regular reversible Rosenberg hypergroups of order less than 5 have been computed, up to isomorphism [15], but not those of bigger order. That is why we propose here another method, consisting in representing every Boolean matrix by a non-negative integer obtained as explained in Section 4. In this way we compute the number of idempotent Rosenberg hypergroups, regular Rosenberg hypergroups and reversible Rosenberg hypergroups of order 6, up to isomorphism.

2. Preliminaries

As we have mentioned before, this paper is a continuation of a previous work [15]. Since we want the paper to be self-contained, we recall here the basic concepts regarding the regular, reversible and Rosenberg hypergroups, that are also included in [15].

Let us briefly recall some basic notions and results about hypergroups; for a comprehensive overview of this subject, the reader is refereed to [4, 8].
Let H be a non-empty set and $P^*(H)$ denote the set of all non-empty subsets of H. Let $\circ$ be a *hyperoperation* (or *join operation*) on H, that is, a function from the cartesian product $H \times H$ into $P^*(H)$. The image of the pair $(a, b) \in H \times H$ under the hyperoperation $\circ$ in $P^*(H)$ is denoted by $a \circ b$. The join operation can be extended in a natural way to subsets of H as follows: for non-empty subsets A, B of H, define $A \circ B = \cup\{a \circ b \mid a \in A, b \in B\}$. The notation $a \circ A$ is used for $\{a\} \circ A$ and $A \circ a$ for $A \circ \{a\}$. Generally, the singleton $\{a\}$ is identified with its element a. The hyperstructure $(H, \circ)$ is called a *semihypergroup* if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in b \circ c} a \circ u = \bigcup_{v \in a \circ b} v \circ c.$$

A semihypergroup $(H, \circ)$ is called a *hypergroup* if the reproduction law holds: $a \circ H = H \circ a = H$, for all $a \in H$. A non-empty subset K of a hypergroup $(H, \circ)$ is called a *subhypergroup* if it is a hypergroup, too. An element e of H is called an *identity* if, for all $a \in H$, $a \in a \circ e \cap e \circ a$ and $a' \in H$ is called an *inverse* of a in H if $e \in a \circ a' \cap a' \circ a$.

Definition 2.1. (i) A hypergroup $(H, \circ)$ is *regular* if it has at least one identity and each element has at least one inverse.

(ii) A regular hypergroup $(H, \circ)$ is called *reversible* if, for any $(x, y) \in H^2$, it satisfies the following conditions:
(1) if $y \in a \circ x$, then there exists an inverse a' of a, such that $x \in a' \circ y$;
(2) if $y \in x \circ a$, then there exists an inverse a'' of a, such that $x \in y \circ a''$.

Definition 2.2. Let $(H, \circ)$ and $(H', \circ')$ be two hypergroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if it satisfies the condition: for any $x, y \in H$,

$$f(x \circ y) \subseteq f(x) \circ' f(y).$$

f is a *good homomorphism* if, for any $x, y \in H$, $f(x \circ y) = f(x) \circ' f(y)$. We say that the two hypergroups are *isomorphic* if there is a good homomorphism between them which is also a bijection.

The Rosenberg partial hypergroupoid $H_\rho = (H, \circ_\rho)$ associated with a binary relation ρ defined on a nonempty set H is constructed as follows. For any $x, y \in H$,

$$x \circ_\rho x = \{z \in H \mid (x, z) \in \rho\} \quad \text{and} \quad x \circ_\rho y = x \circ_\rho x \cup y \circ_\rho y.$$

The set $\mathbb{D}(\rho) = \{x \in H \mid \exists y \in H : (x, y) \in \rho\}$ is called the *domain* of ρ, while $\mathbb{R}(\rho) = \{y \in H \mid \exists x \in H : (x, y) \in \rho\}$ is the *range* of the relation ρ. An element $x \in H$ is called *outer element* of ρ if there exists $h \in H$ such that $(h, x) \notin \rho^2$.

The next theorem gives necessary and sufficient conditions, obtained by Rosenberg, under which the partial hypergroupoid H_ρ is a hypergroup.

Theorem 2.3 ([20])**.** *H_ρ is a hypergroup if and only if*

(i) ρ has full domain: $\mathbb{D}(\rho) = H$;

(ii) ρ has full range: $\mathbb{R}(\rho) = H$;

(iii) $\rho \subset \rho^2$;

(iv) If $(a, x) \in \rho^2$ then $(a, x) \in \rho$, whenever x is an outer element of ρ.

Adding new conditions to the previous ones, Corsini characterized the regular reversible Rosenberg hypergroups.

Theorem 2.4 ([5])**.** *The hypergroup H_ρ is regular if and only if $K = \{e \in H \mid P \subset e \circ_\rho e\} \neq \emptyset$, where e is an identity for H and $P = \{x \in H \mid x \notin x \circ_\rho x\}$.*

Moreover, if $K \neq \emptyset$ and ρ is symmetric, then H_ρ is a regular reversible hypergroup.

The rest of the section gathers properties of regular reversible Rosenberg hypergroups based on the representation of binary relations by Boolean matrices, having all elements 0 or 1 and satisfying the rules in a Boolean algebra: $0 + 1 = 1 + 0 = 1 + 1 = 1$, while $0 + 0 = 0$, and $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$ [21]. Let ρ be a binary relation defined on a finite set $H = \{a_1, ..., a_n\}$.

The associated Boolean matrix $M(\rho) = (a_{ij})$, $i, j \in \{1, 2, \ldots, n\}$, is obtained as follows: $a_{ij} = 1$, if $(a_i, a_j) \in \rho$ and $a_{ij} = 0$, if $(a_i, a_j) \notin \rho$. It is clear that $M(\rho^2) = M^2(\rho)$ and in this way, 2^{n^2} Rosenberg partial hypergroupoids can be defined on every set with n elements.

We recall now some definitions and results from [9, 15].

Definition 2.5 ([9])**.** The matrix $M(\rho)$ is called *very good* if the Rosenberg hypergroupoid H_ρ is a hypergroup.

Definition 2.6 ([15])**.** A Rosenberg hypergroup H_ρ is called *i-Rosenberg hypergroup* if $M(\rho)$ is an idempotent very good matrix (i.e. $M(\rho)^2 = M(\rho)$).

Definition 2.7 ([15])**.** (i) A very good matrix $M(\rho)$ is called *i-very good* only if H_ρ is an i-Rosenberg hypergroup.

(ii) A very good matrix $M(\rho)$ is called *regular* if H_ρ is a regular hypergroup.

(iii) A regular matrix $M(\rho)$ is called *reversible* if H_ρ is a regular reversible hypergroup.

For the transpose matrix of M we use the standard notation M^T. The next theorem characterizes the very good matrices.

Theorem 2.8 ([9])**.** *A matrix $M = M(\rho)$ is a very good matrix if and only if, for any j, with $1 \leq j \leq n$, the following assertions hold:*

(i) $M_j^T \neq (0)$;

(ii) $M_j \neq (0)$;

(iii) *if* $M_j^2 \neq (1)$, *then* $M_j = M_j^2$,

where M_j and M_j^2 are, respectively, the j-column vectors of the matrices $M(\rho)$ and $M(\rho^2)$, while (0) is the zero column vector and (1) is the column vector with all elements equal to 1.

As an immediate consequence, we obtain the following characterization of an idempotent very good matrix.

Proposition 2.9 ([15])**.** *An idempotent matrix $M = M(\rho)$ (i.e. $M^2 = M$) is a very good matrix if and only if, for any j, with $1 \leq j \leq n$, the following assertions hold:*

(i) $M_j^T \neq (0)$;

(ii) $M_j \neq (0)$.

The aim of this paper is to count all non-isomorphic Rosenberg hypergroups satisfying certain conditions. The next result gives the necessary and sufficient conditions such that two Rosenberg hypergroups, having the same support set, are isomorphic. Two matrices $M(\rho)$ and $M(\rho')$ are called *isomorphic* if the associated Rosenberg hypergroups H_ρ and $H_{\rho'}$ are isomorphic.

Theorem 2.10 ([9]). *Let $H = \{a_1, ..., a_n\}$ be a finite set, ρ and ρ' be two binary relations on H and $M(\rho) = (t_{ij}), M(\rho') = (t'_{ij})$ be their associated matrices. The hypergroups H_ρ and $H_{\rho'}$ are isomorphic if and only if $t_{ij} = t'_{\sigma(i)\sigma(j)}$, for σ a permutation of the set $\{1, 2, \ldots, n\}$.*

Now we check when a very good matrix is regular.

Theorem 2.11 ([15]). *Let $M(\rho) = (t_{ij})_{n\times n}$ be a very good matrix. Then $M(\rho)$ is regular if and only if there exists k, $1 \leq k \leq n$, such that, for all i, $1 \leq i \leq n$, we have $t^c_{ii} \leq t_{ki}$, where $t^c_{ij} = 0$ if and only if $t_{ij} = 1$.*

Finally, we present how to combine two idempotent very good matrices in order to obtain another idempotent very good matrix.

Theorem 2.12 ([15]). *Let $M = (t_{ij})_{n\times n}$, $M' = (t'_{ij})_{m\times m}$ be two idempotent very good matrices. Then $M \oplus M' = (m_{ij})_{k\times k}$, where $k = n + m$, and*

$$m_{ij} = \begin{cases} t_{ij}, & \text{if } i \leq n, \quad j \leq n \\ t'_{ij}, & \text{if } n < i, \quad n < j \\ 0 & \text{elsewhere} \end{cases}$$

is an idempotent very good matrix.

Theorem 2.13 ([15]). *Let $M = (t_{ij})_{n\times n}$, $M' = (t'_{ij})_{m\times m}$ be two idempotent very good matrices. Then $M \boxplus M' = (m_{ij})_{k\times k}$, where $k = n + m$, and*

$$m_{ij} = \begin{cases} t_{ij}, & \text{if } i \leq n, \quad j \leq n \\ t'_{ij}, & \text{if } n < i, \quad n < j \\ 1, & \text{if } i \leq n, \quad j > n \\ 0 & \text{if } n < i, \quad j \leq n \end{cases}$$

is an idempotent very good matrix. Moreover, if M is regular then $M \boxplus M'$ is regular, too.

We conclude this section with the following result.

Theorem 2.14 ([15]). *A regular matrix $M = M(\rho)$ is reversible if $M = M^T$.*

3. The main algorithm

Comparing to previous approaches, what makes our method so efficient is that we represent every $n \times n$ Boolean matrix $M = (m_{ij}), i, j \in \{1, 2, \ldots, n\}$ by a non-negative integer as follows. Let $m_{i1}, ..., m_{in}$ be the elements on the i'th row. Then the corresponding integer is given by the formula:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} \times 2^{(i-1)n+j-1}.$$

From now on, we denote the integer by $I(M)$. Let us see one example. For instance consider the following matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

then the corresponding integer is

$$\begin{aligned} I(M) &= \sum_{i=1}^{4}\sum_{j=1}^{4} m_{ij} \times 2^{4(i-1)+j-1} \\ &= \sum_{i=1}^{4}(m_{i1} \times 2^{4(i-1)} + m_{i2} \times 2^{4(i-1)+1} + m_{i3} \times 2^{4(i-1)+2} + m_{i4} \times 2^{4(i-1)+3}) \\ &= 2^0 + 2^3 + 2^4 + 2^5 + 2^9 + 2^{11} + 2^{14} + 2^{15} \\ &= 51769. \end{aligned}$$

Obviously there are 2^{n^2} Boolean matrices of order n, which we denote by $M_1, ..., M_{2^{n^2}}$ and all of them can easily be generated by computer. So computation of vg (Number of very good matrices) and vgi (Number of very good idempotent matrices) is straightforward. The tough part is when we want to enumerate all non-isomorphic matrices, and this is where our integer representation of boolean matrices comes to play. Let A and A_2 are initialized as two $1 \times m$ arrays with all entries equal to 0. Once a Boolean matrix is approved to be a very good matrix, we have to decide weather any of its isomorphic copies has already been approved. Let $I(M)$ be the integer representation of the matrix M. If the $I(M)$'th element of array A_2 is already equal to 1, then this is a copy of another matrix that is already considered and we skip it, otherwise it is new and we increment the variable $co3$ (the number of very good matrices up to isomorphism) by 1. Also we produce all its $n!$ isomorphic copies which are denoted by M_σ where σ is a permutation of $\{1, ..., n\}$ and compute $I(M_\sigma)$ (the set of corresponding integers) and set the corresponding array in A_2 equal to 1. We follow a similar approach in the idempotent case and decide if a very good idempotent matrix is new (up to isomorphism) by looking at corresponding entry in array A and increment the variable $co2$ (the number of very good idempotent matrices up to isomorphism) if the case is approved to be new (up to isomorphism). The Main Algorithm is summarized in the following:

```
co2:=0; co3:=0;vgi:=0;vg:=0; rc:=0; rv:=0;
for j=1 to 2^(n^2)
    A(j) := 0; A_2(j) := 0
end
for j=1 to 2^(n^2)
    M := M_j
    if M ≤ M^2 then
        if M satisfies in conditions (i), (ii), (iii) of Theorem 2.3 then
            if A_2(I(A)) == 0 then
                co3=co3+1;
                for every permutation σ of {1,...,n} set A_2(I(M_σ)) = 1
                if M is regular rc=rc+1;
                if M is reversible rv=rv+1;
            end
            vg=vg+1;
            if M == M^2 then
               vgi=vgi+1;
               if A(I(M)) == 0 then
                   co2=co2+1;
                   for every permutation σ of {1,...,n} set A(I(M_σ)) = 1
               end
            end
        end
    end
end
```

We prepared a C++ code, based on the above algorithm, which will be available upon request from authors.

4. Conclusions

Hypregroups constructed from binary/ternary/n-ary relations have been obtained and studied in [5, 6, 7, 10, 11, 12, 13, 14, 16, 20, 22], as a method to construct new hyperstructures, and their properties are identified based on the properties of the involved relations. The Rosenberg hypergroups represent the subject of several papers [9, 15, 17, 18], dedicated to the computation of their number. In this paper, the number of all non-isomorphic regular reversible Rosenberg hypergroups has been computed, applying a program written in C++. The results of this computation (for $n = 2, 3, 4, 5$ and 6) are summarized in the following table:

vg: Number of very good matrices
co3: Number of very good matrices up to isomorphism
vgi: Number of very good idempotent matrices
co2: Number of very good idempotent matrices up to isomorphism
rc: Number of regular matrices up to isomorphism
rv: Number of reversible matrices up to isomorphism

N=	2	3	4	5	6
vg	6	149	9729	2921442	4578277389
co3	4	33	501	26409	6502030
vgi	4	35	559	14962	636217
co2	3	10	44	239	1668
rc	4	32	466	22672	5000329
rv	3	9	31	147	1095

By using Theorems 2.12 and 2.13 and results presented in above table we can provide a lower bound for very good idempotent matrices of higher orders. For example for order 11 the lower bound would be 14962+636217=651179.

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