$p-[a,b]$-paracompactness in bitopological spaces

Fuad A. Abushaheen*
Basic Science Department
Middle East University
Amman
Jordan
Fshaheen@meu.edu.jo

Hasan Z. Hdeib
Department of Mathematics
University of Jordan
Amman
Jordan
zahdeib@ju.edu.jo

Abstract. In this paper, we introduce a new definition of paracompactness in bitopological spaces, we give a equivalent statements for this notation. Finally a product theorem is given.

Keywords: $p-[a,b]-$paracompact space, $p-$locally$-a$ family, $s-[a,b]$ compact space.

1. Introduction

In 1963, Kelly [4] introduced the concept of bitopological space. A set X with two topologies τ_1, τ_2 is a bitopological space, and denoted by $X = (X, \tau_1, \tau_2)$. A cover $\mathcal{U}$ of a bitopological space $X = (X, \tau_1, \tau_2)$ is called $\tau_1\tau_2-$ open cover (family) [4], if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$, and it is called $p-$ open cover (family) [4],if it is $\tau_1\tau_2-$ open cover and contains at least one nonempty member of τ_1 and one nonempty member of τ_1. A space $X = (X, \tau_1, \tau_2)$ is called $s-[a,b]-(p-[a,b]-)$ compact space [1], if every $\tau_1\tau_2-(p-)$ open cover of X with cardinality $\leq b$ has a subcover with cardinality $< a$.

For a $\tau_1\tau_2-$ open covers $\mathcal{U}$, $\mathcal{V}$ in a bitopological space $X = (X, \tau_1, \tau_2)$, $\mathcal{U}$ is called a parallel refinement of $\mathcal{V}$ [2], if for each $U \in \mathcal{U} \cap \tau_i$ is contained in some $V \in \mathcal{V} \cap \tau_i$ for $i = 1, 2$.

In this paper the letters a, b are infinite regular cardinals, ω_0, ω_1 stands for the cardinality of $\mathbb{N}, \mathbb{R}$, respectively. If $X = (X, \tau_1, \tau_2)$ is a bitopological space and $A \subseteq X$, $int_{\tau_i}(A)$, $\overline{A}^{\tau_i}$ denote the interior and the closure of A in τ_i, respectively for $i = 1, 2$. When $X = (X, \tau_1, \tau_2)$ has a topological property Q this means that both τ_1 and τ_2 have this property. For the concepts not defined here, see [3] and [5].

$*$. Corresponding author

2. Preliminaries

Definition 2.1. A family $\mathcal{A}$ of subsets of a bitopological space $X=(X,\tau_1,\tau_2)$ is called p- locally-a family, if for all $x\in X$ there exists a τ_1- open set U containing x such that U meets less than a members of $\mathcal{A}\cap\tau_2$ or there exists a τ_2- open set V containing x such that V meets less than a members of $\mathcal{A}\cap\tau_1$.

Definition 2.2. A space $X=(X,\tau_1,\tau_2)$ is called $p-[a,b]-$ paracompact if every $p-$ open cover of X with cardinality $\leq b$ has a $p-$ locally$-a$ $p-$ open parallel refinement.

Theorem 2.3. *Let $X=(X,\tau_1,\tau_2)$ be a bitopological spaces. Then X is $p-[a,b]-$ paracompact if and only if each τ_j- open cover of a τ_i- proper closed subset of X with cardinality $\leq b$ has τ_i- locally $-a$ τ_j- open parallel refinement for $i\neq j; i,j=1,2$.*

Proof. $(\Rightarrow)$ Let F be a τ_i-closed proper subset of X. Let $\mathcal{U}=\{U_\alpha|\alpha\in\Delta\}$ be a τ_j-open cover of F with $|\Delta|\leq b$. Now $\{U_\alpha|\alpha\in\Delta\}\cup\{X-F\}$ is $p-$open cover of X, but X is $p-[a,b]-$paracompact, so there exists a $p-$open parallel refinement

$$\mathcal{V}=\{V_\alpha|\alpha\in\Delta^*\subseteq\Delta\}\bigcup\{W_\gamma|\gamma\in\Gamma\}.$$

Now $\{V_\alpha|\alpha\in\Delta^*\subseteq\Delta\}$ is a τ_i-locally$-a$ τ_j- open refinement, for $i\neq j; i,j=1,2$.

$(\Leftarrow)$ Let $\mathcal{U}=\{W_\alpha|\alpha\in\Delta\}\cup\{V_\gamma|\gamma\in\Gamma\}$ be a $p-$open cover of X with $|\Delta\cup\Gamma|\leq b$ where $W_\alpha\in\tau_i$ for all $\alpha\in\Delta$ and $V_\gamma\in\tau_j$ for all $\gamma\in\Gamma$, for $i\neq j; i,j=1,2$. So we have the following cases:

Case (i) If $\bigcup_{\gamma\in\Gamma}V_\gamma=X$, then take $\alpha_0\in\Delta$ such that $W_{\alpha_0}\neq\phi$. Consider the set $F=X-W_{\alpha_0}$, then F is a proper τ_i- closed subset of X and $\{V_\gamma|\gamma\in\Gamma\}$ is a τ_j- open cover of F with $|\Delta|\leq b$, then F has a τ_j- open τ_i- locally $a-$ refinement

$$\mathcal{V}^*=\{V^*_\lambda|\lambda\in\Lambda\}.$$

Finally the family

$$\mathcal{U}^*=\mathcal{V}^*\cup\{W_{\alpha_0}\}$$

is the required $p-$ locally $-a$ $p-$ open parallel refinement.

Case (ii) If $\bigcup_{\gamma\in\Gamma}V_\gamma\neq X$, then

$$E_1=X-\bigcup_{\gamma\in\Gamma}V_\gamma$$

is a τ_j- closed subset of X, and

$$E_1\subseteq\bigcup_{\alpha\in\Delta}W_\alpha,$$

so $\{W_\alpha|\alpha\in\Delta\}$ has τ_i- open τ_j- locally $-a$ parallel refinement

$$\mathcal{W}^*=\{W^*_\lambda|\lambda\in\Lambda\},$$

if $\bigcup_{\lambda\in\Lambda} W^*_\lambda = X$, we are done. if not let

$$E_2 = X - \bigcup_{\lambda\in\Lambda} W^*_\lambda,$$

then E_2 is τ_i- closed and hence there exists a τ_j- open τ_i- locally $-a$ refinement

$$\mathcal{V}^* = \{V^*_\omega | \omega \in \Omega\}.$$

Finally the family

$$\mathcal{U}^* = \mathcal{V}^* \cup \mathcal{W}^*$$

is p- locally $-a$ $p-$ open parallel refinement, hence the result.

□

Theorem 2.4. *Every $p - [\omega_0, \infty]-$ paracompact $p - T_2-$ bitopological space $X = (X, \tau_1, \tau_2)$ is $p - T_4$.*

Proof. Let E and F be disjoint closed sets such that E is a τ_2-closed and F is a τ_1- closed . Let $e \in E$, since X is $p - T_2-$ space, for each $f \in F$ there exists a τ_1- open set U_f and a τ_2- open set V_f such that $e \in U_f$ and $f \in V_f$ with $U_f \cap V_f = \phi$.

Let

$$\mathcal{V} = \{V_f | f \in F\} \bigcup \{X - F\},$$

then $\mathcal{V}$ is a $p-$ open cover of X, and hence there exists a $p-$ locally $-\omega_0$ $p-$ open parallel refinement $\mathcal{V}^*$ such that if $V \in \mathcal{V}^*$ and $V \cap F \neq \phi$, so we have $V \in \tau_2$. Now let

$$V_e = \bigcup \{V | V \in \mathcal{V}^* \quad \text{and} \quad V \cap F \neq \phi\},$$

then V_e is τ_2- open and $F \subseteq V_e$. Let U be a τ_1- open set containing e that intersects $< \omega_0$ of $\mathcal{V}^*$ say $V_1, V_2, ..., V_n$ and $V_k \subseteq V_{f_k}, 1 \leq k \leq n$. Let $U_e = U \cap U_{f_1} \cap U_{f_2} \cap \ldots \cap U_{f_n}$. Now $e \in U_e$ and $F \subseteq V_e$ with $U_e \cap V_e = \phi$, hence X is $p - T_3$.

Now let

$$\mathcal{U} = \{U_e | e \in E\} \cup \{X - E\},$$

then $\mathcal{U}$ is a $p-$ open cover of X, so there exists a $p-$ locally $-\omega_0$ $p-$open parallel refinement $\mathcal{U}^*$, notice that if $U^* \in \mathcal{U}^*$ and $U^* \cap E \neq \phi$, then $U^* \in \tau_1$. Let

$$W = \bigcup \{U^* | U^* \in \mathcal{U}^* \quad \text{and} \quad U^* \cap E \neq \phi\},$$

then W is a τ_1- open and $E \subseteq W$, now for each $f \in F$ there exists τ_2- open set U^*_f that intersects $< \omega_0$ of $\mathcal{U}^*$ say $U^*_1, U^*_2, \ldots, U^*_m$. Now let $U_{e_s} \in \mathcal{U}^*$ with $U^*_s \subseteq U_{e_s}$ for $s = 1, 2, \ldots, m$ and let $U_f = U^*_f \cap Ve_1 \cap Ve_2 \cap \ldots \cap Ve_m$, let $V = \bigcup \{U_f | f \in F\}$. Then $V \in \tau_2$, $F \subseteq V$ and $W \cap F = \phi$, hence X is $p - T_4$.

□

3. $p-[\omega_0,\omega_1]$-paracompact space

Definition 3.1. A family $\mathcal{A}$ of subsets of a bitopological space $X=(X,\tau_1,\tau_2)$ is called $p-$point$-a$ family if for all $x\in X$, x meets $<a$ of $\mathcal{A}\cap\tau_i$, $i=1$ or 2.

Lemma 3.2. *If $X=(X,\tau_1,\tau_2)$ is normal, $p-$ normal space, then for each $p-$open $p-$point$-\omega_0$ cover $\mathcal{G}$ with $|\mathcal{G}|<\omega_1$ has a parallel refinement $\mathcal{V}$ such that $\overline{V_1}^{\tau_i}\cup\overline{W_1}^{\tau_j}\subseteq G$ where $V_1,W_1\in\mathcal{V}\cap\tau_i$ for some $G\in\mathcal{G}\cap\tau_i$,$i\neq j;i,j=1,2$.*

Proof. Let $\mathcal{G}=\{G_k|k\in\Delta,|\Delta|<\omega_1\}$ be a $p-$open $p-$point$-\omega_0$ of X, write $\Delta=\{1,2,\ldots\}$.

Let

$$F_1^i=X-\bigcup_{k>1}(G_k\cap\tau_i),$$

and

$$F_1^j=X-\bigcup_{k>1}(G_k\cap\tau_j),$$

then F_1^i is a τ_i- closed and F_1^j is a τ_j-closed for $i\neq j;i,j=1,2$. Now let

$$F_1=F_1^i\cup F_1^j,$$

then

$$F_1^i\subseteq F_1\subseteq G_1$$

and

$$F_1^j\subseteq F_1\subseteq G_1.$$

Without loss of generality assume that G_1 is a τ_i-open, since X is normal and by [6], there exists τ_i-open sets V_1^i,W_1^i such that

$$F_1^i\subseteq V_1^i\subseteq\overline{V_1^i}^{\tau_i}\subseteq G_1$$

and

$$F_1^j\subseteq W_1^i\subseteq\overline{W_1^i}^{\tau_j}\subseteq G_1,$$

so

$$F_1\subseteq\overline{V_1^i}^{\tau_i}\cup\overline{W_1^i}^{\tau_j}\subseteq G_1,\ \text{ for }\ i\neq j;i,j=1,2.$$

Let

$$V_1=V_1^i\cup W_1^i$$

and

$$F_\alpha^i=X-\big(\bigcup_{\beta<\alpha}V_\beta\big)\cup\big(\bigcup_{\gamma>\alpha}(G_\gamma\cap\tau_i)\big),$$

$$F_\alpha^j=X-\big(\bigcup_{\beta<\alpha}int_{\tau_j}(V_\beta)\big)\cup\big(\bigcup_{\gamma>\alpha}(G_\gamma\cap\tau_j)\big),i\neq j;i,j=1,2,$$

and $F_\alpha = F^i_\alpha \cup F^j_\alpha$. Again without loss of generality assume G_α is a τ_i-open, so

$$F^i_\alpha \cup F^j_\alpha \subseteq G_\alpha$$

and hence there exists τ_i- open sets V^i_α, W^i_α such that

$$F^i_\alpha \subseteq V^i_\alpha \subseteq \overline{V^i_\alpha}^{\tau_i} \subseteq G_\alpha,$$

$$F^j_\alpha \subseteq W^i_\alpha \subseteq \overline{W^i_\alpha}^{\tau_j} \subseteq G_\alpha,$$

and

$$F_\alpha \subseteq \overline{V^i_\alpha}^{\tau_i} \cup \overline{W^i_\alpha}^{\tau_j} \subseteq G_\alpha, i \neq j; i, j = 1, 2.$$

Let $V_\alpha = V^i_\alpha \cup W^i_\alpha$,then $\mathcal{V} = \{V_\alpha | \alpha \in \Delta\}$ is a $p-$open $p-$point$-\omega_0$ parallel refinement of X. For instance, let $x \in X$ x meets $< \omega_0$ of $\mathcal{G} \cap \tau_i$ for $i = 1, 2$ say $G^i_{\alpha_1}, G^i_{\alpha_2}, \ldots, G^i_{\alpha_n}$, let $\alpha = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Now $x \notin G_\alpha$ for any $\gamma > \alpha$ and if $x \notin V_\beta$ for any $\beta < \alpha, V_\beta \in \tau_i$, $x \in F_\alpha \subseteq V_\alpha$, hence $x \in V_\beta$ for some $\beta \leq \alpha$, therefore $\mathcal{V}$ is a $p-$point$-\omega_0$ parallel refinement of X. □

Theorem 3.3. *For a normal, $p-$normal bitopological space $X = (X, \tau_1, \tau_2)$. The following are equivalent:*

(i) *X is $p - [\omega_0, \omega_1]-$ paracompact,*

(ii) *Every $p-$open cover of X has a $p-$point$-\omega_0$ refinement,*

(iii) *Every $p-$open cover $\mathcal{U}$ of X with cardinality $< \omega_1$ has a parallel refinement $\mathcal{V}$ such that for $V_1, W_1 \in \mathcal{V} \cap \tau_i$, we have $\overline{V_1}^{\tau_i} \cup \overline{W_1}^{\tau_j} \subseteq U$ for some $U \in \mathcal{U} \cap \tau_i$,$i \neq j; i, j = 1, 2$,*

(iv) *Given a decreasing sequence of a $p-$closed family $\mathcal{F} = \{F_k | k \in \Delta\}$ with $\bigcap_{k \in \Delta} F_k = \phi$, there exists a sequence of $p-$ open family $\mathcal{G} = \{G_k | k \in \Delta\}$ with $\bigcap_{k \in \Delta} G_k = \phi$, such that $F^i_k \subseteq G^i_k$ and $F^i_k \subseteq G^j_k$, where F^i_k is a τ_i-closed in $\mathcal{F}$, $G^i_k \in \mathcal{G} \cap \tau_i$ and $G^j_k \in \mathcal{G} \cap \tau_j$, $i \neq j; i, j = 1, 2$,*

(v) *Given a decreasing sequence of a $p-$closed family $\mathcal{F} = \{F_k | k \in \Delta\}$ with $\bigcap_{k \in \Delta} F_k = \phi$, there exists a sequence of $p-$closed $(G_\delta)-$ family $\mathcal{A} = \{A_k | k \in \Delta\}$ with $\bigcap_{k \in \Delta} A_k = \phi$, and $F_k \subseteq A_k$.*

Proof. $(i) \rightarrow (ii)$ trivial.

$(ii) \rightarrow (iii)$ Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a $p-$open cover of X with $|\Delta| < \omega_1$, so by (ii), for all $x \in X$, $\mathcal{U}$ has a parallel refinement $\mathcal{S}$ such that x meets $< \omega_0$ of $\mathcal{S} \cap \tau_i, i = 1$ or 2. For each $S \in \mathcal{S}$, let $U(S)$ be the first U_α containing S.

Let

$$G_\alpha = \bigcup_{U(S) = U_\alpha} S,$$

clearly $G_\alpha \in \tau_1 \cup \tau_2, G_\alpha \subseteq U_\alpha$. Let $\mathcal{G} = \{G_\alpha | \alpha \in \Delta\}$ is a $p-$point$-\omega_0$ cover of X which is $p-$open cover (if necessary add $U \in \mathcal{U} \cap \tau_i, i = 1$ or 2), finally the result comes from Lemma 3.2 .

$(iii) \to (iv)$ Let $\mathcal{F} = \{F_k | k \in \Delta||\Delta| < \omega_1\}$ be a sequence of $p-$closed family such that $F_{k+1} \subseteq F_k$, and $\bigcap_{k\in\Delta} F_k = \phi$.

Let

$$U_k^i = X - F_k^i,$$

and

$$U_k^j = X - F_k^j,$$

where F_k^i is a τ_i-closed and F_k^j is a τ_j- closed for $i \neq j; i,j = 1,2$. Then $\mathcal{U} = \{U_k^i | k \in \Delta\} \cup \{U_k^j | k \in \Delta\}$ is a $p-$open cover of X, so by (iii) there exists $V_k, W_k \in \mathcal{V} \cap \tau_i$, where $\mathcal{V}$ is a parallel refinement with $\overline{V_k}^{\tau_i} \cup \overline{W_k}^{\tau_j} \subseteq U_k^i$, we may assume that $U_k^i \in \mathcal{U} \cap \tau_i$. Let

$$G_k^i = X - \overline{V_k}^{\tau_i} \quad \text{and} \quad G_k^j = X - \overline{W_k}^{\tau_j},$$

then

$$\mathcal{G} = \{G_k^i | k \in \Delta\} \cup \{G_k^j | k \in \Delta\},$$

is a $p-$open family and $\bigcap_{k\in\Delta}(G_k^i \cup G_k^j) = \phi$ with $\overline{V_k}^{\tau_i} \subseteq U_k^i$ and $\overline{W_k}^{\tau_i} \subseteq U_k^i$, hence $X - U_k^i \subseteq X - \overline{V_k}^{\tau_i}$ and $X - U_k^i \subseteq X - \overline{W_k}^{\tau_j}$, so we have $F_k^i \subseteq G_k^i$ and $F_k^i \subseteq G_k^j$ for $i \neq j; i,j = 1,2$.

$(iv) \to (v)$ Let $\mathcal{F} = \{F_k, k \in \Delta\}$ be a decreasing sequence of $p-$ closed subset of X with $\bigcap_{k\in\Delta} F_k = \phi$, by (iv) there exists a sequence $\mathcal{G} = \{G_k, k \in \Delta\}$ of $p-$open subsets of X with $\bigcap_{k\in\Delta} G_k = \phi$, such that $F_k^j \subseteq G_k^i$ and $F_k^j \subseteq G_k^j$ where $G_k^i \in \mathcal{G} \cap \tau_i$, $G_k^j \in \mathcal{G} \cap \tau_j$, and F_k^j is τ_j- closed in $\mathcal{F}$, $i \neq j; i,j = 1,2$. So by Urysohn's lemma there exists a continuous function $f_k^j : (X, \tau_j) \to (\mathbb{R}, \tau_u)$ such that $f_k^j(F_k^j) = \{0\}$ and $f_k^j(X - G_k^j) = \{1\}$, and by [5] there exists a $p-$continuous function $g_k^i : (X, \tau_1, \tau_2) \to (\mathbb{R}, \tau_l, \tau_r)$ such that $g_k^i(F_k^j) = \{0\}$ and $g_k^i(X - G_k^i) = \{1\}$.

Let

$$M_{km}^j = \{x | f_k^j(x) < \frac{1}{m}\},$$

and

$$N_{km}^i = \{x | g_k^i(x) < \frac{1}{m}\},$$

then

$$M_k^j = \bigcap_m M_{km}^j = \{x | f_k^j(x) = 0\},$$

and

$$N_k^i = \bigcap_m N_{km}^i = \{x | g_k^i(x) = 0\}.$$

Now $\mathcal{A} = \{M_k^j | k \in \Delta\} \cup \{N_k^i | k \in \Delta\}$ is a $p-$closed $(G_\delta-)$set with $\bigcap_{k\in\Delta} A_k = \phi$ and $F_k \subseteq A_k$.

$(v) \to (i)$ Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta, |\Delta| < \omega_1\}$ be a $p-$open cover of X .

Let

$$F^i_\alpha = X - \bigcup_{\beta \leq \alpha} (U_\beta \cap \tau_i),$$

and

$$F^j_\alpha = X - \bigcup_{\beta \leq \alpha} (U_\beta \cap \tau_j),$$

then

$$F^i_{\alpha+1} \subseteq F^i_\alpha,$$

and

$$F^j_{\alpha+1} \subseteq F^j_\alpha.$$

Let

$$\mathcal{F} = \{F^i_\alpha | \alpha \in \Delta\} \cup \{F^j_\alpha | \alpha \in \Delta\},$$

then

$$F^i_{\alpha+1} \cup F^j_{\alpha+1} \subseteq F^i_\alpha \cup F^j_\alpha,$$

and

$$\bigcap_{\alpha \in \Delta} F_\alpha = \phi,$$

by (v) there exists a sequence $\mathcal{A}$ of $p-$closed $G_\delta-$set with $\bigcap_{\alpha \in \Delta} A_\alpha = \phi$, and $F^i_\alpha \subseteq A_\alpha$, $F^j_\alpha \subseteq A_\alpha$, but $X - A_\alpha$ is a $F_\alpha-$set, let $X - A_\alpha = \bigcup_\alpha B_{\alpha\gamma}$ where each $B_{\alpha\gamma}$ is a τ_i- closed and the family $\mathcal{B} = \{B_{\alpha\gamma} | \alpha \in \Delta\}$ is a $p-$ closed family, since X is normal $p-$normal space, we can assume that

$$B_{\alpha\gamma} \subseteq int_{\tau_i}(B_{\alpha,\gamma+1})$$

and

$$B_{\alpha\gamma} \subseteq int_{\tau_j}(B_{\alpha,\gamma+1}),$$

then

$$X - A_\alpha \subseteq \bigcup_\alpha int_{\tau_i}(B_{\alpha\gamma}),$$

and

$$X - A_\alpha \subseteq \bigcup_\alpha int_{\tau_j}(B_{\alpha\gamma}),$$

$$B_{\alpha\gamma} \subseteq X - A_\alpha \subseteq X - (F^i_\alpha \cup F^j_\alpha) = \bigcup_{\beta \leq \alpha} U_\beta.$$

Let $x \in X$ and U_α be the first element in $\mathcal{U}$ such that $x \in U_\alpha$, so we have two cases:

(i) $U_\alpha \in \tau_i$, let $V^i_\alpha = U_\alpha - \bigcup_{\alpha < \gamma} B_{\alpha\gamma}$ and each $B_{\alpha\gamma}$ is a τ_i- closed.

(ii) $U_\alpha \in \tau_j$, let $V^j_\alpha = U_\alpha - \bigcup_{\alpha < \gamma} B_{\alpha\gamma}$ and each $B_{\alpha\gamma}$ is a τ_j- closed for $i \neq j; i, j = 1, 2$.

Now in each case for $\alpha < \gamma$

$$B_{\alpha\gamma} \subseteq \bigcup_{k<\alpha} U_k \subseteq \bigcup_{k<\gamma} U_k,$$

$$U_\alpha - \bigcup_{k<\gamma} U_k \subseteq V_\gamma^i \cup V_\gamma^j,$$

hence

$$\mathcal{V} = \{V_\alpha^i | \alpha \in \Delta\} \cup \{V_\alpha^j | \alpha \in \Delta\}$$

is a $p-$open parallel refinement cover of X. Finally, we need to show that $\mathcal{V}$ is $p-$ locally $-\omega_0$, let $x \in X$, there exists $\alpha \in \Delta$ such that $x \notin A_\alpha$, so for some $k, x \in int_{\tau_i}(B_{\alpha k})$ and $x \in int_{\tau_j}(B_{\alpha k})$.

If $\gamma > \alpha$ and $\gamma > k$,

$$int_{\tau_i}(B_{\alpha k}) \subseteq B_{\alpha\gamma}$$

and

$$int_{\tau_j}(B_{\alpha k}) \subseteq B_{\alpha\gamma},$$

with

$$int_{\tau_i}(B_{\alpha k}) \cap V_\alpha^i = \phi,$$

and

$$int_{\tau_j}(B_{\alpha k}) \cap V_\alpha^j = \phi.$$

So $int_{\tau_i}(B_{\alpha k})$ is a τ_i- open set contains x and meets $< \omega_0$ of $\mathcal{V} \cap \tau_i$, $int_{\tau_j}(B_{\alpha k})$ is a τ_j- open set contains x and meets $< \omega_0$ of $\mathcal{V} \cap \tau_j$ for $i \neq j; i, j = 1, 2$, hence $\mathcal{V}$ is $p-$locally $-\omega_0$, therefore X is $p-[\omega_0, \omega_1]-$paracompact space.

□

4. A product theorem

Theorem 4.1. *Let $X = (X, \tau_1, \tau_2)$ be a $s-[w_0, \infty]-$ compact space and $Y = (Y, \sigma_1, \sigma_2)$ is a $p-[\omega_0, \omega_1]-$ paracompact space. Then $X \times Y = (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is $p-[\omega_0, \omega_1]-$ paracompact space.*

Proof. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta \, |\Delta| < \omega_1\}$ be a $p-$ open cover of $X \times Y$. Let $x \in X$ and $V_\alpha = \{x \in X | x \times Y \subseteq \bigcup_{\beta \leq \alpha} U_\alpha\}$, for $x \in V_\alpha$ and $y \in Y$ with $(x, y) \in x \times Y$ there exists a set $O_x \times Q_x$ with $x \in O_x \in \tau_1 \cup \tau_2$ and $y \in Q_x \in \sigma_1 \cup \sigma_2$. Let $\mathcal{Q} = \{Q_x | x \in X\}$, then $\mathcal{Q}$ is a $\tau_1\tau_2-$ open cover of Y and hence $\mathcal{Q}$ has a $\tau_1\tau_2-$open subcover with cardinality $< \omega_0$, say $Q_{x_1}, Q_{x_2}, \cdots, Q_{x_n}$.

Let

$$O_1 = \bigcap_m (O_m \cap \tau_1)$$

and

$$O_2 = \bigcap_m (O_m \cap \tau_2) \quad \text{for} \quad 1 \leq m \leq n.$$

Let $O = O_1 \cup O_2$, then $x \in O \subseteq \tau_1 \cup \tau_1$ and

$$O \times Y \subseteq \bigcup_{\beta \leq \alpha} U_\alpha,$$

and

$$x \times Y \subseteq O \times Y \subseteq \bigcup_{\beta \leq \alpha} U_\alpha$$

and hence $V_\alpha \in \tau_1 \cup \tau_1$. Now $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma, |\Gamma| < \omega_1\}$,(if $\mathcal{V} \cap \tau_i = \phi$, add $U_\alpha \in \mathcal{U} \cap \tau_i$ for some $\alpha \in \Delta, i = 1, 2$). Since for $x \in X$, $x \times Y$ is $s - [\omega_0, \infty]-$ compact, so it contains in $< \omega_0$ of $\mathcal{U}$ and hence $x \in V_\alpha$, so $\mathcal{V}$ has a $p-$ locally $-\omega_0$ $p-$ open parallel refinement say $\mathcal{B}$. For $B \in \mathcal{B}$, let $V_B \in \mathcal{V}$ be the first $V_\gamma \in \mathcal{V}$ such that $B \subseteq V_B$ and $G_\gamma = \bigcup_{V_B = V_\gamma} B$, then $G_\gamma \in \tau_1 \cup \tau_2$ and $\mathcal{G} = \{G_\gamma | \gamma \in \Gamma\}$ is a $p-$locally $-\omega_0$ $p-$ open cover of X. If $\alpha \leq \gamma$, let $W_{\gamma\beta} = (G_\gamma \times Y) \cap U_\alpha$, and $\mathcal{W} = \{W_{\gamma\beta} | \alpha \leq \gamma\}$. For $(x, y) \in X \times Y$, $x \in G_\gamma$ for some $\gamma \in \Gamma$ and $(x, y) \in G_\gamma \times Y$, also $x \in G_\gamma \subseteq V_\gamma$, $(x, y) \in X \times Y \subseteq \bigcup_{\beta \leq \alpha} U_\alpha$, so $(x, y) \in W_{\gamma\beta}$, therefore $\mathcal{W}$ is a $p-$ open cover of $X \times Y$. Again for $(x, y) \in X \times Y$, $x \in K$ where $K \in \tau_i$ which meets $< \omega_0$ of $\mathcal{G} \cap \tau_j$ for $i \neq j; i, j = 1, 2$. Now $K \times Y$ meets $< \omega_0$ of $(\mathcal{G} \cap \tau_j) \cap Y$, hence $\mathcal{W}$ is $p-$ locally$-\omega_0$ $p-$ open parallel refinement of $\mathcal{U}$, hence $X \times Y$ is $p - [\omega_0, \omega_1]-$paracompact space.

□

Acknowledgment

The authors are grateful to the Middle East University, Amman, Jordan for the financial support granted to cover the publication fee of this research article.

References

[1] Fuad A. Abushaheen and Hasan Z. Hdeib, *On [a,b] compactness in bitopological spaces,* International Journal of Pure and Applied Mathematics, 110 (2016), 519-535.

[2] M.C. Datta, *Projective bitopological spaces*, J. Austral Math. Soc., 13 (1972), 327-334.

[3] R. Engelking, *General topology*, revised and completed edition, Heldermann Verlag, Berlin, 1989.

[4] P. Fletcher et. al., *The compaison of topologies*, Duke Math. J., 36 (1969), 325-331.

[5] J.C. Kelly, *Bitopological spaces*, Proc. Londan Math. Soc., 13 (1963), 71-89.

[6] L. Reilly, et. al., *On bitopological compactness*, J. Londan Math. Soc., 9, 518-522.

Accepted: 9.11.2018