On TL-fuzzy ideals and their L-fuzzy roughness in lattices

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Abstract. In this paper, we study TL-fuzzy ideals of lattices. First, using a left continuous t-norm T on a complete lattice L, we introduce the concept of TL-fuzzy ideals of lattices and study the corresponding relations between TL-fuzzy ideals of two homomorphic lattices and discuss the image and the inverse image of generated TL-fuzzy ideals. Moreover, we consider the L-fuzzy roughness in lattices based on a complete residuated lattice, which is a generalization of (fuzzy) roughness in lattices. And we study some related properties of TL-fuzzy ideals with respect to T-upper and θ-lower fuzzy rough approximations.
Keywords: TL-fuzzy ideal, TL-fuzzy congruence relation, T-upper fuzzy rough approximation, θ-lower fuzzy rough approximation.

1. Introduction

As is known, an important task of the artificial intelligence is to make the computers simulate human being in dealing with certainty and uncertainty in information. Logic gives a technique for laying the foundations of this task. While information processing dealing with certain information is based on the classical two-valued logic, non-classical logics including logics behind fuzzy reasoning handle information with various facets of uncertainty such as fuzziness, randomness, vagueness, etc. Therefore, non-classical logic has become a formal and useful tool for computer science to deal with uncertain information

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and fuzzy information. And various logical algebras have been proposed as the semantical systems of non-classical logical systems, for instance, MV-algebras [1], BL-algebras [27], etc. Among these logical algebras, lattices are very basic algebraic structures. Filters and ideals are important tools in studying lattice algebraic systems. The sets of provable formulas in corresponding inference systems are described by filters, and from the point of view of uncertain information, by fuzzy filters of algebraic semantics. Moreover, the properties of the sets of filters/ideals have a strong influence on the structure properties of those algebras. Based on these, many authors introduced and studied fuzzy filters of logical algebras [15, 29, 23, 8]. Recently, fuzzy filters based t-norm of BL-algebras has been given in [26], TL-filters of integral residuated l-monoids have been studied in [28], and so on.

An important notion in fuzzy set theory is that of t-norms: t-norms are used to define a generalized intersection. By a fuzzy subset μ in a given universe X we understand a mapping μ: $X \to [0,1]$, the membership degrees $\mu(x)$ can in a natural way be understood as the truth value (in fuzzy logic) of the statement "x belongs to μ". In the same way, the intersection $\mu \cap \nu$ of fuzzy sets μ, ν can be viewed as having the membership degree $(\mu \cap \nu)(x)$ corresponding to the truth degree of the statement "x belongs to μ" AND "x belongs to ν". Here AND refers to a suitably defined conjunction connective, defined according to the different possibilities which one has to determine the membership degrees $(\mu \cap \nu)(x)$. For example, AND can be understood as taking the minimum or as taking the (usual, i.e. algebraic) product, or more generally, it also can be understood as a t-norm. Accordingly, the t-norms were considered as the candidates for generalized conjunction connectives of the background many-valued logic. The fuzzy logic based on t-norms, especially left continuous t-norms, was developed significantly by Hájek, Esteva et al (see [3, 5, **?**, 20]) Also, there are many applications of t-norms in several fields of mathematics and artificial intelligence [10].

Rough sets theory was developed by Pawlak [21] as a formal tool for representing and processing information in data tables-consequently, this formalism acts on partial information. It has been found practical applications in many areas such as knowledge discovery, machine learning, data analysis, approximation classification and so on, see [13, 16, 22]. The fuzzy set theory [39], on the other hand, offers a wide variety of techniques for analyzing imprecise data. It seems therefore natural to combine methods developed within both theories in order to construct hybrid structures capable to deal with both aspects of incompleteness. Such structures, called fuzzy rough sets and rough fuzzy sets, have been proposed in the literature [4]. In [24], Radzikowska investigated fuzzy rough sets taking the interval $[0,1]$ as a basic structure. However, as Goguen [6] pointed out, in some situations it may be impossible to use the linearly ordered set $[0,1]$ to represent degrees of membership. From this reason, the concept of an L-fuzzy set was then introduced.

The integration of the fuzzy set theory and rough set theory is an interesting and valuable research work, which has been studying until now. And it makes sense to study rough sets on fuzzy algebra structures. The initiation and majority of studies on fuzzy rough sets for algebraic structures such as semigroups, groups, rings, and hypermodules have been concentrated on the interval $[0,1]$ as a basic structure, see [35, 9]. The interval $[0,1]$, however, seems to be restrict the application of the generalized rough set model for algebraic sets. To solve this problem, Radzikowska and Kerre [25] proposed the concept of L-fuzzy rough sets. It differs from fuzzy rough sets in that it takes a complete residuated lattice L as its basic structure. This is a fairly wide constructive setting because diverse residuated pairs can be chosen and, in case $L = [0,1]$, the fuzzy rough set theory follows. From this point of view, this paper considers the L-fuzzy roughness in lattices based on a complete residuated lattice, which is a generalization of (fuzzy) roughness in lattices, see [34].

Based on the importance of ideals in studying logic algebras and extensive applications of t-norms and rough sets in fuzzy logic and artificial intelligence. In this paper, we study the relationships among L-fuzzy sets, rough sets and lattices theory. We hope that these links may invoke some new research topics for rough set theory and quantales and provide more applications in the fields such as logic, approximate reasoning and information science. Using a left continuous t-norm T on a complete lattice L, we introduce the notion of TL-fuzzy ideals in lattices and study the corresponding relations between TL-fuzzy ideals of two homomorphic lattices and discuss the image and the inverse image of generated TL-fuzzy ideals of the lattice X. Moreover, we consider the L-fuzzy roughness in lattices based on a complete residuated lattice.

2. Preliminaries

In this section, we recall some notions and definitions that will be used in the sequel.

First, we recall some basic notions about t-norms that we shall use in the following paragraphs.

Let $(L, \wedge, \vee, \leq, 0, 1)$ denote a complete lattice with the top and bottom elements 1 and 0, respectively.

Definition 1 ([18]). *A binary operation T on L is called a t-norm if it satisfies the following conditions: for all $a, b, c \in L$,*

$(T1)$ $aT1 = a$;

$(T2)$ $aTb = bTa$;

$(T3)$ $(aTb)Tc = aT(bTc)$;

$(T4)$ if $b \leq c$, then $aTb \leq aTc$.

Because of associative and commutative properties, for all $a_1, a_2, \cdots, a_n \in L$ $(n \geq 1)$, $a_1Ta_2T\cdots Ta_n$ is well defined and its value is irrelevant to the order of $a_1, a_2, \cdots, a_n$. We write $T_{i=1}^{n} a_i = a_1Ta_2T\cdots Ta_n$. The subset D_T of L is defined as $D_T = \{a \in L | aTa = a\}$. When $T = \wedge$, D_T coincides with L. If $aT(\vee_{i\in I} b_i) = \vee_{i\in I}(aTb_i)$, for all $a, b_i \in L$, where I is an arbitrary nonempty index set, then T is called a left continuous t-norm.

In what follows, let $L = [0, 1]$, there are some examples of the most popular t-norms: for all $x, y \in [0, 1]$,

(1) the Lukasiewicz t-norm: $xT_L y = max\{x + y - 1, 0\}$.

(2) the algebraic product: $xT_P y = xy$.

(3) the standard min operation: $xT_M y = min\{x, y\}$.

Given a t-norm T, the following binary operation on L, for any $a, b \in L$, $a\theta b = \vee\{c \in L | aTc \leq b\}$ is called the implication operator induced by T.

The implication operators induced by T_P, T_M and T_L, respectively, are: for any $x, y \in [0, 1]$,

(1) the Gaines implication: $x\theta_P y = \begin{cases} 1, & x \leq y, \\ \frac{y}{x}, & otherwise. \end{cases}$

(2) the Gödel implication: $x\theta_M y = \begin{cases} 1, & x \leq y, \\ y, & otherwise. \end{cases}$

(3) the Lukasiewicz implication: $x\theta_L y = min\{1, 1 - x + y\}$.

Some basic properties that will be used in the sequel of the implication operator θ are mentioned in the following. For further information about the implication operator θ please refer to [18, 5].

Proposition 1 ([18])**.** *The implication operator θ of a left continuous t-norm T satisfies the following properties: for any $a, b, c, a_i \in L$, $i \in I$,*

($\theta 1$) $a\theta 1 = 1$ *and* $1\theta a = a$;

($\theta 2$) $a \leq b \Longrightarrow c\theta a \leq c\theta b, a\theta c \geq b\theta c$;

($\theta 3$) $(a\theta b)Tc \leq a\theta(bTc)$;

($\theta 4$) $b\theta(\vee_{i\in I} a_i) \geq \vee_{i\in I}(b\theta a_i)$.

Throughout this paper, $(L, \wedge, \vee, \leq, 0, 1)$ always represents a given complete lattice with a left continuous t-norm T and an implication operator θ induced by T. In fact, such kind of lattice is a complete residuated lattice. Unless otherwise stated, the universe $(X, \wedge', \vee', \leq')$ indicates a given lattice.

An L-fuzzy subset of X is a mapping from X to L. The family of all L-fuzzy subsets of X is denoted by $LF[X]$ ([6]). When $L = [0, 1]$, the L-fuzzy subsets of X are known as fuzzy subsets of X ([39]). Let $\mu, \nu \in LF[X]$ be given, Then μ is said to be included in ν and written as $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in X$. And the union, intersection of μ and ν are defined as L-fuzzy subsets of X by $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$, $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$. In general, $(\cup_{i\in I}\mu_i)(x) = \vee_{i\in I}\mu_i(x)$, $(\cap_{i\in I}\mu_i)(x) = \wedge_{i\in I}\mu_i(x)$.

In what follows, we recall some basic notions about (θ, T)-fuzzy rough approximation operators, which will be needed in the following sections.

An L-fuzzy binary relation R on X is referred to an L-fuzzy subset of $X \times X$, where $R(x, y)$ is the degree of relation between x and y, for all $(x, y) \in X \times X$.

Definition 2 ([12]). *An L-fuzzy binary relation R on X is called a TL-fuzzy equivalence relation if it satisfies the following conditions: for any $x, y, z \in X$,*

(1) $R(x, x) = 1$;
(2) $R(x, y) = R(y, x)$;
(3) $R(x, y) \geq R(x, z)TR(z, y)$.

In particular, a TL-fuzzy equivalence relation on X is simply referred to as an L-fuzzy equivalence relation on X when $T = \wedge$. Moreover, when $L = [0, 1]$, an L-fuzzy equivalence relation on X is referred to as a fuzzy equivalence relation on X.

In the sequel, unless otherwise stated, R denotes a TL-fuzzy equivalence relation on X. In this case, the pair (X, R) is called a TL-fuzzy approximation space.

Definition 3 ([33, 31]). *Let (X, R) be a TL-fuzzy approximation space. Define the following two mappings $\overline{R}^T$ and $\underline{R}_\theta$: $LF[X] \to LF[X]$, respectively called T-upper fuzzy rough approximation operator and θ-lower fuzzy rough approximation operator, as follows: for any $\mu \in LF[X]$, $x \in X$, $\overline{R}^T\mu(x) = \vee_{y\in X} R(x, y)T\mu(y)$ and $\underline{R}_\theta\mu(x) = \wedge_{y\in X} R(x, y)\theta\mu(y)$.*

$\overline{R}^T\mu$ and $\underline{R}_\theta\mu$ are called a T-upper fuzzy rough approximation of μ and θ-lower fuzzy rough approximation of μ, respectively. And the pair $R^T_\theta\mu = (\underline{R}_\theta\mu, \overline{R}^T\mu)$ is called a (θ, T)-fuzzy rough set of μ with respect to the TL-fuzzy equivalence relation R if $\underline{R}_\theta\mu \neq \overline{R}^T\mu$.

Remark. In particular, if R is a crisp equivalence relation of X, then it is easy to see that: for any $\mu \in LF[X]$, $x \in X$, $\overline{R}^T\mu(x) = \vee_{y\in[x]_R}\mu(y)$ and $\underline{R}_\theta\mu(x) = \wedge_{y\in[x]_R}\mu(y)$.

Furthermore, if μ is a crisp subset of X, then $\overline{R}^T\mu = \{x \in X | [x]_R \cap \mu \neq \phi\}$ and $\underline{R}_\theta\mu = \{x \in X | [x]_R \subseteq \mu\}$, where $[x]_R$ denotes the equivalence class generated by R.

Let us review several properties of fuzzy rough approximation operators. For further information about (θ, T)-fuzzy rough sets please refer to [33].

Proposition 2 ([33]). *The (θ, T)-fuzzy rough approximation operators $\underline{R}_\theta$, $\overline{R}^T$ satisfy the following properties: for any $\mu, \nu \in LF[X]$,*

$(R1)$ $\underline{R}_\theta\mu \subseteq \mu \subseteq \overline{R}^T\mu$;
$(R2)$ if $\mu \subseteq \nu$, then $\underline{R}_\theta\mu \subseteq \underline{R}_\theta\nu$, $\overline{R}^T\mu \subseteq \overline{R}^T\nu$;
$(R3)$ $\overline{R}^T(\underline{R}_\theta\mu) = \underline{R}_\theta\mu$, $\underline{R}_\theta(\overline{R}^T\mu) = \overline{R}^T\mu$;
$(R4)$ $\overline{R}^T(\overline{R}^T\mu) = \overline{R}^T\mu$, $\underline{R}_\theta(\underline{R}_\theta\mu) = \underline{R}_\theta\mu$.

3. TL-fuzzy ideals of lattices

In this section, we shall introduce the notion of TL-fuzzy ideals in lattices and investigate some related properties of them. Specifically, we give the formula for calculating the TL-fuzzy ideal generated by L-fuzzy subsets and study the lattice structure of TL-fuzzy ideals in lattices. Also, there is an open problem in this section.

First, we shall introduce TL-fuzzy ideals in lattices.

Definition 4. *Let $(X, \wedge', \vee', \leq')$ be a lattice and μ be an L-fuzzy subset of X. Then μ is called a TL-fuzzy ideal of X if it satisfies the following conditions: for all $x, y \in X$,*

(i) $\mu(x \vee' y) \geq \mu(x) T \mu(y)$,

(ii) $\mu(x \wedge' y) \geq \mu(x) \vee \mu(y)$.

We shall denote the set of all TL-fuzzy ideals of the lattice X as $TLFI[X]$.

Now, the following result gives an equivalent version of the concept of TL-fuzzy filters in lattices.

Theorem 1. *Let $(X, \wedge', \vee', \leq')$ be a lattice and μ be an L-fuzzy subset of X. Then μ is a TL-fuzzy ideal of X if and only if it satisfies the following conditions: for all $x, y \in X$,*

(i) $\mu(x \vee' y) \geq \mu(x) T \mu(y)$,

(ii) if $y \leq' x$, then $\mu(y) \geq \mu(x)$.

Proof. The proof is straightforward. □

Let f be a mapping from X_1 into X_2, $\mu \in LF[X_1]$ and $\nu \in LF[X_2]$. The L-fuzzy sets $f(\mu)$ of X_2 and $f^{-1}(\nu)$ of X_1, defined by

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) | f(x) = y, x \in X_1\}, & f^{-1}(y) \neq \phi \\ 0, & f^{-1}(y) = \phi \end{cases}$$

and $f^{-1}(\nu)(x) = \nu(f(x))$, for all $x \in X_1$, $y \in X_2$, are called, respectively, the image of μ under f and the inverse image of ν under f.

Now we discuss properties of TL-fuzzy filters under lattice homomorphisms.

Proposition 3. *Let $f : X_1 \longrightarrow X_2$ be an onto homomorphism from a lattice X_1 to a lattice X_2 and μ be a TL-fuzzy ideal of X_1. Then $f(\mu)$ is also a TL-fuzzy ideal of X_2.*

Proof. Suppose that μ is a TL-fuzzy ideal of X_1.

Since f is surjective, then for all $y_1, y_2 \in X_2$, there are $x_1, x_2 \in X_1$ such that $f(x_1) = y_1, f(x_2) = y_2$. Then we have

$$\begin{aligned} & f(\mu)(y_1 \vee' y_2) = \vee\{\mu(x) | f(x) = y_1 \vee' y_2, x \in X_1\} \\ & \geq \vee\{\mu(x_1 \vee' x_2) | f(x_1 \vee' x_2) = y_1 \vee' y_2, x_1, x_2 \in X_1\} \\ & \geq \vee\{\mu(x_1) T \mu(x_2) | f(x_1) = y_1, f(x_2) = y_2, x_1, x_2 \in X_1\} \\ & = (\vee\{\mu(x_1) | f(x_1) = y_1, x_1 \in X_1\}) T (\vee\{\mu(x_2) | f(x_2) = y_2, x_2 \in X_1\}) \\ & = f(\mu)(y_1) T f(\mu)(y_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&f(\mu)(y_1 \wedge' y_2) = \vee\{\mu(x)|f(x) = y_1 \wedge' y_2, x \in X_1\}\\
&\geq \vee\{\mu(x_1 \wedge' x_2)|f(x_1 \wedge' x_2) = y_1 \wedge' y_2, x_1, x_2 \in X_1\}\\
&\geq \vee\{\mu(x_1) \vee \mu(x_2)|f(x_1) = y_1, f(x_2) = y_2, x_1, x_2 \in X_1\}\\
&= (\vee\{\mu(x_1)|f(x_1) = y_1, x_1 \in X_1\}) \vee (\vee\{\mu(x_2)|f(x_2) = y_2, x_2 \in X_1\})\\
&= f(\mu)(y_1) \vee f(\mu)(y_2).
\end{aligned}$$

Combining the above arguments, we have that $f(\mu)$ is a TL-fuzzy ideal of X_2. □

Definition 5. *Let f be a mapping from X_1 into X_2 and μ be an L-fuzzy subset of X_1. Then μ is said to be $f-$invariant if for any $x_1, x_2 \in X_1$, $f(x_1) = f(x_2)$, then $\mu(x_1) = \mu(x_2)$.*

Obviously, if μ is $f-$invariant, then $f^{-1}(f(\mu)) = \mu$.

Proposition 4. *Let $f : X_1 \longrightarrow X_2$ be a homomorphism from a lattice X_1 to a lattice X_2 and ν be a TL-fuzzy ideal of X_2. Then $f^{-1}(\nu)$ is also a TL-fuzzy ideal of X_1 and $f-$invariant.*

Proof. Assume that ν be a TL-fuzzy ideal of X_2. For any $x_1, x_2 \in X_1$, we have that $f^{-1}(\nu)(x_1 \vee' x_2) = \nu(f(x_1 \vee' x_2)) = \nu(f(x_1) \vee' f(x_2)) \geq \nu(f(x_1))T\nu(f(x_2)) = f^{-1}(\nu)(x_1)Tf^{-1}(\nu)(x_2)$.

Since f is a homomorphism from X_1 into X_2, then f is isotonic. Therefore, if $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$ and $f^{-1}(\nu)(x_1) = \nu(f(x_1)) \geq \nu(f(x_2)) = f^{-1}(\nu)(x_2)$. Thus, we get that $f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1.

Combining the above arguments, we have that $f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1. For any $x_1, x_2 \in X_1$, if $f(x_1) = f(x_2)$, then $f^{-1}(\nu)(x_1) = \nu(f(x_1)) = \nu(f(x_2)) = f^{-1}(\nu)(x_2)$. Therefore, $f^{-1}(\nu)$ is $f-$invariant. □

In order to discuss the corresponding relations between TL-fuzzy ideals of two homomorphic lattices, we recall the following lemmas.

Lemma 1 ([19]). *Let f be a mapping from X_1 to X_2 and μ , ν be L-fuzzy subsets of X_1 and X_2, respectively. Then:*

(1) *$f(f^{-1}(\nu)) \subseteq \nu$ and the equality holds if f is surjective,*

(2) *$f^{-1}(f(\mu)) \supseteq \mu$ and the equality holds if f is injective.*

Based on the discussion above, the following theorem shows the corresponding relations between TL-fuzzy filters of two homomorphic lattices.

Theorem 2. *Let $f : X_1 \longrightarrow X_2$ be an onto homomorphism from a lattice X_1 to a lattice X_2. $TLFI_f[X_1]$ is the set of all TL-fuzzy ideals which are $f-$invariant of X_1, then there is an one to one order-preserving mapping from $TLFI_f[X_1]$ to $TLFI[X_2]$.*

Proof. Let φ: $TLFI_f[X_1] \longrightarrow TLFI[X_2]$ be defined by $\varphi(\mu) = f(\mu)$, for all $\mu \in TLFI_f[X_1]$.

(i) Clearly, φ is a map from $TLFI_f[X_1]$ to $TLFI[X_2]$.

(ii) Let us check that φ is injective. For any $\mu_1, \mu_2 \in TLFI_f[X_1]$, if $\varphi(\mu_1) = \varphi(\mu_2)$, which means $f(\mu_1) = f(\mu_2)$, then $f^{-1}(f(\mu_1)) = f^{-1}(f(\mu_2))$. Since μ_1 and μ_2 are $f-$invariant, it follows that $f^{-1}(f(\mu_1)) = \mu_1$ and $f^{-1}(f(\mu_2)) = \mu_2$. Hence $\mu_1 = \mu_2$. Therefore, φ is injective.

(iii) Now, we prove that φ is surjective. For all $\nu \in TLFI[X_2]$, by Proposition 4, we have that $f^{-1}(\nu)$ is also a TL-fuzzy ideal of X_1 and $f-$invariant. This means that $f^{-1}(\nu) \in TLFI_f[X_1]$. Since f is surjective, we have that $\varphi(f^{-1}(\nu)) = f(f^{-1}(\nu)) = \nu$ by Lemma 1. It follows that φ is surjective.

(iv) We shall check that φ is order-preserving. For any $\mu_1, \mu_2 \in TLFI_f[X_1]$, if $\mu_1 \subseteq \mu_2$, then for $x \in X_1$, $\mu_1(x) \leq \mu_2(x)$. Notice this, for any $y \in X_2$, we have that $f(\mu_1)(y) = \vee\{\mu_1(x)|f(x) = y, x \in X_1\} \leq \vee\{\mu_2(x)|f(x) = y, x \in X_1\} = f(\mu_2)(y)$, which means $f(\mu_1) \subseteq f(\mu_2)$. That is, $\varphi(\mu_1) \subseteq \varphi(\mu_2)$. Therefore, φ is order-preserving.

Combining the above arguments, we have that φ is an one to one order-preserving mapping from $TLFI_f[X_1]$ to $TLFI[X_2]$. □

Definition 6. *Let μ be an L-fuzzy subset in a lattice X. A TL-fuzzy ideal ν of the lattice X is said to be generated by μ, if $\mu \subseteq \nu$ and for any TL-fuzzy ideal ω of X, $\mu \subseteq \omega$ implies $\nu \subseteq \omega$. The TL-fuzzy ideal generated by μ will be denoted by $(\mu]_{TL}$.*

It follows from Definition 6 that $(\mu]_{TL}$ is the smallest TL-fuzzy ideal of the lattice X containing μ. And we can easily get that $(\mu]_{TL} = \cap_{i\in I}\{\mu_i \in TLFI[X]|\mu_i \supseteq \mu, i \in I\}$.

In what follows, we give the formula for calculating the TL-fuzzy ideals generated by L-fuzzy subsets.

Theorem 3. *Let μ be an L-fuzzy subset in the lattice X. Then for any $x \in X$, $(\mu]_{TL}(x) = \vee\{\mu(a_1)T\mu(a_2)T\cdots T\mu(a_n)|x \leq' a_1 \vee' a_2 \vee' \cdots \vee' a_n, a_i \in X, i \in I\}$.*

Proof. The proof is straightforward. □

Now, we discuss the image and the inverse image of the generated TL-fuzzy ideals of the lattice X. In order to get this, we recall the following lemma.

Lemma 2 ([19]). *Let f be a mapping from X_1 to X_2 and $\mu, \mu' \in LF[X_1]$, $\nu, \nu' \in LF[X_2]$, then:*

(1) $\mu \subseteq \mu'$ implies $f(\mu) \subseteq f(\mu')$, which means f is order-preserving. (2) *$\nu \subseteq \nu'$ implies $f^{-1}(\nu) \subseteq f^{-1}(\nu')$, which means f^{-1} is order-preserving, too.*

Now, we arrive at one of our main theorems.

Theorem 4. *Let $f : X_1 \longrightarrow X_2$ be an onto homomorphism from a lattice X_1 to a lattice X_2 and μ be an L-fuzzy subset of X_1. Then $f((\mu]_T) = (f(\mu)]_T$.*

Proof. By Proposition 3, we get that $f((\mu]_T)$ is a TL-fuzzy ideal of X_2. Notice $(\mu]_T \supseteq \mu$, we have $f((\mu]_T) \supseteq f(\mu)$ by Lemma 2. Now, suppose that ν is an any TL-fuzzy ideal of X_2 and $\nu \supseteq f(\mu)$, then $f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1 by Proposition 4. Since $\nu \supseteq f(\mu)$, we have that $f^{-1}(\nu) \supseteq f^{-1}(f(\mu)) \supseteq \mu$ by Lemmas 1 and 2, which implies $f^{-1}(\nu) \supseteq (\mu]_T$. It follows from Lemmas 1 and 2, we get that $\nu \supseteq f(f^{-1}(\nu)) \supseteq f((\mu]_T)$. This means that $f((\mu]_T)$ is the smallest TL-fuzzy ideal of X_2 which contains $f(\mu)$. Therefore, $f((\mu]_T) = (f(\mu)]_T$. □

Theorem 5. *Let $f : X_1 \longrightarrow X_2$ be an isomorphism from a lattice X_1 to a lattice X_2 and ν be a TL-fuzzy subset of X_2. Then $f^{-1}((\nu]_T) = (f^{-1}(\nu)]_T$.*

Proof. By Proposition 4, we get that $f^{-1}((\nu]_T)$ is a TL-fuzzy ideal of X_1. Notice $(\nu]_T \supseteq \nu$, we have $f^{-1}((\nu]_T) \supseteq f^{-1}(\nu)$ by Lemma 2. Now, suppose that ω is an any TL-fuzzy ideal of X_1 and $\omega \supseteq f^{-1}(\nu)$, then $f(\omega)$ is a TL-fuzzy ideal of X_2 by Proposition 3. Since $\omega \supseteq f^{-1}(\nu)$, notice that f is surjective, we have that $f(\omega) \supseteq f(f^{-1}(\nu)) = \nu$ by Lemmas 1 and 2, which implies $f(\omega) \supseteq (\nu]_T$. It follows from Lemmas 1 and 2, we get that $\omega = f^{-1}(f(\omega)) \supseteq f^{-1}((\nu]_T)$. This means that $f^{-1}((\nu]_T)$ is the smallest TL-fuzzy ideal of X_1 which contains $f^{-1}(\nu)$. Therefore, $f^{-1}((\nu]_T) = (f^{-1}(\nu)]_T$. □

4. T-upper and θ-lower fuzzy rough approximations of TL-fuzzy ideals

In this section, based on the discussion in the previous sections, we consider the L-fuzzy roughness in lattices based on a complete residuated lattice, which is a generalization of (fuzzy) roughness in lattices. We will investigate some properties of (θ, T)-fuzzy rough approximations of TL-fuzzy filters in lattices. We start by introducing the concept of TL-fuzzy congruence relation on lattices.

Definition 7. *A TL-fuzzy equivalence relation R on the lattice $(X, \wedge', \vee')$ is called a TL-fuzzy congruence relation if it satisfies the following conditions: for $x_1, x_2, y_1, y_2 \in X$, $R(x_1 \wedge' x_2, y_1 \wedge' y_2) \geq R(x_1, y_1)TR(x_2, y_2)$, $R(x_1 \vee' x_2, y_1 \vee' y_2) \geq R(x_1, y_1)TR(x_2, y_2)$.*

In particular, by a fuzzy congruence relation R on the lattice $(X, \wedge', \vee')$, we mean a TL-fuzzy congruence relation on $(X, \wedge', \vee')$ when $T = T_M$ and $L = [0, 1]$.

Proposition 5. *Let R be a TL-fuzzy congruence relation on the lattice X. If μ is a TL-fuzzy ideals of X, then $\overline{R}^T \mu$ is a TL-fuzzy ideals of X.*

Proof. Suppose that μ is a TL-fuzzy ideal of X. For any $x, y \in X$, we have $\overline{R}^T \mu(x \vee' y) = \vee_{z \in X} R(x \vee' y, z)T\mu(z) \geq \vee_{s,t \in X} R(x \vee' y, s \vee' t)T\mu(s \vee' t) \geq (\vee_{s,t \in X} R(x, s)TR(y, t))T(\mu(s)T\mu(t)) = (\vee_{s \in X} R(x, s)T\mu(s))T(\vee_{t \in X} R(y, t)T\mu(t)) = \overline{R}^T \mu(x)T\overline{R}^T \mu(y)$. On the other hand, we have $\overline{R}^T \mu(x \wedge' y) = \vee_{z \in X} R(x \wedge' y, z)T\mu(z) \geq \vee_{s,t \in X} R(x \wedge' y, s \wedge' t)T\mu(s \wedge' t) \geq (\vee_{s,t \in X} R(x, s)TR(y, t))T(\mu(s) \vee \mu(t)) = (\vee_{s \in X} R(x, s)T\mu(s)) \vee (\vee_{t \in X} R(y, t)T\mu(t)) = \overline{R}^T \mu(x) \vee \overline{R}^T \mu(y)$.

Combining the above arguments, we have that $\overline{R}^T\mu$ is a TL-fuzzy ideal of X. $\square$

Next, let us consider the relationships between T-upper fuzzy rough approximation operator and the generated TL-fuzzy ideal.

Proposition 6. *Let R be a TL-fuzzy congruence relation on the lattice X. If μ is an L-fuzzy subset of X, then $\overline{R}^T(\mu]_T = \overline{R}^T(\overline{R}^T\mu]_T$.*

Proof. Since $\mu \subseteq (\mu]_T$, it follows that $\overline{R}^T\mu \subseteq \overline{R}^T(\mu]_T$. By Proposition 5, we have that $\overline{R}^T(\mu]_T$ is a TL-fuzzy ideal of the lattice X. We conclude that $(\overline{R}^T\mu]_T \subseteq \overline{R}^T(\mu]_T$. Again, by $(R2)$ and $(R4)$ of Proposition 2, we have that $\overline{R}^T(\overline{R}^T\mu]_T \subseteq \overline{R}^T\overline{R}^T(\mu]_T = \overline{R}^T(\mu]_T$. On the other hand, since $\mu \subseteq \overline{R}^T\mu$, it follows that $(\mu]_T \subseteq (\overline{R}^T\mu]_T$. Hence $\overline{R}^T(\mu]_T \subseteq \overline{R}^T(\overline{R}^T\mu]_T$.

Combining the above arguments, we have $\overline{R}^T(\mu]_T = \overline{R}^T(\overline{R}^T\mu]_T$. $\square$

Open problem 2. Whether $\underline{R}_\theta\mu$ is a TL-fuzzy ideal of X if μ is a TL-fuzzy ideal of the lattice X and R is a TL-fuzzy congruence relation on the lattice X.

In what follows, we discuss some homomorphism properties of (θ, T)-fuzzy rough approximations of TL-fuzzy filters in lattices. In order to do this, we recall the following definition [30] and give some lemmas.

Let f be a mapping from X_1 into X_2 let R_1 and R_2 be two L-fuzzy relations on X_1 and X_2, respectively. The L-fuzzy relations $f(R_1)$ on X_2, defined by $f(R_1)(y_1, y_2) = \vee\{R_1(x_1, x_2)|f(x_1) = y_1, f(x_2) = y_2\}$, for any $(y_1, y_2) \in X_2 \times X_2$, if both y_1 and y_2 have inverse elements in X_1. Otherwise, $f(R_1)(y_1, y_2) = 0$. The L-fuzzy relations $f^{-1}(R_2)$ on X_1, is defined by $f^{-1}(R_2)(x_1, x_2) = R_2(f(x_1), f(x_2))$, for any $(x_1, x_2) \in X_1 \times X_1$. The L-fuzzy relations $f(R_1)$ and $f^{-1}(R_2)$ are called, respectively, the image of R_1 under f and the inverse image of R_2 under f.

Lemma 3. *Let $f : X_1 \longrightarrow X_2$ be a homomorphism from a lattice X_1 to a lattice X_2 and let R_1 and R_2 be two TL-fuzzy congruence relations on X_1 and X_2, respectively.*

(1) *If f is an isomorphism, then $f(R_1)$ is a TL-fuzzy congruence relation on X_2.*

(2) *$f^{-1}(R_2)$ is a TL-fuzzy congruence relation on X_1.*

Proof. It is easy to check them. $\square$

Lemma 4. *Let $f : X_1 \longrightarrow X_2$ be an isomorphism from a lattice X_1 to a lattice X_2 and let R_1 be a TL-fuzzy congruence relation on the lattice X_1. Suppose that μ is a L-fuzzy subset of X_1, then $\underline{f(R_1)}_\theta f(\mu) = f(\underline{R_1}_\theta\mu)$, $\overline{f(R_1)}^T f(\mu) = f(\overline{R_1}^T\mu)$.*

Proof. Since $f : X_1 \longrightarrow X_2$ is an isomorphism from a lattice X_1 to a lattice X_2 and R_1 is a TL-fuzzy congruence relation on X_1. By Lemma 3, we have that $f(R_1)$ is a TL-fuzzy congruence relation on X_2. Then, for any $y \in X_2$, we have that

$\underline{f(R_1)}_\theta f(\mu)(y) = \wedge_{y' \in X_2} f(R_1)(y, y')\theta f(\mu)(y') = \wedge_{x' \in X_1} R_1(x, x')\theta\mu(x') = \underline{R_1}_\theta\mu(x) = f(\underline{R_1}_\theta\mu)(y)$, where $f(x) = y, f(x') = y'$. Therefore, $\underline{f(R_1)}_\theta f(\mu) = f(\underline{R_1}_\theta\mu)$.

In a similar way, for any $y \in X_2$, we have that

$\overline{f(R_1)}^T f(\mu)(y) = \vee_{y' \in X_2} f(R_1)(y, y')T f(\mu)(y') = \vee_{x' \in X_1} R_1(x, x')T\mu(x') = \overline{R_1}^T \mu(x) = f(\overline{R_1}^T \mu)(y)$, where $f(x) = y, f(x') = y'$. Therefore, $\overline{f(R_1)}^T f(\mu) = f(\overline{R_1}^T \mu)$. □

Lemma 5. *Let $f : X_1 \longrightarrow X_2$ be a homomorphism from a lattice X_1 to a lattice X_2 and let R_2 be a TL-fuzzy congruence relation on X_2. Suppose that ν is a L-fuzzy subset of X_2, then $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu) \supseteq f^{-1}(\underline{R_2}_\theta\nu)$, $\overline{f^{-1}(R_2)}^T f^{-1}(\nu) \subseteq f^{-1}(\overline{R_2}^T \nu)$. In particular, when f is surjective, then $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)=f^{-1}(\underline{R_2}_\theta\nu)$, $\overline{f^{-1}(R_2)}^T f^{-1}(\nu) = f^{-1}(\overline{R_2}^T \nu)$.*

Proof. Since $f : X_1 \longrightarrow X_2$ is a homomorphism from a lattice X_1 to a lattice X_2 and R_2 is a TL-fuzzy congruence relation on X_2. By Lemma 3, we have that $f^{-1}(R_2)$ is a TL-fuzzy congruence relation on X_1. Then, for any $x \in X_1$, we have that $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)(x) = \wedge_{x' \in X_1} f^{-1}(R_2)(x, x')\theta f^{-1}(\nu)(x') = \wedge_{x' \in X_1} R_2(f(x), f(x'))\theta\nu(f(x')) \geq \wedge_{y \in X_2} R_2(f(x), y)\theta\nu(y) = \underline{R_2}_\theta\nu(f(x)) = f^{-1}(\underline{R_2}_\theta\nu)(x)$. Therefore, we have $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu) \supseteq f^{-1}(\underline{R_2}_\theta\nu)$.

In a similar way, for any $x \in X_1$, we have that $\overline{f^{-1}(R_2)}^T f^{-1}(\nu)(x) = \vee_{x' \in X_1} f^{-1}(R_2)(x, x')T f^{-1}(\nu)(x') = \vee_{x' \in X_1} R_2(f(x), f(x'))T\nu(f(x'))$
$\leq \vee_{y \in X_2} R_2(f(x), y)T\nu(y) = \overline{R_2}^T \nu(f(x)) = f^{-1}(\overline{R_2}^T \nu)(x)$. Therefore, we have $\overline{f^{-1}(R_2)}^T f^{-1}(\nu) \subseteq f^{-1}(\overline{R_2}^T \nu)$.

It is easy to see that the equality "=" in the above proof holds if f is surjective. Then $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)=f^{-1}(\underline{R_2}_\theta\nu)$, $\overline{f^{-1}(R_2)}^T f^{-1}(\nu) = f^{-1}(\overline{R_2}^T \nu)$. □

Combining the above lemmas and theorems, we have the following results.

Theorem 6. *Let $f : X_1 \longrightarrow X_2$ be an isomorphism from a lattice X_1 to a lattice X_2 and R_1 be a TL-fuzzy congruence relation on a lattice X_1. Suppose that μ is a L-fuzzy subset of X_1. Then*

(1) $\underline{R_1}_\theta\mu$ is a TL-fuzzy ideal of X_1 iff $\underline{f(R_1)}_\theta f(\mu)$ is a TL-fuzzy ideal of X_2,

(2) $\overline{R_1}^T \mu$ is a TL-fuzzy ideal of X_1 iff $\overline{f(R_1)}^T f(\mu)$ is a TL-fuzzy ideal of X_2.

Proof. (1) $\Longrightarrow$ Assume that $\underline{R_1}_\theta\mu$ is a TL-fuzzy ideal of X_1. By Proposition 3, we have that $f(\underline{R_1}_\theta\mu)$ is a TL-fuzzy ideal of X_2. Since f is an isomorphism from

X_1 to X_2, we get that $\underline{f(R_1)}_\theta f(\mu) = f(\underline{R_1}_\theta \mu)$ by Lemma 4. Hence, $\underline{f(R_1)}_\theta f(\mu)$ is a TL-fuzzy ideal of X_2.

$\Longleftarrow$ If $\underline{f(R_1)}_\theta f(\mu)$ is a TL-fuzzy ideal of X_2. By Proposition 4, we obtain that $f^{-1}(\underline{f(R_1)}_\theta f(\mu))$ is a TL-fuzzy ideal of X_1. Since f is injective, we can get that $f^{-1}(\underline{f(R)}_\theta f(\mu)) = \underline{f^{-1}(f(R_1))}_\theta f^{-1}(f(\mu)) = \underline{R_1}_\theta \mu$ by use of Lemmas 1 and 5. Therefore, $\underline{R_1}_\theta \mu$ is a TL-fuzzy ideal of X_1.

(2) In a similar way, we can prove that $\overline{R_1}^T \mu$ is a TL-fuzzy ideal of X_1 if and only if $\overline{f(R_1)}^T f(\mu)$ is a TL-fuzzy ideal of X_2.

□

Theorem 7. *Let $f : X_1 \longrightarrow X_2$ be an isomorphism from a lattice X_1 to a lattice X_2 and R_2 be a TL-fuzzy congruence relation on the lattice X_2. Suppose that ν is a L-fuzzy subset of X_2. Then*

(1) *$\underline{R_2}_\theta \nu$ is a TL-fuzzy ideal of X_2 iff $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1,*

(2) *$\overline{R_2}^T \nu$ is a TL-fuzzy ideal of X_2 iff $\overline{f^{-1}(R_2)}^T f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1.*

Proof. (1) $\Longrightarrow$ Suppose that $\underline{R_2}_\theta \nu$ is a TL-fuzzy ideal of X_2. By Proposition 4, we have that $f^{-1}(\underline{R_2}_\theta \nu)$ is a TL-fuzzy ideal of X_1. Since f is an epimorphism from X_1 to X_2, we get $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu) = f^{-1}(\underline{R_2}_\theta \nu)$ by Lemma 5. Hence, $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1.

$\Longleftarrow$ If $\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1. By Proposition 3, we obtain that $f(\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu))$ is a TL-fuzzy ideal of X_2. Since f is surjective, we get that $f(\underline{f^{-1}(R_2)}_\theta f^{-1}(\nu)) = \underline{f(f^{-1}(R_2))}_\theta f(f^{-1}(\nu)) = \underline{R_2}_\theta \nu$ by use of Lemmas 1 and 4. Therefore, $\underline{R_2}_\theta \nu$ is a TL-fuzzy ideal of X_2.

(2) In a similar way, we can prove that $\overline{R_2}^T \mu$ is a TL-fuzzy ideal of X_2 if and only if $\overline{f^{-1}(R_2)}^T f^{-1}(\nu)$ is a TL-fuzzy ideal of X_1. □

5. Conclusion

Based on the importance of ideals in studying logic algebras and extensive applications of t-norms and rough sets in fuzzy logic and artificial intelligence, in this paper, using a left continuous t-norm T on a complete lattice L, we introduce the notion of TL-fuzzy ideals of lattices and investigate some properties of them. We study the relationships among L-fuzzy sets, rough sets and lattices theory. We hope that the results presented in this paper can hopefully provide more insight into and a full understanding of lattices and fuzzy rough sets theory.

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