Schur convexity of generalized geometric Bonferroni mean involving three parameters

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Abstract. In this paper, we discuss the Schur convexity, Schur geometric convexity and Schur harmonic convexity of the generalized geometric Bonferroni mean. At the end of the paper, two inequalities related to the generalized geometric Bonferroni mean are established to illustrate the applications of the obtained results.
Keywords: Schur convexity, Schur geometric convexity, Schur harmonic convexity, generalized geometric Bonferroni means, inequalities.

1. Introduction

The Schur convexity of functions relating to special means is a very significant research subject and has attracted the interest of many mathematicians. There are numerous articles written on this topic in recent years, see [1, 2] and the references therein. As supplements to the Schur convexity of functions, the

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Schur geometrically convex functions and Schur harmonically convex functions were investigated by Zhang and Yang [3], Chu et al. [4, 5, 6], Shi et al. [7, 8], Meng et al. [9], Zheng et al. [10]. These properties of functions have been found to have an important application in discovering and proving the inequalities for special means (see [11, 12, 13, 14]).

Recently, it comes to our attention that a type of means which is symmetrical on n variables $x_1,\ x_2,\ \ldots,\ x_n$ and involves two parameters, it was initially proposed by Bonferroni [15], as follows

$$B^{p,q}(\boldsymbol{x}) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} x_i^p x_j^q \right)^{\frac{1}{p+q}}, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n),\ x_i \geq 0, i = 1, 2, \ldots, n,\ p, q \geq 0$ and $p+q \neq 0$.

$B^{p,q}(\boldsymbol{x})$ is called the Bonferroni mean. It has important application in multi criteria decision-making (see [16, 17, 18, 19, 20, 21]).

Beliakov et al. [22] generalized the Bonferroni mean by introducing three parameters p, q, r, i.e.,

$$B^{p,q,r}(\boldsymbol{x}) = \left(\frac{1}{n(n-1)(n-2)} \sum_{i,j.k=1, i\neq j\neq k}^{n} x_i^p x_j^q x_k^r \right)^{\frac{1}{p+q+r}}, \tag{2}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n),\ x_i \geq 0, i = 1, 2, \ldots, n,\ p, q, r \geq 0$ and $p+q+r \neq 0$.

Motivated by the the Bonferroni mean $B^{p,q}(\boldsymbol{x})$ and the geometric mean $G(\boldsymbol{x}) = \prod_{i=1}^{n} (x_i)^{\frac{1}{n}}$, Xia et al. [23] introduced a new mean which is called the geometric Bonferroni mean, as follows

$$GB^{p,q}(\boldsymbol{x}) = \frac{1}{p+q} \prod_{i,j=1, i\neq j}^{n} (px_i + qx_j)^{\frac{1}{n(n-1)}}, \tag{3}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n),\ x_i > 0, i = 1, 2, \ldots, n,\ p, q \geq 0$ and $p+q \neq 0$.

An extension of the geometric Bonferroni mean was given by Park and Kim in [19], which is called generalized geometric Bonferroni mean, i.e.,

$$GB^{p,q,r}(\boldsymbol{x}) = \frac{1}{p+q+r} \prod_{i,j,k=1, i\neq j\neq k}^{n} (px_i + qx_j + rx_k)^{\frac{1}{n(n-1)(n-2)}}, \tag{4}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n),\ x_i > 0, i = 1, 2, \ldots, n,\ p, q, r \geq 0$ and $p+q+r \neq 0$.

Remark 1. For $r = 0$, it is easy to observe that

$$GB^{p,q,0}(\boldsymbol{x}) = \frac{1}{p+q+0} \prod_{i,j=1, i\neq j}^{n} \Big[\prod_{k=1, i\neq j\neq k}^{n} (px_i + qx_j + 0 \times x_k) \Big]^{\frac{1}{n(n-1)(n-2)}}$$

$$= \frac{1}{p+q} \prod_{i,j=1,i\neq j}^{n} \left[(px_i + qx_j)^{(n-2)}\right]^{\frac{1}{n(n-1)(n-2)}}$$

$$= \frac{1}{p+q} \prod_{i,j=1,i\neq j}^{n} (px_i + qx_j)^{\frac{1}{n(n-1)}}$$

$$= GB^{p,q}(\boldsymbol{x}).$$

Remark 2. If $q = 0, r = 0$, then the generalized geometric Bonferroni mean reduces to the geometric mean, i.e.,

$$GB^{p,0,0}(\boldsymbol{x}) = GB^{p,0}(\boldsymbol{x}) = \frac{1}{p} \prod_{i,j=1,i\neq j}^{n} (px_i)^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} (x_i)^{\frac{1}{n}} = G(\boldsymbol{x}).$$

Remark 3. If $\boldsymbol{x} = (x, x, \ldots, x)$, then

$$GB^{p,q,r}(\boldsymbol{x}) = GB^{p,q,r}(x, x, \ldots, x) = x.$$

For convenience, throughout the paper $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \geq 0,\ i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i > 0,\ i = 1, 2, \ldots, n\}.$$

In a recent paper [24], Shi and Wu investigated the Schur m-power convexity of the geometric Bonferroni mean $GB^{p,q}(\boldsymbol{x})$ and obtained the following results:

Proposition 1. *For fixed positive real numbers p, q, (i) if $m < 0$ or $m = 0$, then $GB^{p,q}(\boldsymbol{x})$ is Schur m-power convex on $\mathbb{R}^n_{++}$; (ii) if $m = 1$ or $m \geq 2$, then $GB^{p,q}(\boldsymbol{x})$ is Schur m-power concave on $\mathbb{R}^n_{++}$.*

In this paper we discuss the Schur convexity, Schur geometric convexity and Schur harmonic convexity of the generalized geometric Bonferroni mean $GB^{p,q,r}(\boldsymbol{x})$. Our main results are as follows.

Theorem 1. *For fixed non-negative real numbers p, q, r with $p + q + r \neq 0$, and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n), n \geq 3$, (i) if $m \geq 1$ then $GB^{p,q,r}(\boldsymbol{x})$ is Schur m-power concave on $\mathbb{R}^n_{++}$; (ii) if $m < 0$, then $GB^{p,q,r}(\boldsymbol{x})$ is Schur m-power convex on $\mathbb{R}^n_{++}$.*

Corollary 1. *For fixed non-negative real numbers p, q, r with $p + q + r \neq 0$, if $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $n \geq 3$, then $GB^{p,q,r}(\boldsymbol{x})$ is Schur concave and Schur harmonic convex on $\mathbb{R}^n_{++}$.*

2. Preliminaries

We introduce some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

Definition 1 ([1]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(ii) Let $\Omega \subset \mathbb{R}^n$,the function φ: $\Omega \to \mathbb{R}$ is said to be Schur convex on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be Schur concave function on Ω if and only if $-\varphi$ is Schur convex function on Ω.

Definition 2 ([1]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega,\ 0 \leq \alpha \leq 1$ implies

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega.$$

Definition 3 ([1]). (i) A set $\Omega \subset \mathbb{R}^n$ is called symmetric, if $\boldsymbol{x} \in \Omega$ implies $\boldsymbol{x}P \in \Omega$ for every $n \times n$ permutation matrix P.

(ii) A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$.

The following proposition is called Schur's condition. It provides an approach for testing whether a vector valued function is Schur convex or not.

Proposition 2 ([1]). *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the Schur convex function (Schur concave function) if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left[\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - \frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0 \ (\leq 0) \tag{5}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

Definition 4 ([25]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+$.

(i) $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$,$\boldsymbol{y} \in \Omega$ and α,$\beta \in [0, 1]$ such that $\alpha + \beta = 1$.

(ii) Let $\Omega \subset \mathbb{R}^n_+$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur geometrically convex function on Ω if $(\log x_1, \log x_2, \ldots, \log x_n) \prec (\log y_1, \log y_2, \ldots, \log y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is Schur geometrically convex function.

Definition 5 ([25])**.** Let $\Omega \subset \mathbb{R}_+^n$.

(*i*) A set Ω is said to be harmonically convex if $\dfrac{\boldsymbol{xy}}{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, where $\boldsymbol{xy} = (x_1y_1, x_2y_2, \ldots, x_ny_n)$ and $\frac{1}{\boldsymbol{\lambda x + (1-\lambda) y}} = (\dfrac{1}{\lambda x_1 + (1-\lambda)y_1}, \dfrac{1}{\lambda x_2 + (1-\lambda)y_2}, \cdots, \dfrac{1}{\lambda x_n + (1-\lambda)y_n})$.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur harmonically convex on Ω if $\dfrac{1}{\boldsymbol{x}} \prec \dfrac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Proposition 3 ([25])**.** *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric and harmonically convex set with inner points, and let $\varphi : \Omega \to \mathbb{R}_+$ be a continuously symmetric function which is differentiable on Ω^0. Then φ is Schur harmonically convex* (*Schur harmonically concave*) *on Ω if and only if*

$$(x_1 - x_2)\left[x_1^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \ge 0 \ (\le 0) \tag{6}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

Proposition 4 ([26])**.** *Let $\Omega \subset \mathbb{R}_{++}^n$ be a symmetric set with nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex (Schur m-power concave) on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{1-m} \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m} \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \ge 0 \ (\le 0), \quad \textit{if} \ \ m \ne 0 \tag{7}$$

and

$$(\log x_1 - \log x_2)\left[x_1 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \ge 0 \ (\le 0), \quad \textit{if} \ \ m = 0 \tag{8}$$

for all $\boldsymbol{x} \in \Omega^\circ$.

Lemma 1 ([12])**.** *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$, $A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i$, $n \ge 2$ and $1 \le \lambda \le \frac{n}{n-1}$. Then*

$$\begin{aligned}&(\lambda A_n(\boldsymbol{x}) + (1-\lambda)x_1, \lambda A_n(\boldsymbol{x}) + (1-\lambda)x_2, \ldots, \lambda A_n(\boldsymbol{x}) + (1-\lambda)x_n)\\ &\prec (x_1, x_2, \ldots, x_n).\end{aligned} \tag{9}$$

Lemma 2 ([2])**.** *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $\sum_{i=1}^n x_i = (n-2)c$, $n \ge 2$. Then*

$$\left(\frac{c + x_1}{n-1}, \frac{c + x_2}{n-1}, \ldots, \frac{c + x_n}{n-1}\right) \prec (c - x_1, c - x_2, \ldots, c - x_n)\,. \tag{10}$$

3. Proof of main results

Proof of Theorem 1.

Note that the generalized geometric Bonferroni mean is defined by

$$GB^{p,q,r}(\boldsymbol{x}) = \frac{1}{p+q+r} \prod_{i,j,k=1, i\neq j\neq k}^{n} (px_i + qx_j + rx_k)^{\frac{1}{n(n-1)(n-2)}},$$

Taking the natural logarithm gives

$$\log GB^{p,q,r}(\boldsymbol{x}) = \log \frac{1}{p+q+r} + \frac{1}{n(n-1)(n-2)} Q,$$

where

$$\begin{aligned} Q = & \sum_{j,k=3, j\neq k}^{n} [\log(px_1 + qx_j + rx_k) + \log(px_2 + qx_j + rx_k)] \\ & + \sum_{i,k=3, i\neq k}^{n} [\log(px_i + qx_1 + rx_k) + \log(px_i + qx_2 + rx_k)] \\ & + \sum_{i,j=3, i\neq j}^{n} [\log(px_i + qx_j + rx_1) + \log(px_i + qx_j + rx_2)] \\ & + \sum_{k=3}^{n} [\log(px_1 + qx_2 + rx_k) + \log(px_2 + qx_1 + rx_k)] \\ & + \sum_{j=3}^{n} [\log(px_1 + qx_j + rx_2) + \log(px_2 + qx_j + rx_1)] \\ & + \sum_{i=3}^{n} [\log(px_i + qx_1 + rx_2) + \log(px_i + qx_2 + rx_1)] \\ & + \sum_{i,j,k=3,\, i\neq j\neq k}^{n} \log(px_i + qx_j + rx_k). \end{aligned}$$

Differentiating $GB^{p,q,r}(\boldsymbol{x})$ with respect to x_1 and x_2 respectively, we have

$$\frac{\partial GB^{p,q,r}(\boldsymbol{x})}{\partial x_1} = \frac{GB^{p,q,r}(\boldsymbol{x})}{n(n-1)(n-2)} \cdot \frac{\partial Q}{\partial x_1}$$

$$= \frac{GB^{p,q,r}(\boldsymbol{x})}{n(n-1)(n-2)} \Bigg[\sum_{j,k=3, j\neq k}^{n} \frac{p}{px_1 + qx_j + rx_k} + \sum_{i,k=3, i\neq k}^{n} \frac{q}{px_i + qx_1 + rx_k}$$

$$+ \sum_{i,j=3, i\neq j}^{n} \frac{r}{px_i + qx_j + rx_1} + \sum_{k=3}^{n} \left(\frac{p}{px_1 + qx_2 + rx_k} + \frac{q}{px_2 + qx_1 + rx_k} \right)$$

$$+\sum_{j=3}^{n}\Big(\frac{p}{px_1+qx_j+rx_2}+\frac{r}{px_2+qx_j+rx_1}\Big)$$
$$+\sum_{i=3}^{n}\Big(\frac{q}{px_i+qx_1+rx_2}+\frac{r}{px_i+qx_2+rx_1}\Big)\Bigg].$$

$$\frac{\partial GB^{p,q,r}(\boldsymbol{x})}{\partial x_2}=\frac{GB^{p,q,r}(\boldsymbol{x})}{n(n-1)(n-2)}\cdot\frac{\partial Q}{\partial x_2}$$
$$=\frac{GB^{p,q,r}(\boldsymbol{x})}{n(n-1)(n-2)}\Bigg[\sum_{j,k=3,j\neq k}^{n}\frac{p}{px_2+qx_j+rx_k}+\sum_{i,k=3,i\neq k}^{n}\frac{q}{px_i+qx_2+rx_k}$$
$$+\sum_{i,j=3,i\neq j}^{n}\frac{r}{px_i+qx_j+rx_2}+\sum_{k=3}^{n}\Big(\frac{q}{px_1+qx_2+rx_k}+\frac{p}{px_2+qx_1+rx_k}\Big)$$
$$+\sum_{j=3}^{n}\Big(\frac{r}{px_1+qx_j+rx_2}+\frac{p}{px_2+qx_j+rx_1}\Big)$$
$$+\sum_{i=3}^{n}\Big(\frac{r}{px_i+qx_1+rx_2}+\frac{q}{px_i+qx_2+rx_1}\Big)\Bigg].$$

It is easy to see that $GB^{p,q,r}(\boldsymbol{x})$ is symmetric on $\mathbb{R}^n_{++}$. Without loss of generality, we may assume that $x_1 \geq x_2 > 0$. Hence, for $m \neq 0$ and $n \geq 3$, we have

$$\Delta=\frac{x_1^m-x_2^m}{m}\left[x_1^{1-m}\frac{GB^{p,q,r}(\boldsymbol{x})}{\partial x_1}-x_2^{1-m}\frac{\partial GB^{p,q,r}(\boldsymbol{x})}{\partial x_2}\right]$$
$$=\frac{(x_1^m-x_2^m)GB^{p,q,r}(\boldsymbol{x})}{mn(n-1)(n-2)}$$
$$\times\Bigg[p\sum_{j,k=3,j\neq k}^{n}\Big(\frac{x_1^{1-m}}{px_1+qx_j+rx_k}-\frac{x_2^{1-m}}{px_2+qx_j+rx_k}\Big)$$
$$+q\sum_{i,k=3,i\neq k}^{n}\Big(\frac{x_1^{1-m}}{px_i+qx_1+rx_k}-\frac{x_2^{1-m}}{px_i+qx_2+rx_k}\Big)$$
$$+r\sum_{i,j=3,i\neq j}^{n}\Big(\frac{x_1^{1-m}}{px_i+qx_j+rx_1}-\frac{x_2^{1-m}}{px_i+qx_j+rx_2}\Big)$$
$$+\sum_{k=3}^{n}\Big(\frac{px_1^{1-m}-qx_2^{1-m}}{px_1+qx_2+rx_k}+\frac{qx_1^{1-m}-px_2^{1-m}}{px_2+qx_1+rx_k}\Big)$$
$$+\sum_{j=3}^{n}\Big(\frac{px_1^{1-m}-rx_2^{1-m}}{px_1+qx_j+rx_2}+\frac{rx_1^{1-m}-px_2^{1-m}}{px_2+qx_j+rx_1}\Big)$$

$$+\sum_{i=3}^{n}\Big(\frac{qx_1^{1-m}-rx_2^{1-m}}{px_i+qx_1+rx_2}+\frac{rx_1^{1-m}-qx_2^{1-m}}{px_i+qx_2+rx_1}\Big)\Bigg],$$

i.e.,

$$\Delta=\frac{(x_1^m-x_2^m)GB^{p,q,r}(\boldsymbol{x})}{mn(n-1)(n-2)}$$
$$\times\Bigg[p\sum_{j,k=3,j\neq k}^{n}\frac{px_1x_2(x_1^{-m}-x_2^{-m})+(qx_j+rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_j+rx_k)(px_2+qx_j+rx_k)}$$
$$+q\sum_{i,k=3,i\neq k}^{n}\frac{qx_1x_2(x_1^{-m}-x_2^{-m})+(px_i+rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_1+rx_k)(px_i+qx_2+rx_k)}$$
$$+r\sum_{i,j=3,i\neq j}^{n}\frac{rx_1x_2(x_1^{-m}-x_2^{-m})+(px_i+qx_j)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_j+rx_1)(px_i+qx_j+rx_2)}$$
$$+\sum_{k=3}^{n}\frac{(p^2+q^2)x_1x_2(x_1^{-m}-x_2^{-m})+2pq(x_1^{2-m}-x_2^{2-m})+(p+q)rx_k(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_2+rx_k)(px_2+qx_1+rx_k)}$$
$$+\sum_{j=3}^{n}\frac{(p^2+r^2)x_1x_2(x_1^{-m}-x_2^{-m})+2pr(x_1^{2-m}-x_2^{2-m})+(p+r)qx_j(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_j+rx_2)(px_2+qx_j+rx_1)}$$
$$+\sum_{i=3}^{n}\frac{(q^2+r^2)x_1x_2(x_1^{-m}-x_2^{-m})+2qr(x_1^{2-m}-x_2^{2-m})+(q+r)px_i(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_1+rx_2)(px_i+qx_2+rx_1)}\Bigg].$$

In view of $x_1^{2-m}-x_2^{2-m}=(x_1^{1-m}-x_2^{1-m})(x_1+x_2)-(x_1^{-m}-x_2^{-m})x_1x_2$, we can rewrite the expression of Δ as

$$\Delta=\frac{(x_1^m-x_2^m)GB^{p,q,r}(\boldsymbol{x})}{mn(n-1)(n-2)}$$
$$\times\Bigg[p\sum_{j,k=3,j\neq k}^{n}\frac{px_1x_2(x_1^{-m}-x_2^{-m})+(qx_j+rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_j+rx_k)(px_2+qx_j+rx_k)}$$
$$+q\sum_{i,k=3,i\neq k}^{n}\frac{qx_1x_2(x_1^{-m}-x_2^{-m})+(px_i+rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_1+rx_k)(px_i+qx_2+rx_k)}$$
$$+r\sum_{i,j=3,i\neq j}^{n}\frac{rx_1x_2(x_1^{-m}-x_2^{-m})+(px_i+qx_j)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_j+rx_1)(px_i+qx_j+rx_2)}$$
$$+\sum_{k=3}^{n}\frac{(p-q)^2x_1x_2(x_1^{-m}-x_2^{-m})+(2pq(x_1+x_2)+(p+q)rx_k)(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_2+rx_k)(px_2+qx_1+rx_k)}$$

$$+\sum_{j=3}^{n}\frac{(p-r)^2x_1x_2(x_1^{-m}-x_2^{-m})+(2pr(x_1+x_2)+(p+r)qx_j)(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_j+rx_2)(px_2+qx_j+rx_1)}$$
$$\left.+\sum_{i=3}^{n}\frac{(q-r)^2x_1x_2(x_1^{-m}-x_2^{-m})+(2qr(x_1+x_2)+(q+r)px_i)(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_1+rx_2)(px_i+qx_2+rx_1)}\right].$$

If $m \geq 1$, by the assumption $x_1 \geq x_2 > 0$, $p \geq 0, q \geq 0, r \geq 0$, one has $x_1^{-m} - x_2^{-m} \leq 0$, $x_1^{1-m} - x_2^{1-m} \leq 0$ and $x_1^m - x_2^m \geq 0$, which implies $\Delta \leq 0$. Using Proposition 4, we conclude that $GB^{p,q,r}(\boldsymbol{x})$ is Schur m-power concave for $\boldsymbol{x} \in \mathbb{R}_{++}^n$.

If $m < 0$, by the assumption $x_1 \geq x_2 > 0$, $p \geq 0, q \geq 0, r \geq 0$, we obtain $x_1^{-m} - x_2^{-m} \geq 0$, $x_1^{1-m} - x_2^{1-m} \geq 0$ and $x_1^m - x_2^m \leq 0$. Thus, $\Delta \geq 0$. From Proposition 4, it follows that $GB^{p,q,r}(\boldsymbol{x})$ is Schur m-power convex for $\boldsymbol{x} \in \mathbb{R}_{++}^n$.

The proof of Theorem 1 is completed.

Remark 4. As a direct consequence of Theorem 1, taking $m = 1$ and $m = -1$ in Theorem 1 respectively, together with the definitions of Schur concave functions and Schur harmonic convex functions, we arrive at the assertions of Corollary 1.

4. Applications

As an application of Theorem 1, we establish the following interesting inequalities for generalized geometric Bonferroni mean.

Theorem 2. *Let p, q, r be non-negative real numbers with $p + q + r \neq 0$, and let $1 \leq \lambda \leq \frac{n}{n-1}$. Then for $\boldsymbol{x} \in \mathbb{R}_{++}^n (n \geq 3)$ we have the inequality*

$$GB^{p,q,r}(\boldsymbol{x}) \leq GB^{p,q,r}\left(\lambda A_n(\boldsymbol{x}) + (1-\lambda)x_1, \lambda A_n(\boldsymbol{x}) + (1-\lambda)x_2, \ldots, \lambda A_n(\boldsymbol{x}) + (1-\lambda)x_n\right). \tag{11}$$

Proof. It follows form Theorem 1 that $GB^{p,q,r}(\boldsymbol{x})$ is Schur concave on $\mathbb{R}_{++}^n$.

Using Lemma 1, one has $(\lambda A_n(\boldsymbol{x})+(1-\lambda)x_1, \lambda A_n(\boldsymbol{x})+(1-\lambda)x_2, \ldots, \lambda A_n(\boldsymbol{x})+(1-\lambda)x_n) \prec (x_1, x_2, \ldots, x_n)$.

Thus, we deduce from Definition 1 that $GB^{p,q,r}(\lambda A_n(\boldsymbol{x})+(1-\lambda)x_1, \lambda A_n(\boldsymbol{x})+(1-\lambda)x_2, \ldots, \lambda A_n(\boldsymbol{x}) + (1-\lambda)x_n) \geq GB^{p,q,r}(x_1, x_2, \ldots, x_n)$.

The Theorem 2 is proved. □

Putting $\lambda = 1$ in Theorem 2 gives

Corollary 2. *Let p, q, r be non-negative real numbers with $p + q + r \neq 0$. Then for $\boldsymbol{x} \in \mathbb{R}_{++}^n$ $(n \geq 3)$ we have the inequality*

$$GB^{p,q,r}(\boldsymbol{x}) \leq A_n(\boldsymbol{x}). \tag{12}$$

Theorem 3. *Let p, q, r be non-negative real numbers with $p+q+r \neq 0$, and let $\sum_{i=1}^{n} x_i = (n-2)c$. Then for $\boldsymbol{x} \in \mathbb{R}_{++}^{n}$ $(n \geq 3)$ we have the inequality*

$$GB^{p,q,r}(c(n-2)^2 + \boldsymbol{x}) \geq (n-1)\, GB^{p,q,r}(c(n-2) - \boldsymbol{x}). \tag{13}$$

Proof. By the relationship of majorization given in Lemma 2, we have

$$\left(\frac{c+x_1}{n-1}, \frac{c+x_2}{n-1}, \ldots, \frac{c+x_n}{n-1}\right) \prec (c-x_1, c-x_2, \ldots, c-x_n)\,.$$

Using the properties of majorization yields

$$\left(\frac{c+x_1}{n-1} + (n-3)c,\ \frac{c+x_2}{n-1} + (n-3)c,\ \ldots,\ \frac{c+x_n}{n-1} + (n-3)c\right)$$
$$\prec (c - x_1 + (n-3)c,\ c - x_2 + (n-3)c,\ \ldots,\ c - x_n + (n-3)c)\,,$$

i.e.

$$\left(\frac{c(n-2)^2 + x_1}{n-1}, \frac{c(n-2)^2 + x_2}{n-1}, \ldots, \frac{c(n-2)^2 + x_n}{n-1}\right)$$
$$\prec (c(n-2) - x_1,\ c(n-2) - x_2,\ \ldots,\ c(n-2) - x_n)\,.$$

It follows from Theorem 1 that

$$GB^{p,q,r}\left(\frac{c(n-2)^2 + x_1}{n-1}, \frac{c(n-2)^2 + x_2}{n-1}, \ldots, \frac{c(n-2)^2 + x_n}{n-1}\right)$$
$$\geq GB^{p,q,r}\left(c(n-2) - x_1,\ c(n-2) - x_2,\ \ldots,\ c(n-2) - x_n\right).$$

Hence

$$GB^{p,q,r}\left(c(n-2)^2 + x_1,\ c(n-2)^2 + x_2,\ \ldots,\ c(n-2)^2 + x_n\right)$$
$$\geq (n-1) GB^{p,q,r}\left(c(n-2) - x_1,\ c(n-2) - x_2,\ \ldots,\ c(n-2) - x_n\right),$$

that is

$$GB^{p,q,r}(c(n-2)^2 + \boldsymbol{x}) \geq (n-1)\, GB^{p,q,r}(c(n-2) - \boldsymbol{x}).$$

This completes the proof of Theorem 3. □

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