Application of scaling group transformation on viscoelastic fluid with Cattaneo Christov heat flux model

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Abstract. In this paper Cattaneo Christov heat flux model is considered to analyze the heat transfer in radiated Maxwell fluid with velocity slip and temperature jump condition over a stretching plate and the Lie symmetry group transformations are used to convert the boundary layer equations into non-linear ordinary differential equations. The dimensionless governing equations are solved numerically using Bvp4c with MATLAB, which is a collocation method equivalent to the fourth order mono-implicit Runge-Kutta method. The effects of some physical parameters, such as elasticity number, velocity slip coefficient, the relaxation time of the heat flux, thermal slip parameter, radiation parameter and the Prandtle number on velocity and temperature fields are analyzed.

Keywords: scaling group transformation, UCM fluid, Cattaneo Christov heat flux.

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1. Introduction

The mechanism of heat transfer in different scenarios is successfully described through a famous heat flux model given by Fourier [1] in 1822. This model is an empirical formula, it gives a way to investigate transfer of heat phenomenon and provide the basis to study heat conduction in the successive two centuries. This model has a drawback that it produces a parabolic energy equation which means that initial disturbance is instantly experienced by the medium under investigation. To minimize this drawback, a modification to Fourier's Law is given by Cattaneo [2] by introducing relaxation time for heat flux and in result it produces hyperbolic energy equation that permits heat transportation through promulgation of thermal waves with fixed speed. The process of such heat transportation has useful applications that span from nanofluid flows to the modeling of skin burn injury (see Tibullo and Zampoli [3]). Christove [4] modified the Maxwell Catteneo model to maintain the material invariant formulation by using the Oldroyd's upper-convected derivative and this modification in literature is known as Cataneo Christov heat flux model. Uniqueness of the solution's for the Cattaneo Christov model is proved by Ciarletta and Straughan [5]. Cattaneo-Christov Model is discussed in the study of slip flow and transfer of heat through Maxwell fluid by Han et al [6]. Cattaneo Christov model in rotating flow with upper convected Maxwell fluid is studied by Mustafa [7]. Recently Hayat et al [8] applied the Cattaneo Christov heat flux model on magneto hydrodynamic flow of Oldroyd B fluid with homogeneous-heterogeneous reactions.

In the last few decades, researchers have given more attention to study Maxwell fluid due to its vast applications in industry. Maxwell fluid is an important class of viscoelastic fluid. An effort has been put by different researchers to investigate the flow of upper convected Maxwell fluid. Choi et al [9] investigated the effects of suction, viscoelasticity and inertia in porous medium. Sadeghy et al [10] studied Sakiadis flow and showed that wall skin friction decreases by increasing the value of Deborah number. Magnetohydrodynamic flows in porous medium were analyzed in [11, 12, 13, 14] and stagnation point flows of UCM model were discussed in [15, 16, 17, 18]. Hayat et al [19] discussed the effects of MHD and chemical reaction on boundary layer flow over a shirking sheet. Akber et al [20] discussed viscous dissipation, thermal radiation and heat transfer of an upper convected Maxwell fluid.

Lie group analysis, also called symmetry analysis, was developed by Sophus Lie to find point transformations which map a given differential equation to itself. This method has been used by many investigators to analyze various flow phenomenon arising in fluid mechanics [21, 22, 23, 24, 25, 26, 27, 28]. Moreover, a special form of Lie group transformations is the scaling group technique and it has been applied by many researchers [29, 30, 31, 32, 33, 34, 35, 36, 37] to study different flow models over different nonlinear dynamical systems, aerodynamics, and some other engineering branches. For example, Rashidi et al [38] investigated free convective flow of a nanofluid over a chemically reacting horizontal plate in a porous media using scaling group of transformation. Cao et al [39] studied the MHD effects on the convectively heated stretching porous surface with the heat source/sink. Aziz et al [40] used this method to studied MHD flow over an inclined radiating plate with the temperature dependent thermal conductivity, variable reactive index and heat generation.

In this paper, we shall focus on the application of scaling group of transformation to investigate the coupled flow and heat transfer of viscoelastic fluid with Cattaneo Christov heat flux model, velocity slip and temperature jump conditions by considering the radiation effects. By implementing appropriate scaling group of transformation, the set of governing partial differential equations are converted into a set of ordinary differential equations along with boundary conditions. The solutions of the transformed system of ordinary differential equations is then solved numerically and shown the results graphically.

2. Formulation

Consider a two dimensional steady boundary layer flow of the Maxwell fluid over a plate. For present problem the governing boundary layer equations for expressing conservation of mass and momentum can be written as [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0, \tag{2.1}$$

$$\overline{u}\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{u}}{\partial \overline{y}} + \lambda_1\left(\overline{u}^2\frac{\partial^2 \overline{u}}{\partial \overline{x}^2} + \overline{v}^2\frac{\partial^2 \overline{u}}{\partial \overline{y}^2} + 2\overline{u}\overline{v}\frac{\partial^2 \overline{u}}{\partial \overline{x}\partial \overline{y}}\right) = \nu\frac{\partial^2 \overline{u}}{\partial \overline{y}^2}. \tag{2.2}$$

The boundary conditions with velocity slip on boundary are

$$\overline{u} = a\overline{x} + \lambda_0\frac{2-\sigma_\nu}{\sigma_\nu}\frac{\partial \overline{u}}{\partial \overline{y}},\ \ and\ \ \ \overline{v} = 0\ at\ \ \ \overline{y} = 0,\ \overline{u} \to 0\ as\ \ \ \overline{y} \to \infty, \tag{2.3}$$

where α is positive constant, σ_ν is the tangential momentum accommodation coefficient and λ_0 is the molecular mean free path. We use the Cattaneo–Christov heat flux model to study heat transfer with radiation effects of the Maxwell fluid and the heat flux model is of the following form [4]

$$\mathbf{q}+\lambda_2\left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{V}.\nabla\mathbf{q} - \mathbf{q}.\nabla\mathbf{V} + (\nabla.\mathbf{V})\mathbf{q}\right) = -k\nabla\overline{T}. \tag{2.4}$$

Where $\mathbf{q}$ is the heat flux, λ_2 is the relaxation time of the heat flux, T is the temperature of the Maxwell fluid and k is the thermal conductivity. $\mathbf{V} = (\overline{u},\ \overline{v})$ is the velocity vector of the Maxwell fluid. When $\lambda_2 = 0$, Eq. (2.4) is simplified to Fourier's law. Since the fluid is incompressible, which satisfies $\nabla.\mathbf{V} = 0$. Eq. (2.4) becomes the following form [3]

$$\mathbf{q}+\lambda_2\left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{V}.\nabla\mathbf{q} - \mathbf{q}.\nabla\mathbf{V}\right) = -k\nabla\overline{T}. \tag{2.5}$$

The energy balance equation for the steady boundary layer flow in the presence of thermal radiation[20] is

$$\rho C_p \mathbf{V}.\nabla\overline{T} = -\nabla.\mathbf{q} - \frac{\partial q_r}{\partial y}. \tag{2.6}$$

where C_p = specific heat at constant pressure; and q_r = radiative heat flux. Using the Rosseland approximation [41] for thermal radiation, the radiative heat flux is simplified as

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial \overline{y}} = -\frac{16\sigma^* T_\infty^3}{3k^*}\frac{\partial \overline{T}}{\partial \overline{y}}. \tag{2.7}$$

Eliminating $\mathbf{q}$ between Eqs. (2.5) and (2.6) and on using eq. (2.7), we can obtain the temperature governing equation for the steady flow as

$$\begin{aligned} &\overline{u}\frac{\partial \overline{T}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{T}}{\partial \overline{y}} + \lambda_2\left(\overline{u}\frac{\partial \overline{u}}{\partial \overline{x}}\frac{\partial \overline{T}}{\partial \overline{x}} + \overline{v}\frac{\partial \overline{v}}{\partial \overline{y}}\frac{\partial \overline{T}}{\partial \overline{y}} + \overline{u}\frac{\partial \overline{v}}{\partial \overline{x}}\frac{\partial \overline{T}}{\partial \overline{y}} + \overline{v}\frac{\partial \overline{u}}{\partial \overline{y}}\frac{\partial \overline{T}}{\partial \overline{x}}\right. \\ &\left. +2\overline{u}\overline{v}\frac{\partial^2 \overline{T}}{\partial \overline{x}\partial \overline{y}} + \overline{u}^2\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \overline{v}^2\frac{\partial^2 \overline{T}}{\partial \overline{y}^2}\right) \\ &= \alpha\frac{\partial^2 \overline{\overline{T}}}{\partial \overline{y}^2} + \frac{16\sigma^* T_\infty^3}{3kk^*}\frac{\partial^2 \overline{T}}{\partial \overline{y}^2}. \end{aligned} \tag{2.8}$$

Where $\alpha = \frac{K}{\rho C_P}$ is thermal diffusivity. The corresponding temperature jump boundary condition [42] is

$$\overline{T} = T_w + k\frac{\partial \overline{T}}{\partial \overline{y}} \quad at \quad \overline{y} = 0, \qquad \overline{T} \to T_\infty \quad as \quad \overline{y} \to \infty. \tag{2.9}$$

The following non dimensional variables can be introduced as

$$x = \frac{\overline{x}}{\sqrt{\nu/a}},\ y = \frac{\overline{y}}{\sqrt{\nu/a}},\ u = \frac{\overline{u}}{\sqrt{a\nu}},\ \ v = \frac{\overline{v}}{\sqrt{a\nu}},\ \theta = \frac{\overline{T} - T_\infty}{T_w - T_\infty}. \tag{2.10}$$

The stream function ψ defined by

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}$$

leads to equations (2.1), (2.2) and (2.8) in the following non dimensional form

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} - \psi_{yyy} + \beta\left[(\psi_y)^2\psi_{xxy} + (\psi_x)^2\psi_{yyy} - 2\psi_x\psi_y\psi_{xyy}\right] = 0, \tag{2.11}$$

$$\begin{aligned} &\psi_y\theta_x - \psi_x\theta_y + \gamma\left[\psi_y\psi_{xy}\theta_x + \psi_x\psi_{xy}\theta_y - \psi_y\psi_{xx}\theta_y - \psi_x\psi_{yy}\theta_x - 2\psi_y\psi_x\theta_{xy}\right. \\ &\left. +(\psi_y)^2\theta_{xx} + (\psi_x)^2\theta_{yy}\right] - (\frac{1}{Pr} + \frac{4}{3}R)\theta_{yy} = 0. \end{aligned} \tag{2.12}$$

The boundary conditions (2.3) and (2.8) can be written as

$$\psi_y = x + b\psi_{yy},\ and \quad \psi_x = 0\ at \quad y = 0,\ \psi_y \to 0\ as \quad y \to \infty, \tag{2.13}$$
$$\theta = 1 + c\,\theta_y \quad at\ \ y = 0, \quad \theta \to 0\ as\ \ y \to \infty,$$

where

$$b = \lambda_0 \frac{2-\sigma_\nu}{\sigma_\nu}\sqrt{\frac{a}{\nu}}, \gamma = \lambda_2 a,\ R = \frac{4\sigma^* T_\infty^3}{kk^*} \text{ and } c = \sqrt{\frac{a}{\nu}}k.$$

3. Lie point symmetries and similarity solutions

The symmetry groups of equations (2.11)-(2.12) are calculated using the classical Lie group approach. The one parameter infinitesimal Lie group of transformations leaving (2.11)-(2.12) invariant is defined as

$$x^* = xe^{\in a_1}, y^* = ye^{\in a_2}, \psi^* = \psi e^{\in a_3}, u^* = ue^{\in a_4}, v^* = ve^{\in a_5}, \theta^* = \theta e^{\in a_6}, \tag{3.1}$$

where the coordinates $(x,\ y,\ \psi,\ u,\ v,\ \theta) \to (x^*,\ y^*,\ \psi^*, u^*, v^*, \theta^*)$. Substituting (3.1) into (2.11) and (2.12), we get

$$\begin{aligned}
&e^{\epsilon(a_1+2a_2-2a_3)}({\psi^*}_{y^*}{\psi^*}_{x^*y^*} - {\psi^*}_{x^*}{\psi^*}_{y^*y^*})\\
&- e^{\epsilon(3a_2-a_3)}{\psi^*}_{y^*y^*y^*} +\ \beta e^{\epsilon(a_1+2a_2-2a_3)}\\
&(({\psi^*}_{y^*})^2{\psi^*}_{x^*x^*y^*} + ({\psi^*}_{x^*})^2{\psi^*}_{y^*y^*y^*} - 2{\psi^*}_{x^*}{\psi^*}_{y^*}{\psi^*}_{x^*y^*y^*}) = 0,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&e^{\epsilon(a_1+a_2-a_3-a_6)}({\psi^*}_{y^*}{\theta^*}_{x^*} - {\psi^*}_{x^*}{\theta^*}_{y^*}) + \gamma e^{\epsilon(2a_1+2a_2-2a_3-a_6)}\big({\psi^*}_{y^*}{\psi^*}_{x^*y^*}{\theta^*}_{x^*}\\
&+ {\psi^*}_{x^*}{\psi^*}_{x^*y^*}{\theta^*}_{y^*} - {\psi^*}_{y^*}{\psi^*}_{x^*x^*}{\theta^*}_{y^*} - {\psi^*}_{x^*}{\psi^*}_{y^*y^*}{\theta^*}_{x^*}\\
&- 2{\psi^*}_{x^*}{\psi^*}_{y^*}{\theta^*}_{x^*y^*} + ({\psi^*}_{y^*})^2{\theta^*}_{x^*x^*}\big) - ({\psi^*}_{x^*})^2{\theta^*}_{y^*y^*}\\
&- (\frac{1}{Pr} + \frac{4}{3}R)e^{\epsilon(2a_2-a_6)}{\theta^*}_{y^*y^*} = 0.
\end{aligned} \tag{3.3}$$

Since the system remains invariant under the group of transformations, we have$a_1 = a_3,\ a_2 = a_4 = a_6 = 0$, the characteristic equations for finding the similarity transformations would be

$$\frac{dx}{x} = \frac{dy}{0} = \frac{d\psi}{\psi} = \frac{d\theta}{0}. \tag{3.4}$$

The corresponding similarity variable and functions are

$$\eta = y,\ \psi = xF(\eta),\ \theta = \theta(\eta), \psi = \psi(\eta). \tag{3.5}$$

Substituting (3.5) into (2.11)-(2.12) obtains the following ordinary equations

$$F'''(\eta)+F(\eta)F''(\eta)-(F'(\eta))^2+\beta(2F(\eta)F'(\eta)F''(\eta)-(F(\eta))^2F'''(\eta))=0, \tag{3.6}$$

$$(\frac{1}{Pr} + \frac{4}{3}R)\theta''(\eta) + F(\eta)\theta'(\eta) - \gamma(F(\eta)F'(\eta)\theta'(\eta) + (F(\eta))^2\theta''(\eta)) = 0. \tag{3.7}$$

And the boundary conditions from Eq. (2.13) are transformed as

$$F'(\eta) = 1 + bF''(\eta),\ F(\eta) = 0\ \ as\ \eta = 0,\ F'(\eta) \to 0,\ \eta \to \infty, \tag{3.8}$$
$$\theta(\eta) = 1 + c\,\theta'(\eta) \quad at \quad \eta = 0, \quad \theta(\eta) \to 0 \quad as \quad \eta \to \infty.$$

4. Results and discussions

In this paper the nonlinear ordinary differential equations (3.6) and (3.7) along with boundary conditions (3.8) are solved numerically using BVP4C for several values of the elasticity number, slip coefficient, the relaxation time of the heat flux, Prandtle number, radiation and temperature jump parameter on velocity and temperature fields. The presence of elastic term in upper convected Maxwell model shows that it has some effects on flow and heat transfer of viscoelastic fluids. The effects of elasticity number β on the velocity and temperature distribution shown in figs. 1-3 which indicates that on increasing the value of elasticity number β increase the elastic force and for $\beta = 0$ shows that the elastic force disappears and the fluid becomes the Newtonian fluid. Figs. 1 and 2 indicates that the value of $F(\eta)$ and $F^{'}(\eta)$ become smaller with the increase of β. This shows that the velocity boundary layer is increasingly thin with the increase of β. Fig. 3 describes that fluid temperature $\theta(\eta)$ increases with increasing the value β. This shows that the elastic force promotes heat transfer of the viscoelastic fluid. The effects of slip coefficient b on velocity distribution $F(\eta)$ and $F^{'}(\eta)$ are shown in figs. 4 and 5. On increasing the value of slip coefficient b, the velocity distribution becomes smaller. Fig. 6 indicates the effects of γ on temperature profile and analyzed that for $\gamma = 0$ corresponding results for the Fourier's law obtained and on enhancing the value of γ temperature decreases which means that temperature in Cattaneo Christov heat flux model is smaller than the Fourier's model. Fig. 7 shows the impact of Prandtl number Pr on the temperature field $\theta(\eta)$. Prandtl number has inverse relationship with thermal diffusivity. Therefore an increase in Prandtl number reduces the temperature and enhances the thermal boundary layer thickness. Fig.8 shows that the increasing values of thermal radiation parameter R , due to this we saw enhancement in the temperature field. Similarly, the influences of temperature jump parameter c upon temperature distribution is depicted in Fig. 9; the higher values of c lead to thinner thickness of thermal boundary layer. The influence of c on the rate of heat transfer is also shown in Fig. 10. It displays that the values of $-\theta'(0)$ increase with decreasing c.

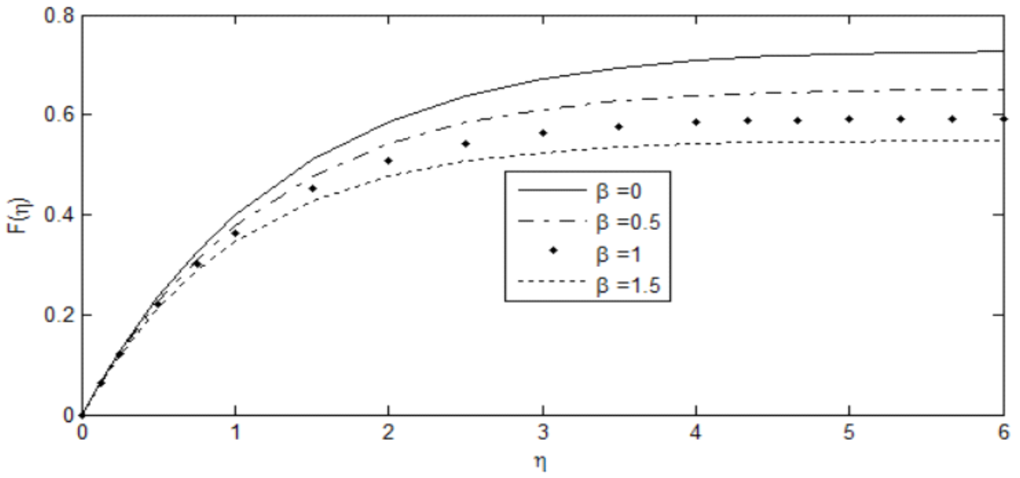

Figure 1: Profile of $F(\eta)$ for different values of β when $b = c = Pr = \gamma = 1, R = 0.2$.

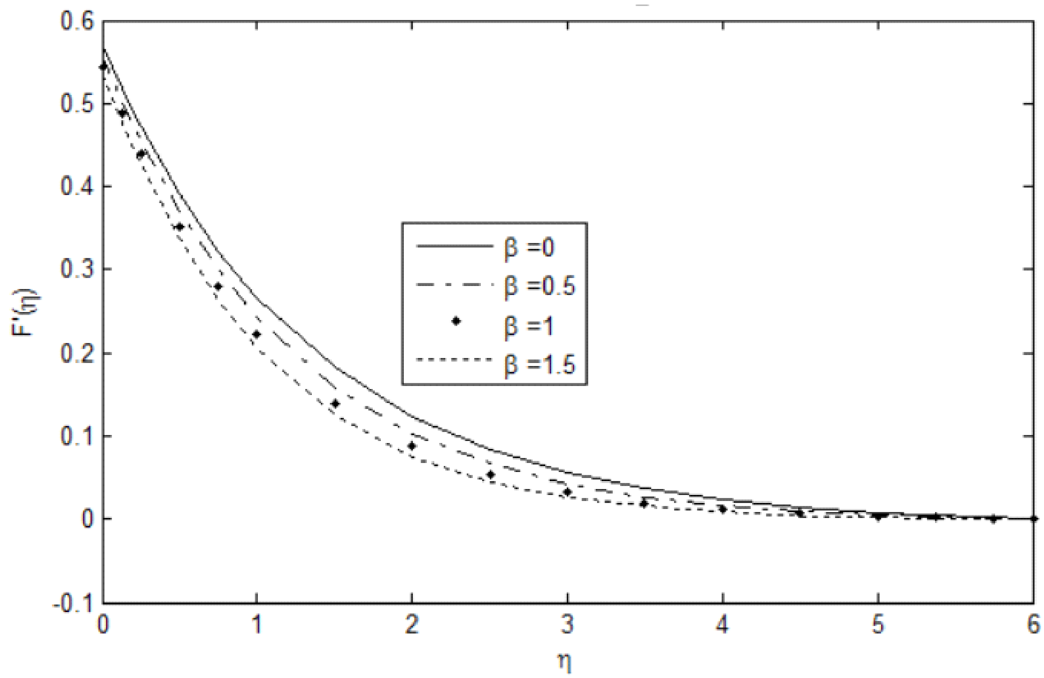

Figure 2: Profile of $F'(\eta)$ for different values of β when $b = c = Pr = \gamma = 1, R = 0.2$.

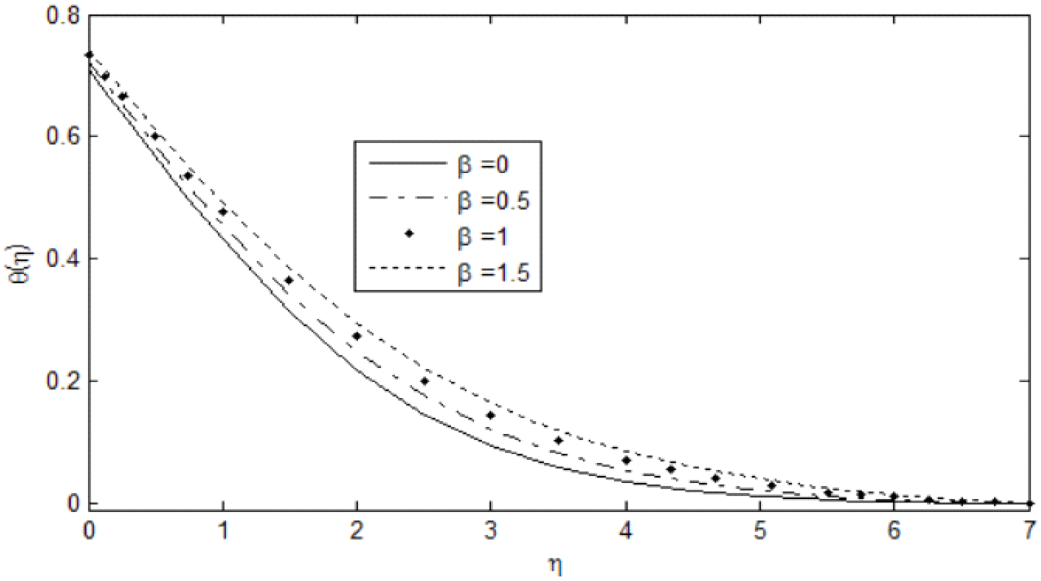

Figure 3: Profile of $\theta(\eta)$ for different values of β when $b = c = Pr = \gamma = 1, R = 0.2$.

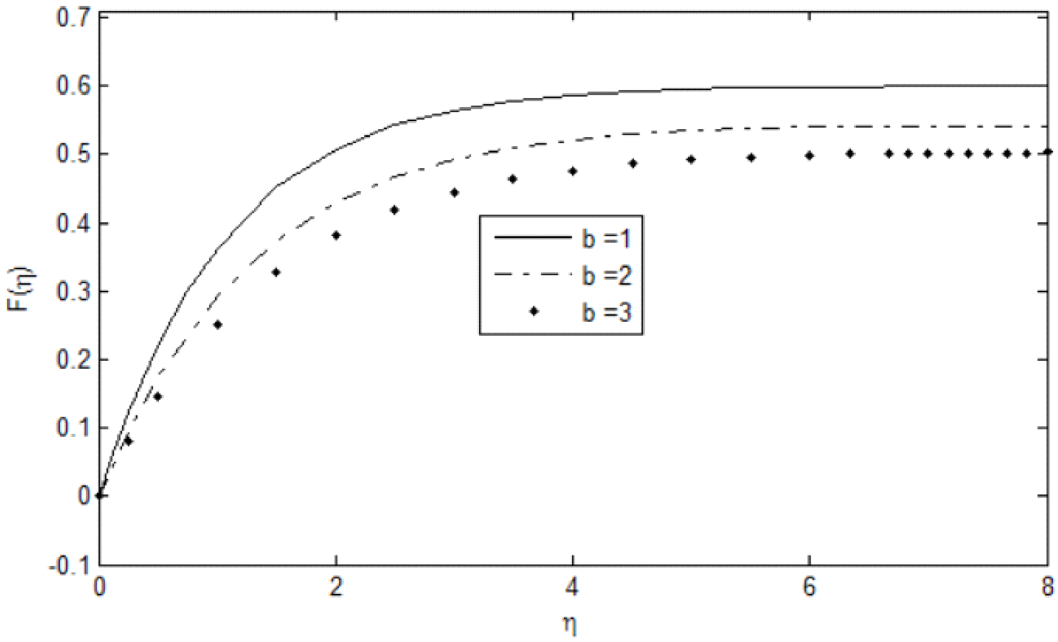

Figure 4: Profile of $F(\eta)$ for different values of b when $\beta = c = Pr = \gamma = 1, R = 0.2$.

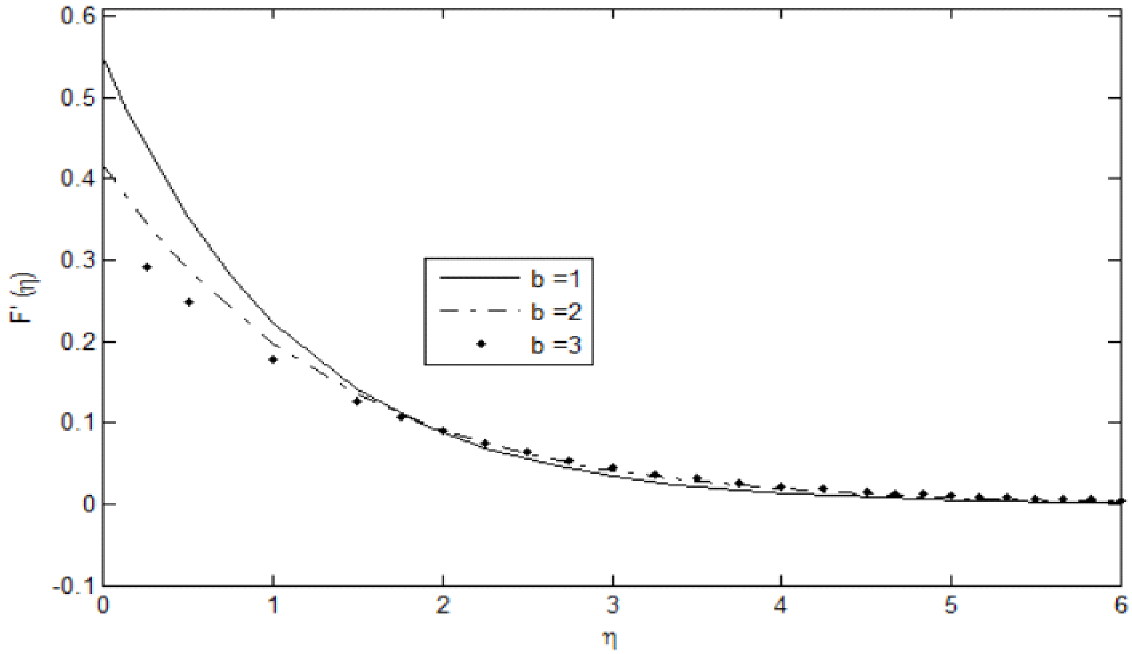

Figure 5: Profile of $F'(\eta)$ for different values of b when $\beta = c = Pr = \gamma = 1, R = 0.2$.

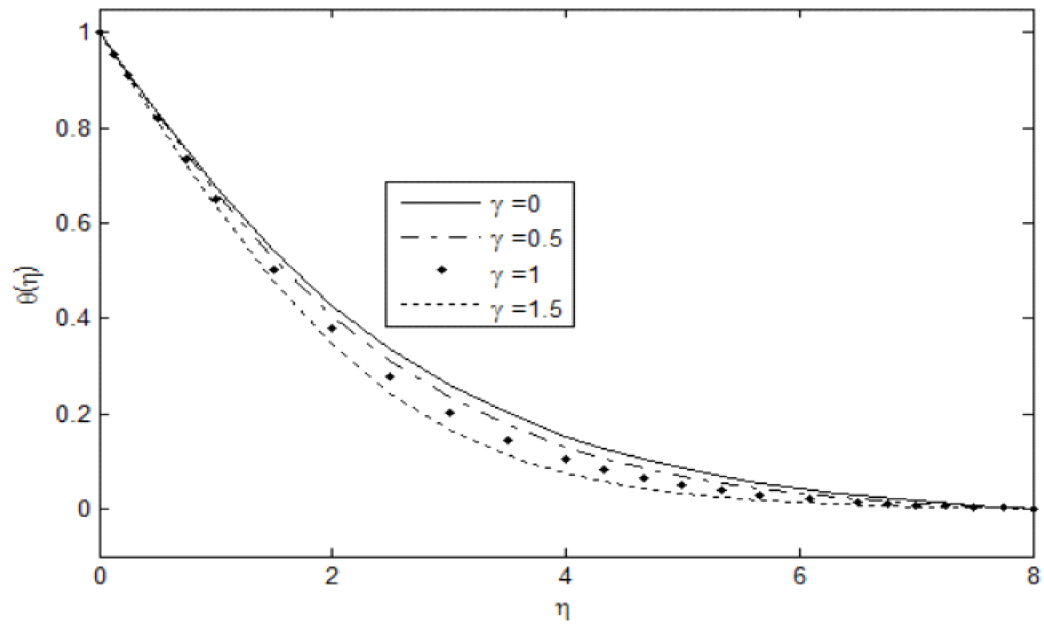

Figure 6: Profile of $\theta(\eta)$ for different values of γ when $\beta = c = Pr = b = 1, R = 0.2$.

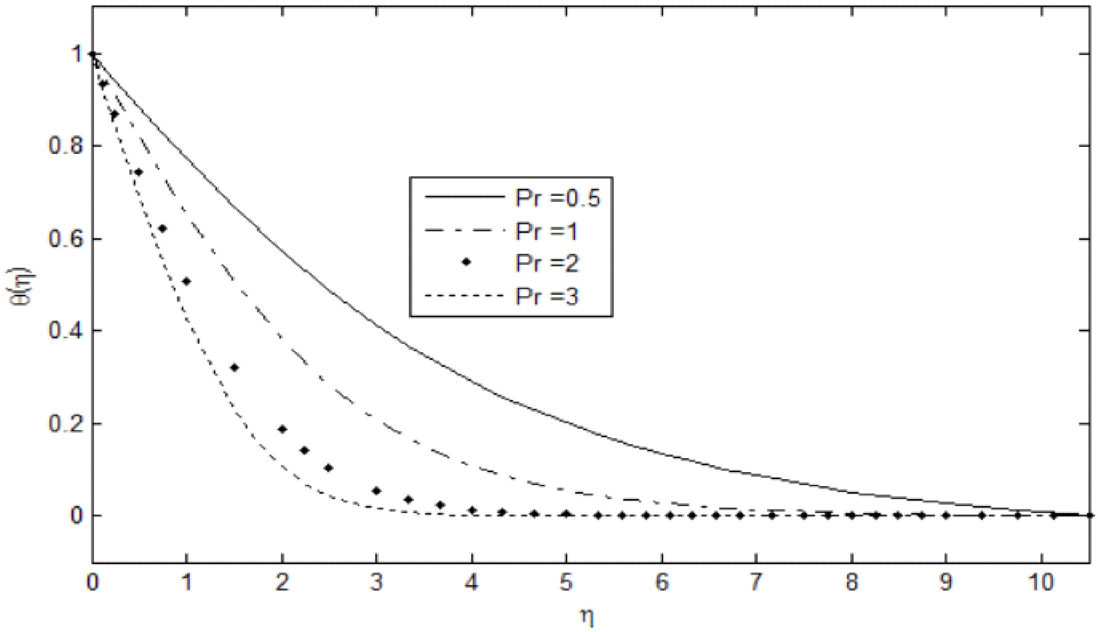

Figure 7: Profile of $\theta(\eta)$ for different values of Pr when $\beta = \gamma = c = b = 1, R = 0.2$.

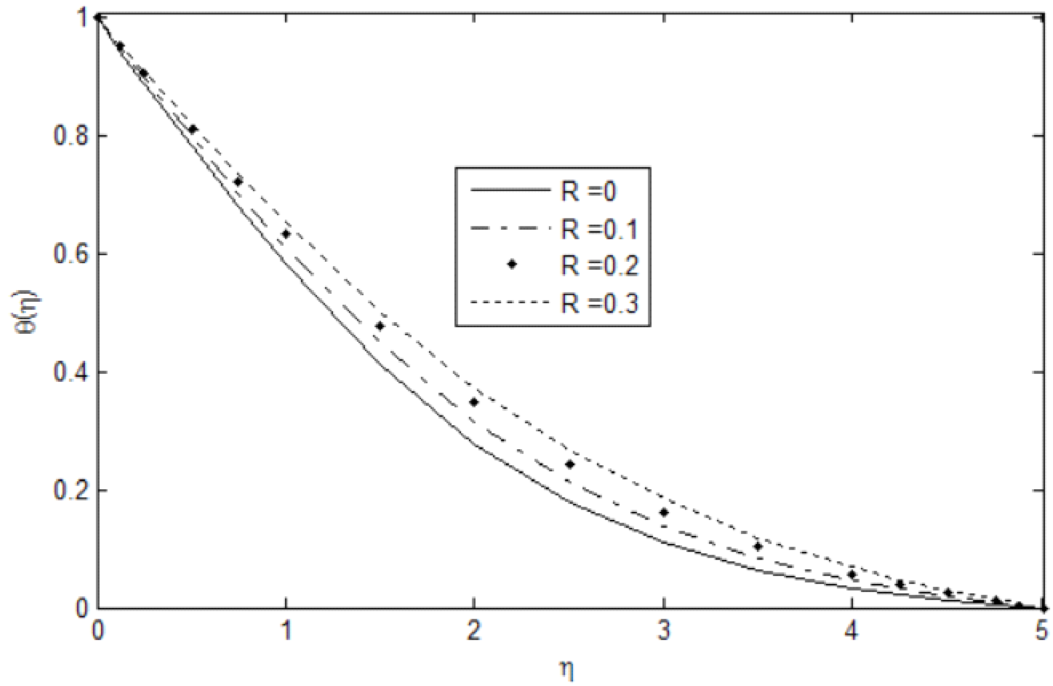

Figure 8: Profile of $\theta(\eta)$ for different values of R when $\beta = \gamma = c = Pr = b = 1$.

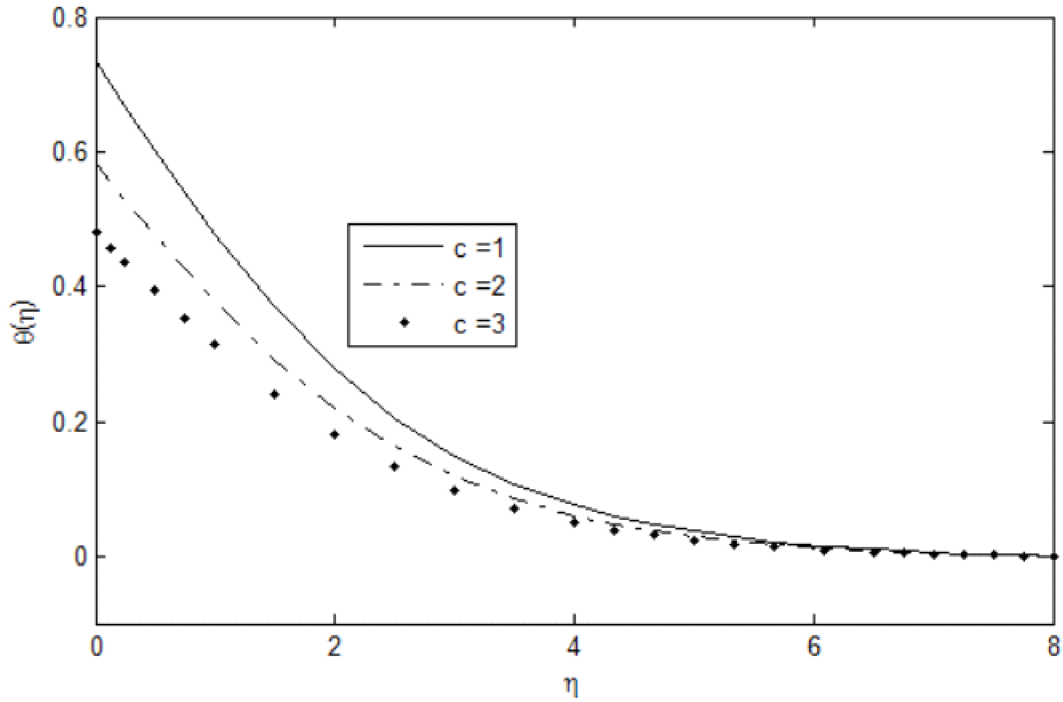

Figure 9: Profile of $\theta(\eta)$ for different values of c when $\beta = \gamma = Pr = b = 1, R = 0.2$.

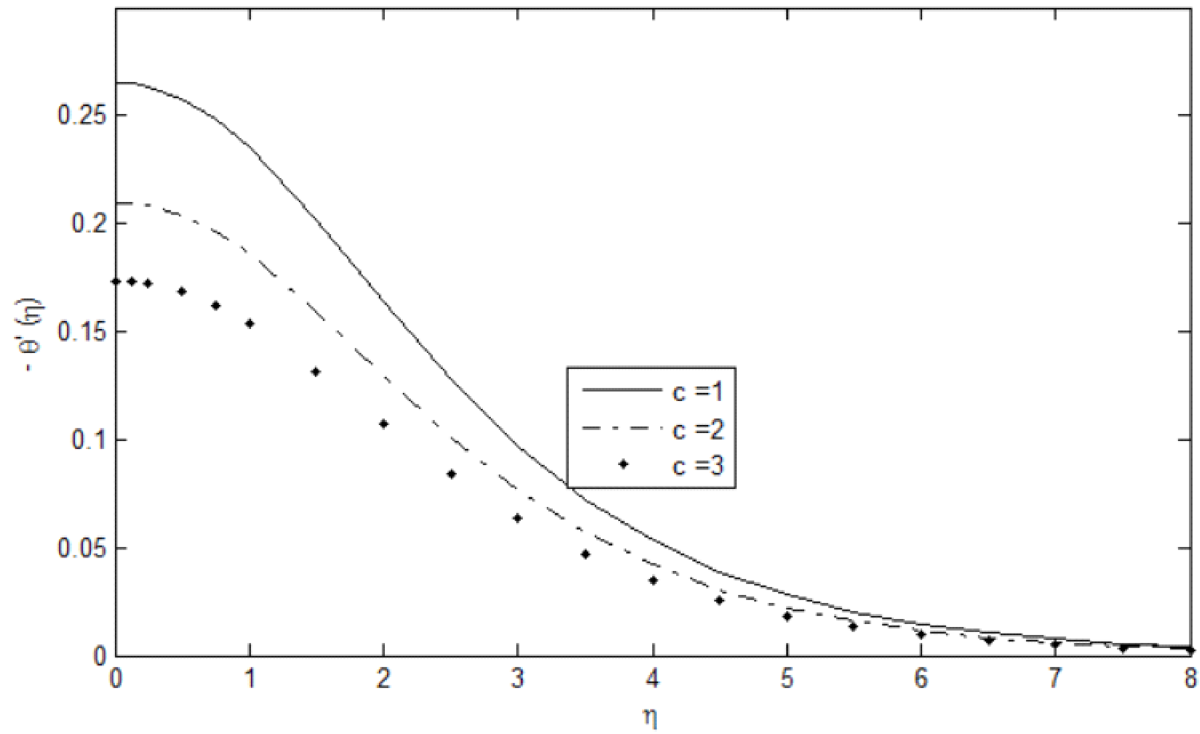

Figure 10: Profile of $-\theta(\eta)$ for different values of c when $\beta = \gamma = Pr = b = 1, R = 0.2$.

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