Coefficient estimates for a subclass of analytic functions using Faber polynomials

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Abstract. In this paper, we introduce and investigate a new subclass $Q_{\Sigma}^{**}(\alpha, \phi)$ of normalized analytic functions defined using convolution in the open unit disk U whose inverse has univalent analytic continuation to U. Estimates of the coefficients of bi-univalent functions belonging to this class are determined by using Faber polynomial techniques.

Keywords: bi-univalent, Faber polynomials, Quasi-convex.

1. Introduction

Let A be the class of all normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n, \tag{1}$$

which are analytic in the unit disk U.

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A function that is regular (holomorphic) in U is said to be univalent in U if it assumes no value more than once in U. Denote by S, the subclass of A, of all univalent functions in U.

Convolution or Hadamard is a mathematical operation on two functions f and g resulting in a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated. For $f(z)$ defined by (1) and $g(z)$ defined by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hadamard product is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n \; b_n \; z^n.$$

Polya and Schoenberg [12] conjectured that the class of convex functions C [13] is preserved under convolution with convex functions:$f, g \in C \Rightarrow f * g \in C$. In 1973, Ruscheweyh and Sheil-Small [14] see also [12] proved the Polya-Schoenberg conjecture. They also proved that the classes of starlike [13] functions and close-to-convex [7] functions are closed under convolution with convex functions. Also it is to be noted that the convolution of two univalent (or starlike) functions need not be univalent.

If $f(z) \in S$ and its inverse has an analytic continuation to $|w| < 1$, then the function $f(z) \in A$ is said to be bi-univalent in U. Let Σ represent the class of all bi-univalent functions.

The concept of bi-univalent functions was introduced by Lewin [8] who proved that if $f(z)$ is bi-univalent, then $|a_2| < 1.51$. Brannan and Clunie [4] improved Lewin's [8] result to $|a_2| \leq \sqrt{2}$. There is a rich literature on the estimates of the initial coefficients of bi-univalence. However not much is known about the estimates of higher coefficients.

The classes $S_{\Sigma}^{*}(\alpha)$ of bi-starlike functions and $C_{\Sigma}(\alpha)$ of bi-convex functions of order α, where $0 \leq \alpha < 1$ were discussed by Brannan and Taha [5].

A function $f(z)$ defined by (1) is in the class $S_{\Sigma}^{*}(\alpha)$, $0 \leq \alpha < 1$ if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, |z| < 1$$

and

$$Re\left(\frac{wg'(w)}{g(w)}\right) > \alpha, |w| < 1$$

The class $C_{\Sigma}(\alpha)$, $0 \leq \alpha < 1$ is the class of all functions of the form (1) satisfying the following conditions:

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, |z| < 1$$

and

$$Re\left(1+\frac{wg''(w)}{g'(w)}\right)>\alpha, |w|<1$$

where $g(w) = (f^{-1})(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - \dots$

Let C^* (see [10], [11]) denote the class of Quasi-convex functions in U, that is if $f(z) \in C^*$, then there exists a function $g \in C$ so that $Re\{\frac{(zf'(z))'}{g'(z)}\} > 0$ in U.

It is possible to approximate an analytic function h on K by polynomials uniformly on K, where K is a simply connected compact set in the Complex plane.

Faber polynomials can be viewed as a.r. (almost regular) formal Laurent series over the Field $L_\infty(u)$, the set of Formal Laurent series over F with an indeterminate u. Faber polynomials introduced by Faber play an important role in various areas of mathematical sciences, especially in geometric function theory. The advantage of using Faber polynomials over the other methods is that we find the nth coefficient that is the general term and use it for our computations.

Using the Faber polynomial expansion for functions $f \in S$ of the form (1), the coefficients of its inverse map $F = f^{-1}$may be expressed as (e.g. see [3], Eq.(1.33), page 185)

$$\begin{aligned} F(w) = f^{-1}(w) &= w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots) w^n \\ &= w + \sum_{n=2}^{\infty} A_n w^n, \end{aligned} \tag{2}$$

$$\begin{aligned} K_{n-1}^{-n} = &\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 \\ &+ \frac{(-n)!}{(-2n+3))!(n-4)!}a_2^{n-4}a_4 \\ &+ \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}[a_5 + (-n+2)a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6 + (-2n+5)a_3a_4] + \sum_{j\geq 7} a_2^{n-j}V_j, \end{aligned}$$

where V_j is a homogeneous polynomial in the variables a_2, a_3, $\dots$, a_n (see [12] and [6]). In particular, the first few terms of K_{n-1}^{-n} are $K_1^{-2} = -2a_2$, $K_2^{-3} = 3(2a_2^2 - a_3)$ and $K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4)$. In general, an expansion of $K_{n-1}^p(a_2, a_3 \dots, a_n)$ is given by

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}D_{n-1}^2 + \frac{p!}{(p-3)!3!}D_{n-1}^3 + \dots + \frac{p!}{(p-n+1)!(n-1)!}D_{n-1}^{n-1},$$

where $D^p_{n-1} = D^p_{n-1}(a_2, a_3, \ldots, a_n)$,

$$D^m_{n-1}(a_2, \ldots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \ldots (a_n)^{\mu_{n-1}}}{\mu_1! \ldots \mu_{n-1}!} \quad \text{for } m \leq n,$$

and the sum is taken over all non-negetive integers $\mu_1, \ldots, \mu_{n-1}$ satisfying $\mu_1 + \mu_2 + \cdots + \mu_{n-1} = m$ and $\mu_1 + 2\mu_2 + \cdots + (n-1)\mu_{n-1} = n-1$. Evidently: $D^{n-1}_{n-1}(a_2, \ldots, a_n) = a_2^{n-1}$ (see [1], [2]).

An analytic function p of the form $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is called a function with positive real part in U if $\Re[p(z)] > 0$ for all $z \in U$. The class of all functions with positive real part is denoted by $\wp$.

Lemma 1.1 ([6]). *The coefficient p_n of a function $p \in \wp$ satisfies the sharp inequality $|p_n| \leq 2, n \geq 1$.*

Motivated by the definition of the class C^* of Quasi-convex functions investigated by Noor and Thomas ([10], [11]), In this paper we introduce the class $Q^{**}_{\Sigma}(\alpha, \phi)$, $0 \leq \alpha < 1$ and obtain the coefficient bounds using Faber polynomial techniques.

2. Main result

Coefficient bounds for functions in the class $Q^{**}_{\Sigma}(\alpha, \phi)$, $0 \leq \alpha < 1$.

Definition 2.1. *Let $f \in A$. Then $f \in Q^{**}_{\Sigma}(\alpha, \phi)$, $0 \leq \alpha < 1$ if $f \in \Sigma$,*

$$Re\left(\frac{(z(f*\phi)'(z))'}{(h*\phi)'(z)}\right) > \alpha$$

and

$$Re\left(\frac{(w((f*\phi)^{-1})'(w))'}{((h*\phi)^{-1})'(w)}\right) > \alpha,$$

where $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ are convex.

Theorem 2.1. *For $0 \leq \alpha < 1$, let the function $f \in Q^{**}_{\Sigma}(\alpha, \phi) in\, U$. If $a_k = b_k = B_k = 0; 2 \leq k \leq n-1$ then*

$$|a_n| \leq \frac{1}{n} + \frac{2(1-\alpha)}{n^2}.$$

Proof. The Faber polynomial expansions are

$$\begin{aligned}\frac{(z(f*\phi)'(z))'}{(h*\phi)'(z)} &= 1 + \sum_{n=2}^{\infty}[(n^2 a_n\phi_n - nb_n\phi_n) \\ &+ \sum_{r=1}^{n-2} k_r^{-1}(2b_2\phi_2, 3b_3\phi_3, \ldots, (r+1)b_{r+1}\phi_{r+1}) \\ &\times ((n-r)^2 a_{n-r}\phi_{n-r} - (n-r)b_{n-r}\phi_{n-r})]z^{n-1},\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{(w((f*\phi)^{-1})'(w))'}{((h*\phi)^{-1})'(w)} &= 1+\sum_{n=2}^{\infty}[(n^2A_n\Phi_n - nB_n\Phi_n) \\ &+\sum_{r=1}^{n-2}K_r^{-1}(2B_2\Phi_2, ...(r+1)B_{r+1}\Phi_{n-r}) \\ &\times((n-r)^2A_{n-r}\Phi_{n-r}-(n-r)B_{n-r}\Phi_{n-r})]w^{n-1}.\end{aligned} \tag{4}$$

On the other hand, since

$$Re\left(\frac{(z(f*\phi)'(z))'}{(h*\phi)'(z)}\right) > \alpha$$

in U, there exists $p(z) = 1+\sum_{n=1}^{\infty} c_n\ z^n \in A$ so that

$$\frac{(z(f*\phi)'(z))'}{(h*\phi)'(z)} = \alpha+(1-\alpha)p(z) = 1+(1-\alpha)\sum_{n=1}^{\infty}c_nz^n. \tag{5}$$

Also as

$$Re\left(\frac{(w((f*\phi)^{-1})'(w))'}{((h*\phi)^{-1})'(w)}\right) > \alpha$$

in U, there exists $q(w) = 1+\sum_{n=1}^{\infty} d_nw^n \in A$ such that

$$\frac{(w((f*\phi)^{-1})'(w))'}{((h*\phi)^{-1})'(w)} = \alpha+(1-\alpha)q(w) = 1+(1-\alpha)\sum_{n=1}^{\infty}d_nw^n. \tag{6}$$

By the Caratheodory lemma, $|c_n| \le 2$ and $|d_n| \le 2$. Comparing the coefficients from (3) and (5), we obtain for $n \ge 2$,

$$\begin{aligned}&\sum_{n=2}^{\infty}[(n^2a_n\phi_n - nb_n\phi_n)+\sum_{r=1}^{n-2}K_r^{-1}(2b_2\phi_2, 3b_3\phi_3, \dots, (r+1)b_{r+1}\phi_{r+1}) \\ &\times((n-r)^2a_{n-r}\phi_{n-r}-(n-r)b_{n-r}\phi_{n-r})] = (1-\alpha)c_{n-1}.\end{aligned} \tag{7}$$

Similarly, from (4) and (6), we obtain:

$$\begin{aligned}&\sum_{n=2}^{\infty}[(n^2A_n\Phi_n - nB_n\Phi_n)+\sum_{r=1}^{n-2}K_r^{-1}(2B_2\Phi_2, ...(r+1)B_{r+1}\Phi_{n-r}) \\ &\times((n-r)^2A_{n-r}\Phi_{n-r}-(n-r)B_{n-r}\Phi_{n-r})] = (1-\alpha)d_{n-1}.\end{aligned} \tag{8}$$

But under the assumption $a_k = b_k = B_k = 0, 2 \le k \le n-1$ and $\phi_k = 0, 2 \le k \le n-1$, equation (7) and (8), respectively yield:

$$n^2a_n\phi_n - nb_n\phi_n = (1-\alpha)c_{n-1}. \tag{9}$$

and

$$-n^2A_n\Phi_n - nB_n\Phi_n = (1-\alpha)d_{n-1}. \tag{10}$$

By the definition of $K_n^p, A_n = -a_n$, Thus

$$-n^2 a_n \Phi_n - nB_n\Phi_n = (1-\alpha)d_{n-1}. \tag{11}$$

Solving either of equations (9) or (10), we obtain

$$|a_n| \leq \frac{1}{n} + \frac{2(1-\alpha)}{n^2}$$

upon noticing that $|b_n| \leq 1$ and $|B_n| \leq 1$.

Remark 2.1. For $n = 2$ equations (9) and (11) yield

$$\begin{aligned} &4a_2\phi_2 - 2b_2\phi_2 = (1-\alpha)c_1, \\ &-4a_2\Phi_2 - 2B_2\Phi_2 = (1-\alpha)d_1. \end{aligned}$$

Solving these two equations, we have $|a_2| \leq \frac{2-\alpha}{2}$.

Remark 2.2. When $\phi(z) = \frac{1}{1-z}$, this class $Q_\Sigma^{**}(\alpha, \phi)$ will reduce to the class of all bi-quasi-convex functions of order α [16] .

Remark 2.3. When $\phi(z) = \frac{1}{1-z}$, as every convex function is starlike, immediate replacements of $(f * \phi)'$ and $(h * \phi)'$ by $(f * \phi)$ and $(h * \phi)$ respectively in a result of Libera [9], the class $Q_\Sigma^{**}(\alpha, \phi)$ will reduce to the class of bi-close-to-convex functions of order α studied by Hamidi and Jahangiri[15].

Remark 2.4. Replacing $\phi(z)$ by $\frac{1}{1-z}$ and f by h, the class $Q_\Sigma^{**}(\alpha, \phi)$ reduces to the class of bi-convex functions of order α discussed by Brannan and Taha [5]. When n=2 and $\alpha = 0$, $|a_2| \leq 1$, which is an improvement of the corresponding estimate of Brannan and Taha [5].

Remark 2.5. Replacing h by f in Remark 2.3, we notice that the class $Q_\Sigma^{**}(\alpha, \phi)$ reduces to the class of bi-starlike functions α discussed by Brannan and Taha [5]. When n=2 and $\alpha = 0$, $|a_2| \leq 1$, which is an improvement of the corresponding estimate of Brannan and Taha [5].

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