Derivations of rings and Banach algebras

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Abstract. We investigate the action of derivation d in a prime ring and semiprime ring R with center $Z(R)$ satisfying (i) $(d(x\circ y))^m-(x\circ y)^n \in Z(R)$ (ii) $(d(x)\circ d(y))^m=(x\circ y)^n$ for all $x, y \in I$, a nonzero ideal of R. Finally, as an application we apply our results to the continuous derivations on Banach algebras.

Keywords: prime and semiprime rings, derivation, martindale ring of quotient, Banach algebra, radical.

1. Introduction

Throughout this paper, unless stated otherwise, R will be a semi(-prime) ring, $Z(R)$ the center of R, Q its Martindale quotient ring and U its Utumi quotient ring. The center of U, denoted by C is called the extended centroid of R (we refer the reader to [4], for the definitions and related properties of these objects). For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stands for the commutator $xy - yx$ and anti-commutator $xy+yx$, respectively. Recall that a ring R is prime if $xRy = \{0\}$ implies either $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds, for all $x, y \in R$. In particular d is an inner derivation induced by an element $q \in R$ if $d(x) = [q, x]$ holds, for all $x \in R$.

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During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations (see [2], [3], [19] and [20] where further references can be found). In [3], Ashraf and Rehman proved that if R is a prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In [1], Argaç and Inceboz generalized the above result and they proved that: If a prime ring R admits a nonzero derivation d with the property $(d(x \circ y))^n = x \circ y$, for all $x, y \in I$, a nonzero ideal of R, where n is a fixed positive integer, then R is commutative.

If R is a ring and $S \subseteq R$, a mapping $f : S \to R$ is called strong commutativity-preserving (scp) on S if $[f(x), f(y)] = [x, y]$, for all $x, y \in S$. In 1994 Bell and Daif [5] initiated the study of strong commutativity-preserving maps and proved that a nonzero right ideal I of a semiprime ring R is central if R admits a derivation which is scp on I. In 2002 Ashraf and Rehman [3] proved that if R is a 2-torsion free prime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d(x) \circ d(y) = x \circ y$, for all $x, y \in I$, then R is commutative. Inspired by above mention results, we here generalized the result obtained in [1] and [3]. Moreover, we continue this line of investigation by examining what happens when a ring R satisfies the identity

1. $(d(x \circ y))^m - (x \circ y)^n \in Z(R)$, for all $x, y \in I$;
2. $(d(x) \circ d(y))^m = (x \circ y)^n$, for all $x, y \in I$.

In the last section of this paper we will consider $\mathfrak{A}$ as a Banach algebra with Jacobson radical $rad(\mathfrak{A})$. Let us introduce some well known and elementary definitions for the sake of completeness. By Banach algebra we shall mean a complex normed algebra $\mathfrak{A}$ whose underlying vector space is a Banach space. The Jacobson radical $rad(\mathfrak{A})$ of $\mathfrak{A}$ is the intersection of all primitive ideals. If the Jacobson radical reduces to the zero element, $\mathfrak{A}$ is called semisimple. In fact any Banach algebra $\mathfrak{A}$ without a unity can be embedded into a unital Banach algebra $\mathfrak{A}_I = \mathfrak{A} \oplus \mathbb{C}$ as an ideal of codimension one. In particular, we may identify $\mathfrak{A}$ with the ideal $\{(x, 0) : x \in \mathfrak{A}\}$ in $\mathfrak{A}_I$ via the isometric isomorphism $x \to (x, 0)$.

The classical result of Singer and Wermer in [22], says that any continuous derivation on a commutative Banach algebra has the range in the Jacobson radical of the algebra. Singer and Wermer also formulated the conjecture that the continuity assumption can be removed. In 1988 Thomas [23], verified this conjecture. It is clear that the same result of Singer and Wermer does not hold in non-commutative Banach algebras (because of inner derivations). Hence in this context a very interesting question is how to obtain the non-commutative version of the Singer-Wermer theorem. A first answer to this problem was obtained by Sinclair in [21]. He proved that every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant. Since then many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebras. Here we will continue the investigation about the relationship between the structure of an algebra $\mathfrak{A}$ and the behaviour of derivations defined on $\mathfrak{A}$. Then we apply our results on prime rings to the study of analogous conditions for continuous derivations on Banach algebras.

2. The results in prime rings

We will make frequent use of the following result due to Kharchenko [13] (see also [15]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two sided ideal of R. Let $f(x_1,\cdots x_n, d(x_1),\cdots d(x_n))$ be a differential identity in I, that is

$$f(r_1,\cdots r_n, d(r_1),\cdots, d(r_n)) = 0 \text{ for all } r_1,\cdots, r_n \in I.$$

One of the following holds:

1. Either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q)(x) = [q, x]$, for all $x \in R$, and I satisfies the generalized polynomial identity
$$f(r_1,\cdots, r_n, [q, r_1],\cdots, [q, r_n]) = 0;$$
2. or, I satisfies the generalized polynomial identity
$$f(x_1,\cdots, x_n, y_1,\cdots, y_n) = 0.$$

Theorem 2.1. *Let R be a prime ring, I a nonzero ideal of R and m, n are fixed positive integers. If R admits a derivation d such that $(d(x\circ y))^m = (x\circ y)^n$, for all $x, y \in I$, then R is commutative.*

Proof. Suppose that $d = 0$, then $(xy + yx)^n = 0$, for all $x, y \in I$. Now if $char(R) \neq 2$, then $(2x^2)^n = 0$, for all $x \in I$. This is a contradiction by Xu [24]. If $char(R) = 2$, then $(xy+yx)^n = 0 = [x, y]^n$, for all $x, y \in I$. Thus, by Herstein [10, Theorem 2], $I \subseteq Z(R)$ and so R is commutative by Mayne [18].

Now we assume that $d \neq 0$. Since R is a prime ring and if R admits a nonzero derivation d, by the given hypothesis, we have

$$(d(xy + yx))^m = (xy + yx)^n, \text{ for all } x, y \in I.$$

Thus I satisfies the differential identity

$$(d(x)y + yd(x) + d(y)x + xd(y))^m = (xy + yx)^n, \text{ for all } x, y \in I.$$

In the light of [13] and [15], we split the proof into two cases:

Case 1. If d is not inner, then by Kharchenko's theorem [13], I satisfies the polynomial identity

$$(sy + ys + tx + xt)^m = (xy + yx)^n, \text{ for all } x, y, s, t \in I.$$

In particular for $y = 0$, I satisfies the blended component $(tx + xt)^m = 0$, for all $x, t \in I$. Then R is commutative, by using the argument presented above.

Case 2. Let d be an inner derivation induced by an element $q \in Q$, i.e., $d(x) = [q, x]$ for all $x \in R$. Then for any $x, y \in I$,

$$([q, x]y + y[q, x] + [q, y]x + x[q, y])^m = (xy + yx)^n$$

By Chuang [7, Theorem 1], I and Q satisfy same generalized polynomial identities (GPIs), we have

$$([q, x]y + y[q, x] + [q, y]x + x[q, y])^m = (xy + yx)^n, \text{ for all } x, y \in Q.$$

In case the center C of Q is infinite, we have

$$([q, x]y + y[q, x] + [q, y]x + x[q, y])^m = (xy + yx)^n,$$

for all $x, y \in Q \otimes_C \overline{C}$, where $\overline{C}$ is algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [9, Theorems 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C ($i.e., RC = R$) which is either finite or algebraically closed and

$$([q, x]y + y[q, x] + [q, y]x + x[q, y])^m = (xy + yx)^n, \text{ for all } x, y \in R.$$

By Martindale [16, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with $\mathcal{D}$ as the associated division ring. Hence by Jacobson's theorem [11, p.75], R is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{D}$ and H consists of the finite rank linear transformations in R. If $\mathcal{V}$ is a finite dimensional over $\mathcal{D}$. Then the density of R on $\mathcal{V}$ implies that $R \cong \mathcal{M}_k(C)$, where $k = dim_{\mathcal{D}}\mathcal{V}$. Assume first that $dim_{\mathcal{D}}\mathcal{V} \geq 3$.

Step 1. We want to show that, for any $v \in \mathcal{V}$, v and qv are linearly $\mathcal{D}$-dependent. If $qv = 0$, then $\{v, qv\}$ is linearly $\mathcal{D}$-dependent. Suppose v and qv are linearly $\mathcal{D}$-independent and as $dim_{\mathcal{D}}\mathcal{V} \geq 3$, then there exists $w \in \mathcal{V}$ such that $\{v, qv, w\}$ is also linearly $\mathcal{D}$-independent. By the density of R, there exist $x, y \in R$ such that

$$xv = 0, \ xqv = w, \ xw = 0, \ \ yv = 0, \ yqv = 0, \ yw = v.$$

These imply that $(-1)^m v = ([q, x]y + y[q, x] + [q, y]x + x[q, y])^m v = (xy + yx)^n v = 0$, a contradiction.

So we conclude that $\{v, qv\}$ is linearly $\mathcal{D}$-dependent, for all $v \in V$.

Step 2. We show here that there exists $\alpha \in \mathcal{D}$ such that $qv = v\alpha$, for any $v \in \mathcal{V}$. Note that the arguments in [6], are still valid in the present situation. For the sake of completeness and clearness we prefer to present it. In fact, choose

$v, w \in \mathcal{V}$ linearly independent. Since $dim_{\mathcal{D}}\mathcal{V} \geq 3$, there exists $u \in \mathcal{V}$ such that v, w, u are linearly independent. By Step 1, there exist $\alpha_v, \alpha_w, \alpha_u \in \mathcal{D}$ such that

$$qv = v\alpha_v, qw = w\alpha_w, qu = u\alpha_u \text{ that is } q(v+w+u) = v\alpha_v + w\alpha_w + u\alpha_u.$$

Moreover $q(v+w+u) = (v+w+u)\alpha_{v+w+u}$, for a suitable $\alpha_{v+w+u} \in \mathcal{D}$. Then

$$0 = v(\alpha_{v+w+u} - \alpha_v) + w(\alpha_{v+w+u} - \alpha_w) + u(\alpha_{v+w+u} - \alpha_u),$$

and, because v, w, u are linearly independent, $\alpha_u = \alpha_w = \alpha_v = \alpha_{v+w+u}$, that is, α does not depend on the choice of v. So there exists $\alpha \in \mathcal{D}$ such that $qv = \alpha v$ for all $v \in \mathcal{V}$. This completes the proof of Step 2. Now for any $r \in R, v \in \mathcal{V}$. By Step 2, $qv = v\alpha$, $r(qv) = r(v\alpha)$, and also $q(rv) = (rv)\alpha$. Thus $0 = [q,r]v$, for any $v \in \mathcal{V}$, that is $[q,r]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible R-module, hence $[q,r] = 0$, for all $r \in R$, i.e., $q \in Z(R)$ and $d = 0$, which contradicts our hypothesis.

Therefore $dim_{\mathcal{D}}\mathcal{V}$ must be ≤ 2. In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [14, Lemma 2], it follows that there exists a suitable field $\mathcal{F}$ such that $R \subseteq \mathcal{M}_k(\mathcal{F})$, the ring of all $k \times k$ matrices over $\mathcal{F}$, and moreover $\mathcal{M}_k(\mathcal{F})$ satisfies the same generalized polynomial identity of R. If we assume $k \geq 3$, then by the same argument as in Steps 1 and 2, we get a contradiction. Obviously if $k = 1$, then R is commutative. Thus we may assume that $k = 2$, i.e., $R \subseteq \mathcal{M}_2(\mathcal{F})$, where $\mathcal{M}_2(\mathcal{F})$ satisfies

$$([q,x]y + y[q,x] + [q,y]x + x[q,y])^m = (xy+yx)^n \text{ for all } x, y \in \mathcal{M}_2(\mathcal{F}).$$

Denote e_{ij} the usual unit matrix with 1 in (i,j)-entry and zero elsewhere. By choosing $x = e_{11}, y = e_{12}$ we get,

$$0 = ([q,x]y + y[q,x] + [q,y]x + x[q,y])^m - (xy+yx)^n = (qe_{12} - e_{12}q)^m - (e_{12})^n.$$

In any case, we can arrive $0 = e_{12}(qe_{12})^m$. Now set $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By calculation, we have $\begin{pmatrix} 0 & q_{21}^m \\ 0 & 0 \end{pmatrix} = 0$ which implies that $q_{21} = 0$. In the same manner, we can see that $q_{12} = 0$. Thus we conclude that q is a diagonal matrix in $\mathcal{M}_2(\mathcal{F})$. Let $\chi \in Aut(\mathcal{M}_2(\mathcal{F}))$. Since

$$\begin{aligned}\big([\chi(q),\chi(x)]\chi(y) + \chi(y)[\chi(q),\chi(x)] &+ [\chi(q),\chi(y)]\chi(x) + \chi(x)[\chi(q),\chi(y)]\big)^m \\ &= (\chi(x)\chi(y) + \chi(y)\chi(x))^n.\end{aligned}$$

So, $\chi(q)$ must be diagonal matrix in $\mathcal{M}_2(\mathcal{F})$. In particular, let $\chi(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$. Then $\chi(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $\mathcal{M}_2(\mathcal{F})$, which leads to $d = 0$, a contradiction. This completes the proof. □

Theorem 2.2. *Let R be a prime ring with characteristic deferent from 2, I be a nonzero ideal of R and m, n be the two fixed positive integers. If R admits a derivation d such that $(d(x \circ y))^m - (x \circ y)^n \in Z(R)$, for all $x, y \in I$, then R is commutative.*

Proof. If $d = 0$, then $(xy + yx)^n \in Z(R)$, for all $x, y \in I$. By Chuang [7], this polynomial identity is also satisfied by Q and hence R as well. Note this is a polynomial identity and hence there exists a field $\mathcal{F}$ such that $R \subseteq \mathcal{M}_k(\mathcal{F})$, the ring of $k \times k$ matrices over field $\mathcal{F}$ where $k \geq 1$. Moreover, R and $\mathcal{M}_k(\mathcal{F})$ satisfy the same polynomial identity, that is $[(xy+yx)^n, w] = 0$ for all $x, y, w \in \mathcal{M}_k(\mathcal{F})$. But by choosing $x = e_{12},\ y = e_{21}$, we get $(xy + yx)^n = e_{11} + e_{22} \in Z(\mathcal{M}_k(\mathcal{F}))$, a contradiction.

Now we assume that $d \neq 0$. If $(d(xy + yx))^m = (xy + yx)^n$ for all $x, y \in I$, then R is commutative by Theorem 2.1. Otherwise we have $I \cap Z(R) \neq 0$ by our assumptions. Let J be a nonzero two sided ideal of R_Z, the ring of the central quotients of R. Since $J \cap R$ is an ideal of R, then $J \cap R \cap Z(R) \neq 0$. Hence J contains an invertible element in R_Z, and so R_Z is simple with 1. By the hypothesis for any $x, y \in I$ and $w \in I$, thus I satisfies the differential identity

$$[(d(x)y + yd(x) + d(y)x + xd(y))^m - (xy + yx)^n, w] = 0. \tag{2.1}$$

Since I and Q satisfy the same differential identities [7, Theorem 1], we may assume that Q satisfies (2.1). If d is not an inner derivation of R, then Q satisfies

$$[(sy + ys + tx + xt)^m - (xy + yx)^n, w] = 0.$$

In particular $x = 0$, we have $[(sy + ys)^m, w] = 0$, for all $y, s, w \in Q$. This is a polynomial identity and hence there exists a field $\mathcal{F}$ such that $Q \subseteq \mathcal{M}_k(\mathcal{F})$ with $k > 1$ and Q, $\mathcal{M}_k(\mathcal{F})$ satisfy the same polynomial identity [14]. Using same technique as above we get a contradiction. Let d be an inner derivation induced by an element $q \in Q$, that is, $q \in Q$ such that $d(x) = [q, x]$ for all $x \in R$. Then by (2.1) we have

$$[([q, x]y + y[q, x] + [q, y]x + x[q, x])^m - (xy + yx)^n, w] = 0. \tag{2.2}$$

for all $x, y \in I$ and $w \in R$. By Chuang [7], Q satisfy (2.2). By localizing R at $Z(R)$ it follows that

$$([q, x]y + y[q, x] + [q, y]x + x[q, x])^m - (xy + yx)^n \in Z(R_Z),$$

for all $x, y \in R_Z$. Since R and R_Z satisfy the same polynomial identities, by our assumption, we have that R_Z is not commutative. Thus, replacing R with R_Z, we may assume that R is a simple ring with 1. By, Martindale theorem [16], R is a primitive ring with minimal right ideal, whose commuting ring $\mathcal{D}$ is a division ring which is finite dimensional over $Z(R)$. However, since R is a simple with 1, R must be Artinian. Hence $R = \mathcal{D}_s$, the $s \times s$ matrices over $\mathcal{D}$, for some $s \geq 1$. By [14, Lemma 2], there exists a field $\mathcal{F}$ such that $R \subseteq \mathcal{M}_k(\mathcal{F})$, the ring of $k \times k$ matrices over field $\mathcal{F}$, with $k > 1$, and $\mathcal{M}_k(\mathcal{F})$ satisfies (2.2) that is,

$$([q,x]y + y[q,x] + [q,y]x + x[q,x])^m - (xy + yx)^n \in Z(\mathcal{M}_k(\mathcal{F})) = \mathcal{F}.\mathcal{I}_k.$$

If $k \geq 2$, now let $q = (q_{ij})_{k\times k}$. By assumption, for all $x, y \in R$,

$$([q,x]y + y[q,x] + [q,y]x + x[q,x])^m - (xy + yx)^n$$

is zero or invertible. We choose $x = e_{ii}, y = e_{ij}$ for any $i \neq j$. Then we have

$$([q,x]y + y[q,x] + [q,y]x + x[q,x])^m - (xy + yx)^n = [q, e_{ij}]^m$$

Since rank of $[q, e_{ij}]^m$ is ≤ 2, it cannot be invertible in R and so $[q, e_{ij}]^m = 0$. By solving above and left multiplying by e_{ij}, one can get $0 = e_{ij}(qe_{ij})^m = e_{ij}q_{ji}^m$ implying $q_{ji} = 0$. Thus, for any $i \neq j$, $q_{ji} = 0$ that is q is diagonal. Now set $q = \sum_t q_{tt}e_{tt}$ with $q_{tt} \in \mathcal{F}$. For any $\mathcal{F}$-automorphism χ of R, we have

$$\begin{aligned}([\chi(q), \chi(x)]\chi(y) + \chi(y)[\chi(q), \chi(x)] + [\chi(q), \chi(y)]\chi(x) + \chi(x)[\chi(q), \chi(x)])^m \\ - (\chi(x)\chi(y) + \chi(y)\chi(x))^n\end{aligned}$$

is zero or invertible, for every $x, y \in R$. By above argument $\chi(q)$ must be diagonal. Therefore, for each $j \neq i$, we have $\chi(q) = (1 + e_{ij})q(1 - e_{ij}) = \sum_{i=1}^{k} q_{ii}e_{ii} + (q_{jj} - q_{ii})e_{ij}$ is diagonal. Therefore, $q_{jj} = q_{ii}$ and so $q \in \mathcal{F}.\mathcal{I}_k$, and hence $d = 0$, a contradiction. This completes the proof of theorem. □

Theorem 2.3. *Let R be a prime ring, I be a nonzero ideal of R and m, n be the two fixed positive integers. If R admits a derivation d such that $(d(x) \circ d(y))^m = (x \circ y)^n$, for all $x, y \in I$, then R is commutative.*

Proof. If $d = 0$, then $(xy + yx)^n = 0$, for all $x, y \in I$, and hence R is commutative as already done in Theorem 2.1. Hence, onward we will assume that $d \neq 0$ and $(d(x) \circ d(y))^m = (x \circ y)^n$, for all $x, y \in I$. Now we divide our proof into two cases:

Case 1. If d is not an inner derivation of R, then I satisfies the polynomial identity

$$(s \circ t)^m = (xy + yx)^n, \text{ for all } x, y, s, t \in I.$$

In particular for $s = 0$, I satisfies the blended component $(xy + yx)^n = 0$, for all $x, y \in I$, and hence R is commutative by Theorem 2.1.

Case 2. Let d is an inner derivation induced by an element $q \in Q$, that is, $d(x) = [q, x]$ for all $x \in R$. It follows that, $([q, x] \circ [q, y])^m = (x \circ y)^n$, for all $x, y \in I$. Using the same argument presented in the proof of Theorem 2.1, we see that

$$([q,x] \circ [q,y])^m = (xy + yx)^n \text{ for all } x, y \in R, \tag{2.3}$$

where R is a primitive ring with $\mathcal{D}$ as the associated division ring. If $\mathcal{V}$ is finite dimensional over $\mathcal{D}$, then the density of R implies that $R \cong \mathcal{M}_k(C)$, where $k = dim_{\mathcal{D}}\mathcal{V}$. Suppose that $dim_{\mathcal{D}}\mathcal{V} \geq 2$, otherwise we are done. Now, we want to show that v and qv are linearly $\mathcal{D}$-dependent for all $v \in \mathcal{V}$. If $qv = 0$, then $\{v, qv\}$ is linearly $\mathcal{D}$-dependent. On contrary, suppose that v and qv are linearly $\mathcal{D}$-independent for some $v \in \mathcal{D}$.

If $q^2v \notin Span_{\mathcal{D}}\{v, qv\}$, then $\{v, qv, q^2v\}$ is linearly $\mathcal{D}$-independent. By the density of R there exist $x_0, y_0 \in R$ such that

$$x_0v = v, \quad x_0qv = 0, \quad x_0q^2v = 0,$$
$$y_0v = 0, \quad y_0qv = v, \quad y_0q^2v = v.$$

The application of (2.3) implies that $(-1)^m v = ([q, x_0] \circ [q, y_0])^m v = (xy + yx)^n v = 0$, and we arrive at a contradiction.

If $q^2v \in Span_{\mathcal{D}}\{v, qv\}$, then $q^2v = v\alpha + qv\gamma$ for some $\alpha, 0 \neq \gamma \in \mathcal{D}$. In view of the density of R, there exist $x_0, y_0 \in R$ such that

$$x_0v = v, \quad x_0qv = 0,$$
$$y_0v = 0, \quad y_0qv = v.$$

It follows from the relation (2.3) that $0 = ([q, x_0] \circ [q, y_0])^m v = (xy + yx)^n v = (-1)^m v\gamma^m \neq 0$, and we arrive at a contradiction. So we conclude that v and qv are linearly $\mathcal{D}$-dependent for all $v \in \mathcal{V}$. Therefore, for each $v \in \mathcal{V}$, $qv = v\beta_v$ for some $\beta_v \in \mathcal{D}$. By a standard arguments, it is easy to see that β_v is independent of the choice of $v \in \mathcal{V}$. Thus we can write $qv = v\beta$ for all $v \in \mathcal{V}$ and a fixed $\beta \in \mathcal{D}$. Using the same techniques as in the proof of the Theorem 2.1, we conclude that $d = 0$, a contradiction. This completes the proof of theorem. □

The following examples shows that the main results are not true in the case of arbitrary rings.

Example 2.1. Let S be any ring.

(i) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}$. We define a map $d : R \to R$ by $d(x) = e_{11}x - xe_{11}$. Then it is easy to see that d is a nonzero derivation and I is a nonzero ideal of R such that for all positive integers m, n, d satisfies the requirements of Theorems 2.1,2.2 and 2.3. However, R is not commutative. Hence, the hypothesis of primeness is crucial.

(ii) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}$. Define a map $d : R \to R$ by $d(x) = [x, e_{11} + e_{12}]$. Then R is a ring under usual operations. It is easy to see that d is a nonzero derivation and I is a nonzero ideal of R. Further, for any $x, y \in I$, the following conditions (1) $(d(x \circ y))^m = (x \circ y)^n$ and (2) $(d(x) \circ d(y))^m = (x \circ y)^n$ are satisfied but R is not commutative. Hence, in Theorems 2.1,2.2 and 2.3, the hypothesis of primeness can not be omitted.

3. The results in semiprime rings

From onwards R will be a semiprime ring, U is the left Utumi quotient ring of R. In order to prove the main results of this section we will make use of the following facts:

Fact 3.1 ([4, Proposition 2.5.1])**.** Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U, and so any derivation of R can be defined on the whole U.

Fact 3.2 ([8, p.38])**.** If R is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.

Fact 3.3 ([8, p.42])**.** Let B be the set of all the idempotents in C, the extended centroid of R. Assume R is a B-algebra orthogonal complete. For any maximal ideal P of B, PR forms a minimal prime ideal of R, which is invariant under any nonzero derivation of R.

Theorem 3.1. *Let R be a semiprime ring with center $Z(R)$ and m, n are fixed positive integers. If R admits a nonzero derivation d such that $(d(x \circ y))^m = (x \circ y)^n$, for all $x, y \in R$, then R is commutative.*

Proof. Since R is semiprime and we are given that $(d(x\circ y))^m = (x\circ y)^n$, which is rewritten as

$$(d(x)y + yd(x) + d(y)x + xd(y))^m = (xy + yx)^n,$$

for all $x, y \in R$. By Beidar [4], any derivation of a semiprime ring R can be defined on the whole U, the Utumi quotient ring of R. In view of Lee [15], R and U satisfy the same differential identities, hence

$$(d(x)y + yd(x) + d(y)x + xd(y))^m = (xy + yx)^n,$$

for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. Due to Chuang [8, p.42], U is an orthogonal complete B-algebra and by Fact 3.3, MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. Therefore $\overline{d}$ has in $\overline{U}$ the same property as d on U. It is obvious that $\overline{U}$ is prime. Therefore, by Theorem 2.1, $\overline{U}$ is commutative. This implies that, for any maximal ideal M of B, $[U, U] \subseteq MU$ and hence $[U, U] \subseteq \bigcap_M MU = 0$, where MU runs over all prime ideals of U. In particular, $[R, R] = 0$ and so R is commutative. This completes the proof of theorem. □

Using arguments similar to those used in the proof of the above theorem, we may conclude with the following (we omit the proof brevity). We can prove

Theorem 3.2. *Let R be a semiprime ring and m, n are fixed positive integers. If R admits a nonzero derivation d such that $(d(x) \circ d(y))^m = (x \circ y)^n$, for all $x, y \in R$, then R is commutative.*

We come now to our last result of this section:

Theorem 3.3. *Let R be a semiprime ring of characteristic different from 2 and m, n are fixed positive integers. If R admits a nonzero derivation d such that $d(x \circ y)^m - (x \circ y)^n \in Z(R)$, for all $x, y \in R$, then R is commutative.*

Proof. Since R is semiprime and we are given that $d(x \circ y)^m - (x \circ y)^n \in Z(R)$, that is

$$((d(x)y + yd(x) + d(y)x + xd(y))^m - (xy + yx)^n \in Z(R),$$

for all $x, y \in R$. by Fact 3.2, $Z(U) = C$, the extended centroid of R, and, by Fact 3.1, d can be uniquely defined on the whole U. By Lee [15], R and U satisfy the same differential identities. Then

$$((d(x)y + yd(x) + d(y)x + xd(y))^m - (xy + yx)^n \in C,$$

for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. As already pointed out in the proof of Theorem 3.1, U is a B-algebra orthogonal complete and by Fact 3.3, MU is a prime ideal of U, which is d-invariant. Let $\overline{d}$ is the derivation induced by d on $\overline{U} = U/MU$. Since $Z(\overline{U}) = (C + MU)/MU = C/MU$, then

$$((d(x)y + yd(x) + d(y)x + xd(y))^m - (xy + yx)^n \in (C + MU)/MU,$$

for any $x, y \in \overline{U}$. Moreover $\overline{U}$ is prime, hence we may conclude, by Theorem 2.2, either $\overline{U}$ is commutative or $\overline{d} = 0$ in $\overline{U}$. This implies that, for any maximal ideal M of B, we have $[U, U] \subseteq \bigcap MU$. Hence $[U, U] \subseteq_M \bigcap MU = 0$. This completes the proof. $\square$

4. Applications on Banach algebras

In this section, we apply the purely algebraic results which are obtained in section 2 and reveal the conditions that every continuous derivation on a Banach algebra maps into the radical. The proofs of the results rely on a Sinclair's Theorem [21] which states that every continuous derivation d of a Banach algebra $\mathfrak{A}$ leaves the primitive ideals of $\mathfrak{A}$ invariant. As we have mentioned before, Thomas [23], has generalized the Singer-Wermer theorem by proving that any derivation on a commutative Banach algebra maps the algebra into its radical. This result leads to the question whether the theorem can be proven without any commutativity assumption. There are many papers that the theorem holds without commutativity assumption [16, 17, 21, 22]. We also obtain that every derivation maps into its radical with some property, but without any commutativity assumption. Our first result in this section is about continuous derivations on Banach algebras:

Theorem 4.1. *Let $\mathfrak{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and $d : R \to R$ be a nonzero continuous derivation of $\mathfrak{A}$. If $(d(x \circ y))^m - (x \circ y)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, where m, n are fixed positive integers, then d maps into the radical of $\mathfrak{A}$.*

Proof. We have given that $(d(x\circ y))^m-(x\circ y)^n \in rad(\mathfrak{A})$, which can be rewritten as $(d(xy+yx))^m - (xy+yx)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$. Under the assumption that d is nonzero continuous derivation with Jacobson radical $rad(\mathfrak{A})$. In [21], Sinclair proved that any continuous derivation of a Banach algebra leaves the primitive ideals invariant. Since the Jacobson radical $rad(\mathfrak{A})$ is the intersection of all primitive ideals, we have $d(rad(\mathfrak{A})) \subseteq rad(\mathfrak{A})$, which means that there is no loss of generality in assuming that $\mathfrak{A}$ is semisimple. Since d leaves all primitive ideals invariant, one can introduce for any primitive ideal $P \subseteq \mathfrak{A}$, a nonzero derivation $d_P : \mathfrak{A}/P \to \mathfrak{A}/P$ by $d_P(x + P) = d(x) + P$, $x \in \mathfrak{A}$ where $\mathfrak{A}/P = \overline{A}$ is a factor Banach algebra. Moreover, by $(d(xy+yx))^m - (xy+yx)^n = 0$ for all $x, y \in \mathfrak{A}/P$.

Thus, by Theorem 2.1, it is immediate that either $\overline{\mathfrak{A}}$ is commutative or $d_P = 0$ on $\mathfrak{A}/P$. Consequently $d(\mathfrak{A}) \subseteq P$ for any primitive ideal P. Since the radical $rad(\mathfrak{A})$ of $\mathfrak{A}$ is the intersection of all primitive ideals in $\mathfrak{A}$, we get the required conclusion. □

Using arguments similar to those used in the proof of the above theorem, we can prove

Theorem 4.2. *Let $\mathfrak{A}$ be a non-commutative Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and $d : R \to R$ be a nonzero continuous derivation of $\mathfrak{A}$. If $(d(x) \circ d(y))^m - (x \circ y)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, where m, n are fixed positive integers, then d maps into the radical of $\mathfrak{A}$.*

In order to prove our last theorem we will use the following well-known result concerning semisimple Banach algebra contained in [12].

Lemma 4.1. *Every nonzero derivation on a semisimple Banach algebra is continuous.*

In view of the above Lemma 4.1 and Theorem 4.1, we may prove the following theorem in the case when $\mathfrak{A}$ is a semisimple Banach algebra.

Corollary 4.1. *Let $\mathfrak{A}$ be a non-commutative semisimple Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and d be a nonzero continuous derivation of $\mathfrak{A}$. If $(d(x \circ y))^m - (x \circ y)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, where m, n are fixed positive integers, then $d(\mathfrak{A}) = 0$.*

Proof. By the hypothesis d is continuous. In view of the above Lemma 4.1, every nonzero derivation on a semisimple Banach algebra is continuous. Thus, every nonzero derivation on a semisimple Banach algebra leaves the primitive ideals of the algebra invariant. Now by using the same argument as used in the proof of the Theorem 4.1 and the fact that $rad(\mathfrak{A}) = 0$, we get the required result. □

Corollary 4.2. *Let $\mathfrak{A}$ be a non-commutative semisimple Banach algebra with Jacobson radical $rad(\mathfrak{A})$ and d be a nonzero continuous derivation of $\mathfrak{A}$. If $(d(x) \circ d(y))^m - (x \circ y)^n \in rad(\mathfrak{A})$, for all $x, y \in \mathfrak{A}$, where m, n are fixed positive integers, then $d(\mathfrak{A}) = 0$.*

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