Fuzzy H_v-semigroups

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Abstract. In this paper, we introduce the notion of fuzzy H_v-semigroups, fuzzy H_v-groups and fuzzy H_v-homomorphisms and establish connections between fuzzy H_v-semigroups and H_v-semigroups. Also, we define and analyze the concept of fuzzy (strong) hypercongruences on fuzzy H_v-semigroups.

Keywords: hyperstructure, H_v-semigroup, fuzzy H_v-semigroup, fuzzy homomorphism, fuzzy (strong) hypercongruence.

1. Introduction

The notion of a hypergroup introduced by Marty in 1934 [12] at the 8th congress of Scandinavian Mathematicians. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, (see [1-8, 17]). Vougiouklis [18] introduced a new class of hyperstructures, the so-called H_v-structures.

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The H_v-structures are hyperstructures where equality is replaced by non-empty intersection.

The notion of a fuzzy subset introduced by Zadeh in 1965 [19] as a function from a nonempty set H to unit real interval $I = [0,1]$. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld [15], there have been a number of generalizations of this fundamental concept (see [14]). The study of fuzzy hyperstructures is an interesting research topic of fuzzy sets. There is a considerable amount of work to do on the connections between fuzzy sets and hyperstructures (see [3, 9, 13]). Recently, Davvaz has applied fuzzy sets to the study of algebraic hyperstructures. He defined fuzzy H_v-subgroups and fuzzy H_v-submodules which are generalizations of Rosenfeld's fuzzy subgroups and fuzzy submodules (see [6, 7]).

In [16] Sen, Ameri and Chowdhury introduced and analyzed fuzzy semihypergroups. The fuzzy hyperring notion is defined and studied in [10]. In [11] Leoreanu-Fotea defined and studied fuzzy hypermodules notion and connections with hypermodules.

The paper is organized as follows: in Section 2, some fundamental definitions on H_v-structures are explored. In Section 3, we consider the concept of fuzzy H_v-semigroups and fuzzy H_v-groups. In Section 4, we study connections between the fuzzy H_v-semigroups(H_v-groups) and H_v-semigroups(H_v-groups) and then establish some useful theorems. Finally, in Section 5, we define the notion of fuzzy (strong) hypercongruence on fuzzy H_v-semigroups and present a few results in this respect.

2. Preliminaries

First of all, we recall some notions and results (see [4, 5, 8, 17]) that we shall use in the following paragraphs.

Let H be a nonempty set and let $\wp^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \longrightarrow \wp^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid* (or hyperstructure).

If A and B are nonempty subsets of H, then we denote

$$A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b, x \circ A = \{x\} \circ A \quad \text{and} \quad A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H, we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, we have $x \circ H = H \circ x = H$.

A hyperstructure $(H, \circ)$ is called an H_v*-semigroup* if

$$((x \circ y) \circ z) \cap (x \circ (y \circ z)) \neq \emptyset,$$

for all $x, y, z \in H$.

Definition 2.1. *An H_v-semigroup $(H, \circ)$ is called a H_v-group if for all $x \in H$, we have $x \circ H = H \circ x = H$.*

A H_v-subgroup $(K, \circ)$ of $(H, \circ)$ is a nonempty set K, such that for all $k \in K$, we have $x \circ K = K \circ x = K$.

An H_v-group $(H, +)$ is called a *weak commutative H_v-group* if $(x+y) \cap (y+x) \neq \emptyset$ for all $x, y \in H$.

It is natural to speak now about homomorphisms between H_v-groups.

Definition 2.2. *Let $(H_1, \circ_1)$, $(H_2, \circ_2)$ be two H_v-groups. A map $f : H_1 \longrightarrow H_2$ is called an H_v-homomorphism or a weak homomorphism if*

$$f(x \circ_1 y) \cap f(x) \circ_2 f(y) \neq \emptyset; \text{ for all } x, y \in H_1.$$

f is called an inclusion homomorphism if

$$f(x \circ_1 y) \subseteq f(x) \circ_2 f(y); \text{ for all } x, y \in H_1.$$

Finally, f is called a strong homomorphism if

$$f(x \circ_1 y) = f(x) \circ_2 f(y); \text{ for all } x, y \in H_1.$$

If f is onto, one to one and strong homomorphism, then it is called an isomorphism.

Many examples of hypergroups and hyperrings can be found in [4], [5], [8] and [17]. We present here only two examples of H_v-groups.

Example 2.3. 1) Let $(G, \cdot)$ be a group and R an equivalence relation on G. In G/R consider the hyperoperation $\circ$ defined by $\overline{x} \circ \overline{y} = \{\overline{z} | z \in \overline{x} \cdot \overline{y}\}$, where $\overline{x}$ denotes the equivalence class of the element x. Then $(G, \circ)$ is an H_v-group which is not always a hypergroup.

2) On the set Z_{mn} consider the hyperoperation $\oplus$ defined by setting $0 \oplus m = \{0, m\}$ and $x \oplus y = x + y$ for all $(x, y) \in Z_{mn}^2 - \{(0, m)\}$. Then $(Z_{mn}, \oplus)$ is an H_v-group.

Let $(H, \circ)$ be a hypergroupoid. If $\{A, B\} \in \wp^*(H)$ and ρ is an equivalence relation on H, then we denote $A\overline{\rho}B$ if

1) $\forall a \in A, \exists b \in B$, such that $a\rho b$ and

2) $\forall b \in B, \exists a \in A$, such that $a\rho b$.

We denote $A\overline{\overline{\rho}}B$ if $\forall a \in A, \forall b \in B$, we have $a\rho b$.

3. Fuzzy H_v-semigroups

The fuzzy hypergroup notion was introduced and studied in [16]. Let S be a nonempty set and $F(S)$ denote the set of all fuzzy subsets of S. A *fuzzy hyperoperation* on S is a map $\circ : S \times S \longrightarrow F(S)$ and the image of (a, b) is denoted by $a \circ b$.

The set S together with a *fuzzy hyperoperation* " $\circ$ " is called a *fuzzy hypergroupoid*. We say that $(S, \circ)$ is commutative if for all $a, b \in S$, we have $a \circ b = b \circ a$. A fuzzy hypergroupoid $(S, \circ)$ is called a *fuzzy semihypergroup* if for all $a, b, c \in S$, we have $(a \circ b) \circ c = a \circ (b \circ c)$, where for any $\mu \in F(S)$, we have

$$(a \circ \mu)(r) = \bigvee_{t \in S} ((a \circ t)(r) \wedge \mu(t)) \text{ and } (\mu \circ a)(r) = \bigvee_{t \in S} (\mu(t) \wedge (t \circ a)(r)) \text{ for all } r \in S.$$

If A is a nonempty subset of S and $x \in S$, then for all $t \in S$ we have

$$(A \circ x)(t) = \bigvee_{a \in A} (a \circ x)(t) \text{ and } (x \circ A)(t) = \bigvee_{a \in A} (x \circ a)(t).$$

A fuzzy semihypergroup $(S, \circ)$ is called a *fuzzy hypergroup* if for all $x \in S$, we have $x \circ S = S \circ x = \chi_S$, where χ_S is the characteristic function of the set S.

We start by defining the notion of fuzzy H_v-semigroup.

Definition 3.1. *A fuzzy hypergroupoid $(S, \circ)$ is called a fuzzy H_v-semigroup if*

$$\exists x \in S,\ ((a \circ b) \circ c)(x) > 0 \textit{ and } (a \circ (b \circ c))(x) > 0 \textit{ for all } a, b, c \in S.$$

Now, suppose that $(H, \circ)$ is an H_v-semigroup and $a, b, c, \in H$. Then, $((a \circ b) \circ c) \cap (a \circ (b \circ c)) \neq \emptyset$ if and only if there exists $x \in H$ such that $x \in ((a \circ b) \circ c) \cap (a \circ (b \circ c))$ if and only if there exists $x \in H$ such that $(\chi_{(a \circ b) \circ c} \wedge \chi_{a \circ (b \circ c)})(x) = 1$ if and only if there exists $x \in H$ such that $(\chi_{(a \circ b) \circ c} \wedge \chi_{a \circ (b \circ c)})(x) \neq 0$ if and only if there exists $x \in H$ such that $\chi_{(a \circ b) \circ c}(x) > 0$ and $\chi_{a \circ (b \circ c)}(x) > 0$. As we show that, the notion of fuzzy H_v-semigroups generalizes the concept of H_v-semigroup.

The following two theorems provide us some examples of fuzzy H_v-semigroups:

Theorem 3.2. *Let S be a semigroup and $\mu(a) \neq 0$ for all $a \in S$ be a fuzzy semigroup on S. Let $a, b \in S$ and define a fuzzy hyperoperation $\circ$ on S by*

$$(a \circ b)(t) = \begin{cases} \mu(a) \wedge \mu(b), & \text{if } t = ab \\ 0, & \text{otherwise.} \end{cases}$$

Then $(S, \circ)$ is a fuzzy H_v-semigroup.

Proof. According to Examples 2.7 [16], we have

$$((a \circ b) \circ c)(abc) = (a \circ (b \circ c))(abc) = \mu(a) \wedge \mu(b) \wedge \mu(c) > 0 \text{ for all } a, b, c \in S.$$

Then $(S, \circ)$ is a fuzzy H_v-semigroup. □

Theorem 3.3. *Let S be a semigroup and $\mu(a) \neq 0$ for all $a \in S$ be a fuzzy semigroup on S. Let $a, b \in S$ and define a fuzzy hyperoperation $\circ$ on S by*

$$(a \circ b)(t) = \begin{cases} \mu(a) \vee \mu(b), & \text{if } t = ab \\ 0, & \text{otherwise.} \end{cases}$$

Then $(S, \circ)$ is a fuzzy H_v-semigroup.

Proof. Since $\mu(a) \vee \mu(b) \geq \mu(a) \wedge \mu(b)$, for all $a, b \in S$. By the above theorem we have $((a \circ b) \circ c)(abc) = (a \circ (b \circ c))(abc) \geq \mu(a) \wedge \mu(b) \wedge \mu(c) > 0$, for all $a, b, c \in S$. Then $(S, \circ)$ is a fuzzy H_v-semigroup. □

Example 3.4. Let S be a nonempty set and $a, b \in S$. We define a fuzzy hyperoperation $\circ$ on S by

$$(a \circ b)(t) = \begin{cases} \frac{1}{2}, & \text{if } t = \{a, b\} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(S, \circ)$ is a fuzzy H_v-semigroup.

Proof. Since $((a \circ b) \circ c)(a) = (a \circ (b \circ c))(a) = \frac{1}{2} > 0$ for all $a, b, c \in S$. Then $(S, \circ)$ is a fuzzy H_v-semigroup. □

We now give an example of a fuzzy H_v-semigroups that is not fuzzy semigroups.

Example 3.5. Let $S = \{a, b\}$ and $(a \circ a)(a) = 0.1,\ (a \circ a)(b) = 0.2,\ (b \circ b)(a) = 0.3,\ (b \circ b)(b) = 0.4,\ (a \circ b)(a) = 0.5,\ (a \circ b)(b) = 0.6,\ (b \circ a)(a) = 0.7$ and $(b \circ a)(b) = 0.8$. Clearly, $((a \circ b) \circ c)(d) > 0$ and $(a \circ (b \circ c))(d) > 0$ for all $a, b, c, d \in S$. It follows that $(S, \circ)$ is a fuzzy H_v-semigroup, but $0.6 = ((a \circ b) \circ a)(a) \neq (a \circ (b \circ a))(a) = 0.5$. Thus $(S, \circ)$ is not a fuzzy semigroup.

Definition 3.6. *A fuzzy H_v-semigroup $(S, \circ)$ is called a fuzzy H_v-group if*

$$a \circ S = S \circ a = \chi_S \textit{ for all } a \in S.$$

Theorem 3.7. *Let $(S, \circ)$ be a fuzzy hypergroup and $(a \circ b)(x) > 0$ for all $a, b, x \in S$. Then $(S, \circ)$ is a fuzzy H_v-group.*

Proof. Straightforward. □

The following three theorems provide us some examples of fuzzy H_v-groups:

Theorem 3.8. *Let S be a nonempty set and define a fuzzy hyperoperation $\circ$ on S by $a \circ b = \chi_{\{a,b\}}$ for all $a, b \in S$. Then $(S, \circ)$ is a fuzzy H_v-semigroup as well as a fuzzy H_v-group.*

Proof. Let $a, b, c \in S$. If $t \in \{a, b, c\}$, by Examples 2.5 [16], we have $((a \circ b) \circ c)(t) = (a \circ (b \circ c))(t) = 1 > 0$. Then $(S, \circ)$ is a fuzzy H_v-semigroup and also a fuzzy H_v-group. □

According to Example 2.6 [16], we have the following result.

Theorem 3.9. *Let $(S, .)$ be a semigroup and define a fuzzy hyperoperation $\circ$ on S by $a \circ b = \chi_{\{a.b\}}$ for all $a, b \in S$. Then $(S, \circ)$ is a fuzzy H_v-semigroup. If $(S, .)$ be a group, then $(S, \circ)$ is a fuzzy H_v-group.*

Theorem 3.10. *Let μ be a fuzzy subgroup of an abelian group S such that $\mu(e) \neq 0$ and let define a fuzzy hyperoperation $\circ$ on S by $(a \circ b)(t) = \mu(abt^{-1})$ for all $a, b \in S$. Then $(S, \circ)$ is a fuzzy H_v-group.*

Proof. Let $a, b, c \in S$. By Theorem 5.16 [16], we have $((a \circ b) \circ c)(c^{-1}b^{-1}a^{-1}) = (a \circ (b \circ c))(c^{-1}b^{-1}a^{-1}) = \mu(e) > 0$. Then $(S, \circ)$ is a fuzzy H_v-group. □

Definition 3.11. *[16] Let μ, ν be two fuzzy subsets of a fuzzy hypergroupoid $(S, \circ)$, then we define $(\mu \circ \nu)$ by $(\mu \circ \nu)(t) = \bigvee_{p,q \in S}(\mu(p) \wedge (p \circ q)(t) \wedge \nu(q))$, for all $t \in S$.*

Theorem 3.12. *Let $(S, \circ)$ be a fuzzy H_v-semigroup. Then we have*

(*i*) $\chi_a \circ \chi_b = a \circ b$ for all $a, b \in S$,

(*ii*) $\chi_S \circ \chi_a = S \circ a$ and $(ii)'$ $\chi_a \circ \chi_S = a \circ S$ for all $a \in S$,

(*iii*) $\chi_S \circ \mu = S \circ \mu$ and $(iii)'$ $\mu \circ \chi_S = \mu \circ S$ for all $\mu \in F(S)$.

Proof. Straightforward. □

Theorem 3.13. *Let $(S, \circ)$ be a fuzzy H_v-semigroup. Then $a \circ b \neq 0$, for all $a, b \in S$.*

Proof. Let $a, b, c \in S$, and $a \circ b = 0$. Then $((a \circ b) \circ c)(r) = \bigvee_{t \in S}((a \circ b)(t) \wedge (t \circ c)(r)) = 0$, for all $r \in S$, which is absurd. Hence, $a \circ b \neq 0$, for all $a, b \in S$. □

Corollary 3.14. *Let $(S, \circ)$ be a fuzzy H_v-group. Then $a \circ b \neq 0$, for all $a, b \in S$.*

4. Connections between fuzzy H_v-semigroups and H_v-semigroups

Let S be a nonempty set, endowed with a fuzzy hyperoperation $\circ$ and for all $a, b \in S$, consider the p-cuts $(a \circ b)_p = \{t \in S | \ (a \circ b)(t) \geq p\}$ of $a \circ b$, where $p \in [0, 1]$. For all $p \in [0, 1]$, we define the following crisp hyperoperation on $S : a \circ_p b = (a \circ b)_p$.

Theorem 4.1 ([11]). *For all $a, b, c, t \in S$ and for all $p \in [0, 1]$ the following equivalence holds:*

$(a \circ (b \circ c))(t) \geq p \Longleftrightarrow t \in a \circ_p (b \circ_p c)$ *and* $((a \circ b) \circ c)(t) \geq p \Longleftrightarrow t \in (a \circ_p b) \circ_p c)$.

Theorem 4.2. *$(S,\circ)$ is a fuzzy H_v-semigroup if and only if $\forall p \in (0,1]$, $(S,\circ_p)$ is a H_v-semigroup.*

Proof. Let $(S,\circ)$ be a fuzzy H_v-semigroup and $p \in (0,1]$. Then for all $a,b,c \in S$ there exists $t \in S$ such that $((a \circ b) \circ c)(t) = m > 0$ and $(a \circ (b \circ c))(t) = n > 0$. Hence, if $m \wedge n = p$, then $((a \circ b) \circ c)(t) \geq p$ and $(a \circ (b \circ c))(t) \geq p$ if and only if for all $a,b,c \in S$ there exists $t \in S$ such that $t \in (a \circ_p (b \circ_p c)) \cap ((a \circ_p b) \circ_p c)$. Therefore $(S,\circ_p)$ is a H_v-semigroup.

Conversely, if $p \in (0,1]$ and $(S,\circ_p)$ be a H_v-semigroup, then for all $a,b,c \in S$ there exists $t \in S$ such that $t \in (a\circ_p(b\circ_p c))\cap((a\circ_p b)\circ_p c)$. Hence $((a \circ b) \circ c)(t) \geq p > 0$ and $(a \circ (b \circ c))(t) \geq p > 0$. Therefore $(S,\circ_p)$ is a fuzzy H_v-semigroup. □

Theorem 4.3 ([11])**.** *For all $a \in S$, the following equivalence holds:*

$$a \circ S = \chi_S \Longleftrightarrow p \in [0,1],\ a \circ_p S = S.$$

Corollary 4.4. *$(S,\circ)$ is a fuzzy H_v-group if and only if $\forall p \in (0,1]$, $(S,\circ_p)$ is a H_v-group.*

According to [16], we can associate a hyperoperation on a fuzzy hypergroup $(G,\circ)$, as follows:

$$\forall\ a,b \in G,\ a * b = \{x \in G|\ (a \circ b)(x) > 0\}.$$

Theorem 4.5 ([16])**.** *Let $(G,\circ)$ be a fuzzy hypergroup. Then $(G,*)$ is a hypergroup.*

Theorem 4.6. *$(S,*)$ is a fuzzy H_v-semigroup if and only if $(S,\circ)$ is a H_v-semigroup.*

Proof. Let $a,b,c \in S$. Then $(S,*)$ be a H_v-semigroup if and only if $\exists x \in S, x \in (a*(b*c)) \cap ((a*b)*c) \Longleftrightarrow \exists x,p \in S, p \in b*c,\ x \in a*p$ and $\exists x,q \in S, q \in a*b, x \in q*c \Longleftrightarrow \exists x,p \in S, (b \circ c)(p) > 0, (a \circ p)(x) > 0$ and $\exists x,q \in S, (a\circ b)(q) > 0, (q\circ c)(x) > 0 \Longleftrightarrow \exists x \in S, \bigvee_{t\in S}((b\circ c)(t) \wedge (a\circ t)(x)) > 0$ and $\bigvee_{t\in S}((a \circ b)(t) \wedge (t \circ c)(x)) > 0 \Longleftrightarrow \exists x \in S,\ ((a \circ b) \circ c)(x) > 0$ and $(a \circ (b \circ c))(x) > 0$.

If and only if $(S,\circ)$ is a fuzzy H_v-semigroup. □

According to Theorem 4.5 and Theorem 4.6, we have the following result.

Theorem 4.7. *Let $(G,\circ)$ be a fuzzy H_v-group. Then $(G,*)$ is a H_v-group.*

Theorem 4.8 ([13])**.** *Let $(G,*)$ be a H_v-group. Then $a*b \neq \emptyset$, for all $a,b \in G$.*

According to [16], we can define a fuzzy hyperoperation on a hypergroup $(G,*)$, as follows:

$$\forall\ a,b \in G,\ a \circ b = \chi_{a*b}.$$

Theorem 4.9 ([16]). *Let $(G,*)$ be a hypergroup. Then $(G,\circ)$ is a fuzzy hypergroup.*

Theorem 4.10. *Let $(G,*)$ be a H_v-semigroup. Then $(G,\circ)$ is a fuzzy H_v-semigroup.*

Proof. Let $(G,*)$ be a H_v-semigroup and $a,b,c \in G$. Then for all $a,b,c \in G$ there exists $t_0 \in G$ such that $t_0 \in (a*(b*c)) \cap ((a*b)*c)$. On the other hand, for all $t \in G$ we have

$$((a\circ b)\circ c)(t) = \begin{cases} 1, & \text{if } t \in ((a*b)*c) \\ 0, & \text{otherwise.} \end{cases}$$

and

$$(a\circ (b\circ c))(t) = \begin{cases} 1, & \text{if } t \in (a*(b*c)) \\ 0, & \text{otherwise.} \end{cases}$$

Hence $((a\circ b)\circ c)(t_0) = (a\circ(b\circ c))(t_0) = 1 > 0$. Therefore $(S,\circ)$ is a fuzzy H_v-semigroup. □

By Theorem 4.9 and Theorem 4.10, we have the following result.

Theorem 4.11. *Let $(G,*)$ be a H_v-group. Then $(G,*)$ is a fuzzy H_v-group.*

By Theorems 4.7 and 4.11, we have the following facts.

Denote by $\mathcal{H}_v\mathcal{G}$ the class of all H_v-groups and by $\mathcal{FH}_v\mathcal{G}$ class of all fuzzy H_v-groups, then we can consider two maps:

(1) $\phi : \mathcal{H}_v\mathcal{G} \longrightarrow \mathcal{FH}_v\mathcal{G}$, $\phi(G,*) = (G,\circ)$, where for all $a,b \in G$ we have $a\circ b = \chi_{a*b}$;

(2) $\psi : \mathcal{FH}_v\mathcal{G} \longrightarrow \mathcal{H}_v\mathcal{G}$, $\psi(G,\circ) = (G,*)$, where for all $a,b \in G$ we have $a*b = \{x \in G|\ (a\circ b)(x) > 0\}$.

Let $\mu,\lambda \in F(G)$. Then we say that μ includes λ and we denote $\mu \subseteq \lambda$ if and only if for all $x \in G$, we have $\mu(x) \leq \lambda(x)$ (see [15]).

Let f be a mapping from a set X into a set Y. Let μ be a fuzzy set in X. Then the *image* $f(\mu)$ of μ is the fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \bigvee\{\mu(x)|x \in X, f(x) = y\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for all $y \in Y$ (see [19]).

Now, we introduce the notion of the fuzzy H_v-group (inclusion, weak) homomorphism, as follows.

Definition 4.12. *Let $(S, \circ_1)$ and $(T, \circ_2)$ be two fuzzy H_v-groups. A map $f : S \longrightarrow T$ is called an H_v-homomorphism or a weak homomorphism if*

$$f(x \circ_1 y) \cap f(x) \circ_2 f(y) \neq \emptyset; \text{ for all } x, y \in S.$$

f is called an inclusion homomorphism if

$$f(x \circ_1 y) \subseteq f(x) \circ_2 f(y); \text{ for all } x, y \in S.$$

Finally, f is called a strong homomorphism if

$$f(x \circ_1 y) = f(x) \circ_2 f(y); \text{ for all } x, y \in S.$$

Remark 4.13. If $f : S \longrightarrow T$ is a map and for all $a \in S$, then $f(\chi_a) = \chi_{f(a)}$.

The next two theorems show connections between the inclusion (weak) homomorphism of fuzzy H_v-groups and the inclusion (weak) homomorphism of corresponding H_v-groups.

Theorem 4.14. *Let $(S, \circ_1)$ and $(T, \circ_2)$ be two fuzzy H_v-groups and $(S, *_1) = \psi(S, \circ_1)$, $(T, *_2) = \psi(T, \circ_2)$ be their associated H_v-groups, respectively. If $f : S \longrightarrow T$ is an inclusion (weak) homomorphism of fuzzy H_v-groups, then f is an inclusion (weak) homomorphism of the associated H_v-groups.*

Proof. Suppose that $f : S \longrightarrow T$ is an inclusion homomorphism of fuzzy H_v-groups. Then we have $f(a \circ_1 b) \subseteq f(a) \circ_2 f(b)$ for all $a, b \in S$. This means that $(f(a \circ_1 b))(y) \leq (f(a) \circ_2 f(b))(y)$ for all $y \in T$. If $x \in a *_1 b$, then $(a \circ_1 b)(x) > 0$ and let $y = f(x)$. It follows that

$$(f(a \circ_1 b))(y) = \bigvee_{r \in f^{-1}(y)} (a \circ_1 b)(r) \geq (a \circ_1 b)(x) > 0.$$

Thus $(f(a) \circ_2 f(b))(y) > 0$. Hence $y = f(x) \in f(a) *_2 f(b)$. So we obtain $f(a *_1 b) \subseteq f(a) *_2 f(b)$. Therefore, f is an inclusion homomorphism of the associated H_v-groups.

Now, let $f : S \longrightarrow T$ be a weak homomorphism of fuzzy H_v-groups. Then we have

$$\exists y \in T,\ y \in f(a \circ_1 b)(y) > 0 \text{ and } (f(a) \circ_2 f(b))(y) > 0,\ \forall a, b \in S.$$

If $y \in f(a \circ_1 b)(y) > 0$, then $\bigvee_{r \in f^{-1}(y)} (a \circ_1 b)(r) > 0$, implies that there exists $x \in S$, $f(x) = y$ such that $(a \circ_1 b)(x) > 0$. Thus $x \in a *_1 b$, this means that $y = f(x) \in f(a *_1 b)$.

If $(f(a) \circ_2 f(b))(y) > 0$, then $y \in f(a) *_2 f(b)$. Hence, $y \in f(a *_1 b) \cap f(x) *_2 f(y) \neq \emptyset$; for all $a, b \in S$. Therefore, f is a weak homomorphism of the associated H_v-groups. □

Theorem 4.15. *Let $(S, *_1)$ and $(T, *_2)$ be two H_v-groups and $(S, \circ_1) = \phi(S, *_1)$, $(T, \circ_2) = \phi(T, *_2)$ be their associated fuzzy H_v-groups, respectively. Then, $f : S \longrightarrow T$ is an inclusion (weak) homomorphism of H_v-groups if and only if f is an inclusion (weak) homomorphism of the associated fuzzy H_v-groups.*

Proof. " $\Longleftarrow$ " Since f is an inclusion homomorphism of fuzzy H_v-groups, we have $f(a \circ_1 b) \subseteq f(a) \circ_2 f(b)$ for all $a, b \in S$. This means that $(f(a \circ_1 b))(y) \leq (f(a) \circ_2 f(b))(y)$ for all $y \in T$. Then, we have $\chi_{f(a *_1 b)}(y) = \bigvee_{r \in f^{-1}(y)} \chi_{a *_1 b}(r) = \bigvee_{r \in f^{-1}(y)} (a \circ_1 b)(r) = (f(a \circ_1 b))(y) \leq (f(a) \circ_2 f(b))(y) = \chi_{f(a) *_2 f(b)}(y)$.

This means that $f(a *_1 b) \subseteq f(a) *_2 f(b)$.

" $\Longrightarrow$ " Let f be an inclusion homomorphism of H_v-groups and $a, b \in S$. If $y \notin \mathrm{Im} f$, then $(f(a \circ_1 b))(y) = 0 \leq (f(a) \circ_2 f(b))(y)$. If $y \in \mathrm{Im} f$, then we have

$$(f(a \circ_1 b))(y) = \chi_{f(a *_1 b)}(y) \leq \chi_{f(a) *_2 f(b)} = (f(a) \circ_2 f(b))(y).$$

Which means that $f(a *_1 b) \subseteq f(a) *_2 f(b)$. Therefore, f is an inclusion homomorphism of the associated fuzzy H_v-groups.

Now, $f : S \longrightarrow T$ is a weak homomorphism of H_v-groups if and only if there exists $y \in T$ such that $(f(a \circ_1 b))(y) > 0$ and $(f(a) \circ_2 f(b))(y) > 0$ for all $a, b \in S$. Since $\chi_{f(a *_1 b)}(y) = (f(a \circ_1 b))(y) > 0$ and $\chi_{f(a) *_2 f(b)}(y) = (f(a) \circ_2 f(b))(y) > 0$, if and only if $\chi_{f(a *_1 b)}(y) = 1$ and $\chi_{f(a) *_2 f(b)}(y) = 1$, if and only if there exists $y \in T$ such that $y \in f(a *_1 b) \cap f(a) *_2 f(b)$ for all $a, b \in S$. If and only if f is a weak homomorphism of the associated fuzzy H_v-groups. □

Let us consider now the algebraic H_v-substructures of fuzzy H_v-groups.

Definition 4.16. *Let $(G, \circ)$ be a fuzzy H_v-group and $(G, *) = \psi(G, \circ)$ be the associated H_v-group. A nonempty subset H of G is called a fuzzy H_v-subgroup if for all $x, y \in G$, the following conditions hold:*

(i) $(x \circ y)(z) > 0$, *then* $z \in H$;

(ii) $x \circ H = H \circ x = \chi_H$.

Theorem 4.17. *(i) Let $(H, \circ)$ be a fuzzy H_v-subgroup of $(G, \circ)$. Then $(H, *) = \psi(H, \circ)$ is an H_v-subgroup of $(G, *) = \psi(G, \circ)$;*

(ii) Let $(H, *)$ be an H_v-subgroup of $(G, *)$. Then $(H, \circ) = \phi(H, *)$ is a fuzzy H_v-subgroup of $(G, \circ) = \psi(G, *)$.

5. Fuzzy (strong) hypercongruence on fuzzy H_v-semigroups

Fuzzy regular relations are introduced in the context of fuzzy semihypergroups in [16]. Now, we introduce the hypercongruence on H_v-semigroups and the fuzzy (strong) hypercongruence on fuzzy H_v-semigroups and discuss the relations of them. We recall some definitions:

Let $(G, \circ)$ be a fuzzy H_v-semigroup and ρ be an equivalence relation on G. If $\mu, \lambda \in F(G)$, then we say that $\mu \overline{\rho} \lambda$ if the following two conditions hold:

(1) $\forall a \in G$, if $\mu(a) > 0$, then there exists $b \in G$, such that $\lambda(b) > 0$ and $a\rho b$;

(2) $\forall x \in G$, if $\lambda(x) > 0$, then there exists $y \in G$, such that $\mu(y) > 0$ and $x\rho y$.

Now, we can introduce the notion of the hypercongruence on fuzzy H_v-semigroups, in the following manner.

Definition 5.1. *An equivalence relation ρ on a fuzzy H_v-semigroup $(G, \circ)$ is called a fuzzy regular relation (or a fuzzy hypercongruence) on G if, for all $a, b, c \in G$, the following implication holds:*

$$a\rho b \implies (a \circ c)\overline{\rho}(b \circ c) \text{ and } (c \circ a)\overline{\rho}(c \circ b).$$

*Clearly, the condition is equivalent to $a\rho a'$*and *$b\rho b'$ implies $(a \circ b)\overline{\rho}(a' \circ b')$ for all $a, b, a', b' \in S$.*

The next theorem shows a fuzzy homomorphism of fuzzy H_v-semigroups can induce a fuzzy hypercongruence on a fuzzy H_v-semigroup.

Theorem 5.2. *Let $(S, \circ_1)$ and $(T, \circ_2)$ be two fuzzy H_v-semigroups. Let $f : S \longrightarrow T$ be a fuzzy homomorphism. Then $\rho = \ker f = \{(a, b) \in S^2 |\ f(a) = f(b)\}$ is a fuzzy hypercongruence on S.*

Proof. Clearly, ρ is an equivalence relation on S. Let $a, b, a', b' \in S$, $a\rho a'$ and $b\rho b'$. Then we have $f(a) = f(a')$ and $f(b) = f(b')$. Since f is a homomorphism of fuzzy H_v-semigroups, we have

$$f(a \circ_1 b) = f(a) \circ_2 f(b) = f(a') \circ_2 f(b') = f(a' \circ_1 b').$$

Now, let $x \in S$ and $(a \circ_1 b)(x) > 0$. Which means that

$$f(a \circ_1 b)(f(x)) = \bigvee_{x' \in S,\ f(x') = f(x)} (a' \circ_1 b')(x') \geq (a \circ_1 b)(x) > 0.$$

This means that there exists $x' \in S$ such that $(a' \circ_1 b')(x') > 0$ and $f(x') = f(x)$. Hence $(a' \circ_1 b')(x') > 0$ and $x\rho x'$.

Similarly, for any $s \in S$ and $(a' \circ_1 b')(s) > 0$, there exists $t \in S$ such that $(a \circ_1 b)(t) > 0$ and $s\rho t$. Hence $(a \circ_1 b)\overline{\rho}(a' \circ_1 b')$. Therefore, $\rho = \ker f$ is a fuzzy hypercongruence on S. □

In what follows, we shall see a relation between the fuzzy hypercongruence on fuzzy H_v-semigroups and the hypercongruence on corresponding H_v-semigroups.

Let $(S, \circ)$ be a fuzzy H_v-semigroup and $(S, *) = \psi(S, \circ)$ be the associated H_v-semigroup, where for any $a, b \in S$, we have $a * b = \{x \in |\ (a \circ b)(x) > 0\}$. Then we can obtain:

Theorem 5.3. *An equivalence relation ρ is a fuzzy hypercongruence on a fuzzy H_v-semigroup $(S, \circ)$ if and only if ρ is a hypercongruence on corresponding H_v-semigroup $(S, *)$.*

Proof. Set $a\rho b$ and $a'\rho b'$, where $a, b, a', b' \in S$. We have that $(a \circ a')\overline{\rho}(b \circ b')$ if and only if the following conditions hold:

(1) if $(a \circ a')(x) > 0$, then there exists $y \in S$, such that $(b \circ b')(y) > 0$ and $x\rho y$;

(2) if $(b \circ b')(s) > 0$, then there exists $t \in S$, such that $(a \circ a')(t) > 0$ and $s\rho t$.

These conditions are equivalent to the following ones:

(1) if $x \in a * a'$, then there exists $y \in b * b'$ and $x\rho y$;

(2) if $s \in b * b'$, then there exists $t \in a * a'$ and $s\rho t$.

Which show that $(a * a')\overline{\rho}(b * b')$. Therefore, ρ is a fuzzy hypercongruence on $(S, \circ)$ if and only if ρ is a hypercongruence on $(S, *)$. □

A hypercongruence plays an important role in studying the quotient structure of H_v-semigroups, as the following theorem shows.

Theorem 5.4. *Let $(S, *)$ be an H_v-semigroup and ρ be an equivalence relation on S. If we define a hyperoperation $\oplus$ on the quotient set S/ρ: for all $a\rho, b\rho, c\rho \in S/\rho$, $a\rho \oplus b\rho = \{c\rho : \ c \in a * b\}$, then ρ is a hypercongruence on S if and only if $(S/\rho, \oplus)$ is an H_v-semigroup.*

Proof. " $\Longrightarrow$ " Suppose that ρ is a hypercongruence on S. Let $a, b \in S$, since $a * b$ is nonempty set, implies $a\rho \oplus b\rho$ is nonempty set. Clear, $\oplus$ is well-defined. Since $(S, *)$ is an H_v-semigroup, implies there exists $x \in S$ such that $x \in (a * (b * c)) \cap ((a * b) * c)$ for all $a, b, c \in S$. Now, if $x \in a * (b * c)$, then $x \in a * d$, for some $d \in b * c$, implies that $x\rho \in a\rho \oplus d\rho$, for some $d\rho \in b\rho \oplus c\rho$. Therefore $x\rho \in a\rho \oplus (b\rho \oplus c\rho)$.

Similarly, if $x \in (a * b) * c$, we can prove that $x\rho \in (a\rho \oplus b\rho) \oplus c\rho$. Therefore, $(S/\rho, \oplus)$ is an H_v-semigroup.

" $\Longleftarrow$ " Suppose that $a, b \in S$, since $a\rho \oplus b\rho$ is nonempty set, then $a * b$ is nonempty set. In the other hand, since $(S/\rho, \oplus)$ is an H_v-semigroup, implies there exists $x \in S$ such that $x\rho \in (a\rho \oplus (b\rho \oplus c\rho)) \cap ((a\rho \oplus b\rho) * c\rho)$ for all $a, b, c \in S$. If $x\rho \in a\rho \oplus (b\rho \oplus c\rho)$, then $x\rho \in a\rho \oplus d\rho$, for some $d\rho \in b\rho \oplus c\rho$, implies that $x \in a * d$, for some $d \in b * c$. Therefore $x \in a * (b * c)$.

Similarly, if $x\rho \in (a\rho \oplus b\rho) * c\rho$, we can prove that $x \in (a * b) * c$. Therefore, ρ is a hypercongruence on S. □

The H_v-semigroup $(S/\rho, \oplus)$ is called the quotient H_v-semigroup induced by the fuzzy H_v-semigroup $(S, *)$ and the fuzzy hypercongruence ρ on $(S, *)$.

Let $(G, \circ)$ be a fuzzy H_v-semigroup and ρ be an equivalence relation on G. If $\mu, \lambda \in F(G)$, then we say that $\mu\overline{\overline{\rho}}\lambda$ if the following two conditions hold:

(1) $\forall a \in G$, if $\mu(a) > 0$, then there exists $b \in G$, such that $\lambda(b) > 0$, $\mu(a) = \lambda(b)$ and $a\rho b$;

(2) $\forall x \in G$, if $\lambda(x) > 0$, then there exists $y \in G$, such that $\mu(y) > 0$, $\mu(x) = \lambda(y)$ and $x\rho y$.

Definition 5.5. *An equivalence relation ρ on a fuzzy H_v-semigroup $(G, \circ)$ is called a fuzzy strong hypercongruence on G if, for all $a, b, c, d \in G$, the following implication holds:*

$$a\rho b,\ c\rho d \implies (a \circ c)\overline{\overline{\rho}}(b \circ d).$$

Theorem 5.6. *Let $(S, \circ)$ be a fuzzy H_v-semigroup and ρ be a fuzzy strong hypercongruence on S. If we define a fuzzy hyperoperation $\odot$ on S/ρ by $(x\rho \odot y\rho)(z\rho) = \bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho}(x' \circ y')(z')$, for all $x\rho, y\rho, z\rho \in S/\rho$, then S/ρ is a fuzzy H_v-semigroup.*

Proof. First, we prove that fuzzy hyperoperation $\odot$ is well-defined. Suppose that $x, y, x', y' \in S$, such that $x\rho = x'\rho, y\rho = y'\rho$. Since ρ is a fuzzy strong hypercongruence on S. Hence $(x \circ y)\overline{\overline{\rho}}(x' \circ y')$. Then, for all $z\rho \in S/\rho$, we have $(x\rho \odot y\rho)(z\rho) = \bigvee_{x_1 \in x\rho, y_1 \in y\rho, z_1 \in z\rho}(x_1 \circ y_1)(z_1) = \bigvee_{x_2 \in x'\rho, y_2 \in y'\rho, z_2 \in z'\rho}(x_2 \circ y_2)(z_2) = (x'\rho \odot y'\rho)(z'\rho)$, it follows that $x\rho \odot y\rho = x'\rho \odot y'\rho$. Therefore, $\odot$ is well-defined.

Now, let $x, y, z \in S$, since $(S, \circ)$ is a fuzzy H_v-semigroup, then there exists $t \in S$, such that $((x \circ y) \circ y)(t) > 0$ and $(x \circ (y \circ z))(t) > 0$.

In the other hand, we have

$$\begin{aligned}((x\rho \odot y\rho) \odot z\rho)(t\rho) &= \bigvee_{s\rho \in S/\rho} ((x\rho \odot y\rho)(s\rho) \wedge (s\rho \odot z\rho)(t\rho)) \\ &= \bigvee_{s \in S} (\bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho, t' \in t\rho} ((x' \circ y')(s') \wedge (s' \circ z')(t'))) \\ &= \bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho, t' \in t\rho} ((x' \circ y') \circ z')(t')).\end{aligned}$$

It follows that

$$\begin{aligned}((x\rho \odot y\rho) \odot z\rho)(t\rho) &= \bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho, t' \in t\rho} ((x' \circ y') \circ z')(t')) \\ &\geq ((x \circ y) \circ z)(t) > 0.\end{aligned}$$

In a similar way, we can show that

$$\begin{aligned}(x\rho \odot (y\rho \odot z\rho))(t\rho) &= \bigvee_{x' \in x\rho, y' \in y\rho, z' \in z\rho, t' \in t\rho} (x' \circ (y' \circ z'))(t')) \\ &\geq (x \circ (y \circ z))(t) > 0.\end{aligned}$$

Therefore, S/ρ is a fuzzy H_v-semigroup. □

Theorem 5.7. *Let $(S, \circ)$ be a fuzzy H_v-semigroup and ρ be a fuzzy strong hypercongruence on S. If we define a mapping $f : S \longrightarrow S/\rho$ by $f(a) = a\rho$, for all $a \in S$, then f is a surjective strong homomorphism from a fuzzy H_v-semigroup $(S, \circ)$ to a fuzzy H_v-semigroup $(S/\rho, *)$.*

Proof. Clearly, the map $f : S \longrightarrow S/\rho$ is surjective. Suppose that $a, b \in S$, since ρ is a fuzzy strong hypercongruence of fuzzy H_v-semigroups $(S, \circ)$, then for all $x\rho \in S/\rho$, we have $(f(a \circ b))(x\rho) = \bigvee_{f(t)=x\rho}(a \circ b)(t) = \bigvee_{t\rho=x\rho}(a \circ b)(t) = \bigvee_{t\in x\rho}(a \circ b)(t) = \bigvee_{t'\in x\rho, a'\in a\rho, b'\in b\rho}(a' \circ b')(t') = (a\rho * b\rho)(x\rho) = (f(a) * f(b))(x\rho)$, thus $f(a \circ b) = f(a) * f(b)$. Therefore, f is a surjective strong homomorphism from a fuzzy H_v-semigroup $(S, \circ)$ to a fuzzy H_v-semigroup $(S/\rho, *)$. □

Corollary 5.8. *Let $(S, \circ)$ be a fuzzy H_v-semigroup and ρ be a fuzzy hypercongruence on S. If we define a mapping $f : S \longrightarrow S/\rho$ by $f(a) = a\rho$, for all $a \in S$, then f is a surjective inclusion homomorphism from a fuzzy H_v-semigroup $(S, \circ)$ to a fuzzy H_v-semigroup $(S/\rho, *)$.*

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