# An iterative solution for a class of optimization problems

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**Abstract.** In this paper, we will describe an iterative approximation method to approach the optimal solution of a linear programming problem by providing an algorithm for resolution with an analysis of its convergence and its complexity.
**Keywords:** linear programming, optimal solution, primal-dual problem, iterative approximation method.

## 1. Introduction

Linear programming models are used in all natural sciences disciplines such as biology, physics, chemistry, earth science, and engineering disciplines such as computer science, logistics, mechanics, artificial intelligence, as well as in the social sciences such as economics, sociology, political science, see [15].

We can deal with problems of management, electoral politics, molecular biology, engineering, logistics, etc.., Whether strategic (choice to invest or not, choosing a location, the design of a fleet of vehicles or real estate ...) or operational (including scheduling, inventory management, sales forecasts ..), see [15].

Linear programming is the methods of optimizations concentrate on achieving the best outcome in a mathematical model whose requirements are represented by linear relationships. That was first solved by Dantzig [3] using Simplex algorithm, how still the most widely used in the literature. The linear programming problem was first shown to be solvable in polynomial time, but a larger theoretical and practical breakthrough in the field came in 1984 when Karmarkar [8, 9] introduced a new interior point algorithm for solving this type

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of problems. After that, Many algorithms and methods have been used to solve this kind of optimization problem with the same idea, as an example, Interior point methods [1, 2], Logarithmic barrier function [7], Path-following algorithm [11], or as Interior-Exterior point algorithm [4].

In this paper, we will describe an iterative approximation method to approach the optimal solution of a linear programming problem by providing an algorithm for resolution with an analysis of its convergence and its complexity.

## 2. Statement problem

We study an iterative approximation method for solving the linear programming problem

$$(P): \min_{x\in I\!R^n} \{c^T x: \; Ax = b, \; x \geq 0\},$$

The logarithmic barrier function method have for but to solve the family of problems

$$\min_{x>0} \{c^T x - \mu \sum_{j=1}^{n} \ln x_j : \; Ax = b\},$$

with $\mu > 0$ called the barrier penalty coefficient.

Such problem can be solved by considering the family of penalized subproblems [4, 12]

$$(P_\mu): \min_{x>0} \{f_\mu(x) = c^T x - \mu \sum_{j=1}^{n} \ln x_j + \frac{1}{2\mu} \sum_{i=1}^{m} (a_{i.}x - b_i)^2\},$$

where $x_j$ is the component of $x$ number $j$ also $a_{i.}$ is the row of the matrix $A$ number $i$.

The family problems $(P_\mu)$ have an optimal solution for all $\mu > 0$ while they are strictly convex add to some standard assumptions see [11] .

From other side $\omega(\mu) = (x(\mu), y(\mu), z(\mu))$ of the system $(P_\mu)$ corresponding to the Karush-Kuhn-Tucker (K-K-T) conditions gives a dual and primal $\mu$-feasible solution.

Finally, every accumulation point $\omega^*$ of the sequence $\omega(\mu)$ is such that $x^*$ is an optimal solution of the primal problem

$$(P): \min\{c^T x: \; Ax = b, \; x \geq 0\}$$

and $(y^*, z^*)$ is an optimal solution of the dual

$$(D): \max\{b^T y: \; A^T y = z = c, \; z \geq 0\}.$$

To solve the primal problem $(P)$, we consider the following penalized problem:

$$(P_\mu): \min_{x>0} \{f_\mu(x) = c^T x - \mu \sum_{j=1}^{n} \ln x_j + \frac{1}{2\mu} \sum_{i=1}^{m} (a_i x - b_i)^2\}, \text{ where } \mu > 0.$$

Observe that $f_\mu$, the objective function of problem $(P_\mu)$, is a strictly convex function.

This implies that problems $(P_\mu)$ has at most one minimum, and that this minimum, if it exists, is global and completely characterized by the K-K-T optimality conditions

$$c - \mu X^{-1}e + \frac{1}{\mu}A^T(Ax-b) = 0 \quad \text{and} \quad x > 0,$$

where $e$ denotes the $\mathbb{R}^n$ unit vector i.e. $e = (1,...,1)^T$ see [12].

By introducing two vectors $y \in \mathbb{R}^m$ et $z \in \mathbb{R}^n$, this system can be rewritten in an equivalent way as:

$$(S_\mu)\begin{cases} ZXe - \mu e = 0 \\ Ax + \mu y - b = 0,\ x > 0 \\ A^T y + z - c = 0 \end{cases}.$$

We denote the unique solution of $(S_\mu)$ by $\omega(\mu) = (x(\mu), y(\mu), z(\mu))$.

In the next section, we describe our methods. The fourth section provide an algorithm of solution with an analysis of its convergence and its complexity.

## 3. Methods

From theoretical point of view, it is very difficult to find an exact solution of $(P_\mu)$. Which justify our utilization of a constructive method leads to an ideal approximative solution of our initial problem. In the next, We will explain how to formulate a solution of the system $(S_\mu)$ which allow the best approximation of $(x(\mu), y(\mu), z(\mu))$. The obvious idea is to determine a solution of $(S_\mu)$ using the Newton method. But this idea is to be avoided, noting the Newton method necessary iterations to find a satisfactory approximative solution.

Consequently, we consider the definition

**Definition 3.1.** *For $\mu > 0$. A point $\omega(\mu) = (x, y, z)$ is called $\mu-$feasible if it verify:*

$$Ax + \mu y - b = 0,\ x > 0,$$
$$A^T y + z - c = 0,\ z > 0.$$

*Such solution $x$ is called the primal $\mu-$feasible in relation with $y$.*

The main idea of this section is the following. For every iteration, among the $\mu-$feasible solutions (i.e. solutions satisfying the two last equations of the system $(S_\mu)$), we take the ones which violate the first equation of $(S_\mu)$. This well known technique is usually used in the "path following" category of interior point algorithms. The direction in these last algorithms coincides with the Newton direction, while they are different in our case.

**Proposition 3.2.** *For a choose parameter $\mu > 0$, and each $(y, z) \in I\!R^m \times I\!R^n$, we consider the point*

$$x_y^\mu = \arg\min_x \left\{ \left\| \frac{Zx}{\mu} - e \right\| : \; Ax + \mu y - b = 0, \; x \in I\!R^n \right\}.$$

*We have $x_y^\mu = \mu Z^{-1}e + \mu Z^{-2}A^T d_y^\mu$ where $d_y^\mu = -(\mu AZ^{-2}A^T)^{-1}(\mu AZ^{-1}e + \mu y - b)$.*

**Proof.** First, we remark that $d_y^\mu$ is well defined because $Z$ is positive definite and $A$ is of full rank. Which imply that $AZ^{-2}A^T$ is invertible. From other side, first-order optimality conditions of the problem

$$\min_x \left\{ \left\| \frac{Zx}{\mu} - e \right\| : \; Ax + \mu y - b = 0, \; x \in I\!R^n \right\}$$

give $Z\left(\frac{Zx_y^\mu}{\mu} - e\right) = A^T d_y^\mu$ with $d_y^\mu \in I\!R^n$, and we deduce that

$$x_y^\mu = \mu Z^{-1}e + \mu Z^{-2}A^T d_y^\mu.$$

Since $Ax_y^\mu + \mu y - b = 0$, we have

$$\mu AZ^{-1}e + \mu AZ^{-2}A^T d_y^\mu + \mu y - b = 0$$

also

$$d_y^\mu = -(\mu AZ^{-2}A^T)^{-1}(\mu AZ^{-1}e + \mu y - b).$$

□

We use the Euclidean norm of $\frac{Zx_y^\mu}{\mu} - e$ to measure the distance between the given point $y$ and $y(\mu)$ on the central trajectory. This norm is a dual version of the measure used by Roos and Vial [13].

We take this distance as measure noted $m(y, \mu)$ and we write

$$m(y, \mu) = \left\| \frac{Zx_y^\mu}{\mu} - e \right\|.$$

It is important to remark that $m(y(\mu), \mu) = 0$ and $y_{y(\mu)}^\mu = x(\mu)$.

Next theorem prove that if we take the direction $d_y^\mu$ at a point $y$ such that $m(y, \mu) < 1$, we converge quadratically to $y(\mu)$.

**Theorem 3.3.** *Let $\widehat{y} = y + d_y^\mu$ and $\widehat{z} = c - A^T\widehat{y}$, for $m(y, \mu) < 1$, we have the following results*

1. $\widehat{z} > 0$ *i.e* $\widehat{y}$ *is strictly feasible solution for* $(D)$.
2. $m(\widehat{y}, \mu) \leq m(y, \mu)^2$.

**Proof.** 1. We remark that,

$$\widehat{z} = z - Z\left(\frac{Zx_y^\mu}{\mu} - e\right) = 2z - \frac{Z^2 x_y^\mu}{\mu}$$

then $\widehat{z}_j = (2 - \frac{Zx_y^\mu}{\mu})_j z_j$ where $u_j$ is the $j$th component of vector $u$.

From other side, $m(y,\mu) < 1 \Rightarrow \|\frac{Zx_y^\mu}{\mu} - e\| < 1 \Rightarrow 0 < \left(\frac{Zx_y^\mu}{\mu}\right) < 2$.

Consequently, since $z_j > 0$, we deduce that $\widehat{z}_j > 0 \forall j$.

2. Let the vector $u = \frac{Zx_y^\mu}{\mu}$, and the definition of $x_y^\mu$ implies that

$$m(\widehat{y},\mu) = \left\|\frac{\widehat{Z}x_{\widehat{y}}^\mu}{\mu} - e\right\| \leq \left\|\frac{\widehat{Z}x_y^\mu}{\mu} - e\right\| = \|\widehat{Z}Z^{-1}u - e\|.$$

Furthermore $\widehat{Z}Z^{-1}u = (2Z - ZU)Z^{-1}u = 2u - Uu$.

Consequently

$$\begin{aligned}
&m(\widehat{y},\mu) \leq \|2u - Uu - e\| \Rightarrow m(\widehat{y},\mu)^2 \\
&\leq \sum_{j=1}^n (2u_j - u_j^2 - 1)^2 = \sum_{j=1}^n (u_j - 1)^4 \\
&\leq \left[\sum_{j=1}^n (u_j - 1)^2\right]^2 = \|u - e\|^4 \\
&= \|\frac{Zx_y^\mu}{\mu} - e\|^4 \\
&= m(y,\mu)^4.
\end{aligned}$$

Which completes the second statement of the proof. □

**Proposition 3.4.** *For $\widehat{u} = (1-\theta)\mu$, with $0 < \theta < 1$, we have*

$$m(y,\widehat{\mu}) \leq \frac{1}{1-\theta}\Big(m(y,\mu) + \theta\sqrt{n}\Big).$$

**Proof.** The definition of $x_y^\mu$, gives

$$m(y,\widehat{\mu}) = \left\|\frac{Zx_y^{\widehat{\mu}}}{\widehat{\mu}} - e\right\| \leq \left\|\frac{Zx_y^\mu}{\widehat{\mu}} - e\right\| = \left\|\frac{u}{1-\theta} - e\right\| \leq \frac{1}{1-\theta}(\|u - e\| + \theta\|e\|).$$

Adding the fact that $\|u - e\| = m(y,\mu)$ and $\|e\| = \sqrt{n}$, complete the result of proposition. □

**Theorem 3.5.** *Suppose that $m(y,\mu) \leq \frac{1}{2}$ and $\theta = \frac{1}{4\sqrt{n+2}}$, then $m(\widehat{y},\widehat{\mu}) \leq \frac{1}{2}$.*

**Proof.** From Proposition (3.4), we have

$$m(\widehat{y},\widehat{\mu}) \leq \tfrac{1}{1-\theta}\Big(m(\widehat{y},\mu)+\theta\sqrt{n}\Big).$$

And using Theorem (3.3), we obtain

$$m(\widehat{y},\widehat{\mu}) \leq \frac{1}{1-\theta}\Big(m(y,\mu)^2+\theta\sqrt{n}\Big).$$

Furthermore

$$m(\widehat{y},\widehat{\mu}) \leq \frac{1}{1-\theta}\Big(\frac{1}{4}+\theta\sqrt{n}\Big) \leq \frac{1+4\theta\sqrt{n}}{4(1-\theta)} = \frac{1}{2}.$$

From theorem (3.5), if we take $(\mu_k)_{k\in I\!N}$ such that

$$\mu_{k+1} = (1-\theta)\mu_k,\ \mu_0 > 0,$$

with $\theta = \frac{1}{4\sqrt{n}+2}$ and, starting from a strictly dual feasible point $y^0$ such

$$m(y^0,\mu_0) \leq \frac{1}{2}.$$

We can found a sequence of strictly dual feasible points $\{y^k\}$ such that

$$m(y^k,\mu_k) \leq \frac{1}{2}.$$

□

We define the pseudo-gap at point $\omega(\mu)$ by

$$\triangle(\mu) = |c^T x(\mu) - b^T y(\mu)| + \frac{1}{\mu}\|Ax(\mu)-b\|^2.$$

The next theorem gives an upper bound on the pseudo-gap calculated at $(x_y^\mu, y, z)$. This allows us to present an algorithm for the solution and to give evaluation its complexity.

**Theorem 3.6.** *Suppose that $m(y,\mu) \leq 1$, we have*

1. *$x_y^\mu$ is the primal $\mu-$feasible in relation to $y$.*
2. *The pseudo-gap $\triangle(x_y^\mu, y)$ calculated at $(x_y^\mu, y, z)$ verify*

$$\triangle(x_y^\mu, y) \leq \mu(\beta(y,\mu)\sqrt{n}), \quad \text{with} \ \ \beta = \sup(2\alpha, 2n).$$

**Proof.** 1. The definition of $x_y^\mu$, gives $Ax_y^\mu + \mu y - b = 0$. from other side,

$$\left\| \frac{Zx_y^\mu}{\mu} - e \right\| = m(y,\mu) < 1 \;\Rightarrow\; Zx_y^\mu > 0 \;\Rightarrow\; x_y^\mu > 0$$

since $z > 0$. Which prove the first statement of theorem (3.6).

2. We have

$$\begin{aligned} \triangle(x_y^\mu, y) &= |c^T x_y^\mu - b^T y| + \tfrac{1}{\mu}\|Ax_y^\mu - b\|^2 \\ &= |z^T x_y^\mu - \mu\|y(\mu)\|^2| + \mu\|y(\mu)\|^2. \end{aligned}$$

Then

$$\triangle(x_y^\mu, y) \;=\; \begin{cases} z^T x_y^\mu, & \text{if } z^T x_y^\mu \geq \mu\|y(\mu)\|^2 \\ \mu\left(2\|y(\mu)\|^2 - \frac{z^T x_y^\mu}{\mu}\right), & \text{else .} \end{cases}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\left| \frac{z^T x_y^\mu}{\mu} - n \right| \leq \left\| \frac{z^T x_y^\mu}{\mu} - e \right\| \|e\| = m(y,\mu)\sqrt{n}.$$

It follows that

$$\mu(n - m(y,\mu)\sqrt{n}) \leq z^T x_y^\mu \leq \mu(n + m(y,\mu)\sqrt{n}).$$

Consequently $z^T x_y^\mu \leq \mu(\beta - n + m(y,\mu))$ since $\beta \geq 2n$.

And whereas $\beta \geq 2\alpha$, we obtain

$$\begin{aligned} 2\mu\|y(\mu)\|^2 - z^T x_y^\mu - z^T x_y^\mu &\leq 2\mu\alpha - \mu(n - m(y,\mu)\sqrt{n}) \\ &\leq \mu(\beta - n + m(y,\mu)\sqrt{n}). \end{aligned}$$

This achieve the proof of theorem (3.6). □

## 4. Results and discussion

We establish the following algorithm for the solution.

### 4.1 Algorithm

**Initialization**

- Let $\mu_0 > 0$, $(y^0, z^0) \in I\!R^m \times I\!R^n$ with $m(y^0, \mu_0) \leq \frac{1}{2}$, and $\theta = \frac{1}{4\sqrt{n}+2}$.
- Define threshold accuracy $\epsilon > 0$ for the pseudo-gab.
- Set $k := 0$.

**Main step**

$y^k$, $z^k$ and $\mu_k$, are given at iteration $k$.

- If $(\beta - n)\mu_k < \epsilon$ calculate $x^k$ by
  $x^k = (Z^k)^{-2}A^T(A(Z^k)^{-2}A^T)^{-1}[b - \mu_k y^k - \mu_k A(Z^k)^{-1}e] + \mu_k(Z^k)^{-1}e$
- Else $(\beta - n)\mu_k \geq \epsilon$, then
  - Compute $d^k = \frac{1}{\mu_k}(A(Z^k)^{-2}A^T)^{-1}[b - \mu_k y^k - \mu_k A(Z^k)^{-1}e]$.
  - Let $y^{k+1} := y^k + d^k$ and $z^{k+1} = c - A^T y^{k+1}$.
  - Put $\mu_{k+1} := (1-\theta)\mu_k$ and $k := k+1$ and repeat the main step.

### 4.2 Convergence and complexity

The following theorem study the convergence of the previous algorithm and its complexity.

**Theorem 4.1.** *1. Let $v = -\ln\varepsilon$ and $v_0 = -\ln(\beta-n)\mu_0$, the algorithm stops after $(4\sqrt{n}+2)(v-v_0)$ steps.*

*2. The last generated points $(y,z)$ and $x_y^\mu$ are respectively strictly dual feasible and strictly primal $\mu-$feasible in relation to $y$.*

*Moreover, the pseudo-gap verify*

$$\triangle(x_y^\mu, y) \leq \Big(1 + \frac{1}{2\sqrt{n}}\Big)\varepsilon.$$

**Proof.** 1. For every iteration, the generated point $y$ by the algorithm will be strictly feasible for $(D)$, and $\mu > 0$, while $m(y,\mu) \leq \frac{1}{2}$, due to Theorem (3.3) and Theorem (3.5).

We have $\mu_k = (1-\theta)^k\mu_0$, and the algorithm stops if

$$(1-\theta)^k e^{-v_0} < e^{-v}. \tag{1}$$

Since $-\ln(1-\theta) > \theta$, Inequality (1) will hold if $k\theta > v - v_0$ which is equivalent to

$$k > (4\sqrt{n}+2)(v-v_0).$$

2. Theorem (3.6) implies that $x_y^\mu$ is primal $\mu-$feasible in relation to $y$ and

$$\begin{aligned}\triangle(x_y^\mu, y) &\leq \mu(\beta - n + m(y,\mu)\sqrt{n}) \leq \mu(\beta-n) + \mu m(y,\mu)\sqrt{n} \\ &\leq \varepsilon + \tfrac{1}{2}\tfrac{\beta-n}{\sqrt{n}}\mu \leq \Big(1+\tfrac{1}{2\sqrt{n}}\Big)\varepsilon.\end{aligned}$$

Which completes the proof. □

### 4.3 Discussion

1. The computational effort for every iteration requires $\mathcal{O}(n^3)$ arithmetic operations. Since the algorithm terminates at most in $\mathcal{O}(\sqrt{n}v)$ iterations, the complexity is $\mathcal{O}(n^{3.5}v)$.

2. To start the algorithm, we have assumed that a feasible solution $(y^0, z^0)$, such that $m(y^0,\mu_0) \leq \frac{1}{2}$, is available.

   In general, it is not the case. However, it is necessary to provide a means for determining an $(y^0,z^0) \in I\!R^m \times I\!R^n$ so that the algorithm can be initiated.

3. For the initialization of the algorithm, the method suggested by Monteiro and Adler in [11] can be easily adapted to our approach.

## 5. Conclusion

The problems $(P_\mu)$ are strictly convex and under some standard assumptions, have an optimal solution for all $\mu > 0$. The solution $\omega(\mu) = (x(\mu), y(\mu), z(\mu))$ of the system corresponding to the Karush-Kuhn-Tucker conditions gives a feasible dual solution and a primal $\mu$-feasible solution; moreover, every accumulation point of the sequence $\{\omega(\mu)\}$ is such that $x^*$ is an optimal solution of the primal problem and $(y^*, z^*)$ is an optimal solution of the dual problem. The presented algorithm is polynomial and its complexity is $\mathcal{O}(n^{3.5}L)$.

## Competing interests

The authors declare that they have no competing interests.

## Authors contributions

Both authors worked in coordination. Both authors carried out the proof, read and approved the final version of the manuscript.

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