Common fixed point theorems for generalized contraction conditions involving rational expressions in complex valued metric spaces

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Abstract. The aim of this paper is to prove common fixed point theorems for a pair of self maps satisfying general contraction conditions involving rational expressions having a point dependent control functions as coefficients in complex valued metric spaces. Our results extend and generalize the results of Nashine et.al., [5] and some known results in the literature. To show validity of our results some illustrative examples are also furnished.

Keywords: complex valued metric spaces, self maps, common fixed points, point dependent control functions.

1. Introduction

The benefit of metric spaces is a natural growth of functional analysis. Several researchers attempted various generalizations of this notion in recent past years such as generalized metric space, D-metric space, quasi-metric space, dislocated metric space, fuzzy metric space, ordered metric spaces *etc*.

Recently Azam et.al. [1] introduced a new notion namely complex valued metric space, which is one of the most attractive research in the field of fixed point theory and established sufficient conditions for the existence of common fixed points for a contractive condition involving rational expressions for a pair of self maps. The idea of complex metric space offers numerous research activities in mathematical analysis. Many researchers have contributed with different concepts in these spaces, for example we refer [2, 4, 5, 6, 7, 8, 9, 10].

We begin this section with notations and definitions in complex valued metric space due to Azam et al. [1] which are useful to prove our main results.

Let $\mathcal{C}$ be the set of complex numbers and $z_1,\ z_2 \in \mathcal{C}$. Define a partial order on C as follows:

$z_1 \preceq z_2$ if and only if $Re(z_1) \preceq Re(z_2)$ and $Im(z_1) \preceq Im(z_2)$.

Thus we can say that $z_1 \preceq z_2$ if one of the following condition holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;

(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we write $z_1 \preceq z_2$ if only (4) is satisfied. Note that $0 \preceq z_1 \preceq z_2$ implies $|z_1| \preceq |z_2|$.

Definition 1.1 (Azam et al. [1])**.** Let X be a nonempty set. A function $d : X \times X \to C$ is called a *complex valued metric on* X if for all $x, y, z \in X$, the following conditions are satisfied:

$(CM1)$ $0 \preceq d(x, y)$ and $d(x, y) = 0$ iff $x = y$;
$(CM2)$ $d(x, y) = d(y, x)$;
$(CM3)$ $d(x, y) \preceq d(x, z) + d(y, z)$

Then the pair (X, d) is called a complex valued metric space.

Example 1.2. Let $X = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$ and we define from $d : X \times X \to C$ by $d(z_1, z_2) = 2|x_1 - x_2| + 3i|y_1 - y_2|$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Clearly, (X, d) is a complex valued metric space.

Definition 1.3 (Azam et al. [1])**.** Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. We say that:

(i) the sequence $\{x_n\}$ *converges to* $x \in X$ if for each $c \in C$ with $0 \prec c$ there is a $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) the sequence $\{x_n\}$ *is a Cauchy sequence* if for each $c \in C$ with $0 \prec c$ there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, where $m \in N$.

(iii) (X, d) is *complete complex valued metric space* if every Cauchy sequence in X is convergent to a point in X.

Lemma 1.4 (Azam et al. [1])**.** *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Proposition 1.5 (Sithikul and Saejung [10])**.** *Let (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X and $h \in [0, 1)$. If $a_n = |d(x_n, x_{n+1})|$ satisfies $a_n \leq h\ a_{n-1}$ for all $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

In [3], Jaggi and Dass generalized Banach contraction principle using rational type of contraction.

Theorem 1.6 (Jaggi and Dass ([3])**.** *Let T be a selfmap of a metric space (X, d) which satisfy*:

(i) *for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, such that*

$$d(Tx, Ty) \leq \alpha \frac{d(x,\ Tx)d(y,\ Ty)}{d(x,\ Ty) + d(y,\ Tx) + d(x,\ y)} + \beta d(x, y), \tag{1.6.1}$$

for all $x,\ y \in X$, $x \neq y$;

(ii) there exists a point $x_0 \in X$ such that $\{T^n x_0\}$ has a convergent subsequence with limit z in X.

Then z is the unique fixed point of T in X.

Using the analog of condition (1.6.1) Nashine et al. [5] proved the following theorem in complex valued metric spaces.

Theorem 1.7 (Nashine et al. [5])**.** *Let (X, d) be a complete complex valued metric space and suppose that S and T be mappings on X satisfy either*
(i)

$$d(Sx, Ty) \precsim \alpha d(x, y) + \beta \frac{d(x, Sx)d(y, Ty)}{d(x, Ty) + d(y, Sx) + d(x, y)}, \tag{1.7.1}$$

where $\alpha + \beta < 1$ for all $x,\ y \in X,\ x \neq y$ and $d(x, Ty) + d(y, Sx) + d(x, y) \neq 0$ or

(ii) $d(Sx, Ty) = 0$ if $d(x, Ty) + d(y, Sx) + d(x, y) = 0$.

Then S and T have a unique common fixed point in X.

The following proposition was proved by Naval Singh et al. [6].

Proposition 1.8 (Naval Singh et. al. [6])**.** *Let (X, d) be a complex valued metric space and $S, T : X \to X$. Let $x_0 \in X$ and define a sequence $\{x_n\} \in X$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all $n = 0, 1, 2, 3, \ldots.$*

Assume that there exists a mapping $\alpha : X \times X \times X \to [0, 1)$ such that

$$\alpha(TSx, y, a) \leq \alpha(x, y, a) \text{ and } \alpha(x, STy, a) \leq \alpha(x, y, a),$$

for all $x, y \in X$ for any fixed $a \in X$ and $n = 0, 1, 2, 3, \ldots$.

Then $\alpha(x_{2n}, y, a) \leq \alpha(x_0, y, a)$ and $\alpha(x, x_{2n+1}, a) \leq \alpha(x, x_1, a)$.

The aim of this paper is to prove common fixed point theorems for a pair of self maps satisfying general contraction conditions involving rational expressions having a point dependent control functions as coefficients in complex valued metric spaces. Our results extend and generalize the results of Nashine et al. [5] (Theorem 2.1) and some known results in the literature. To show validity of our results some illustrative examples are also furnished.

2. Common fixed point theorems for a pair of maps using control functions

We start this section with our main result.

Theorem 2.1. *Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If there exist mappings $\alpha, \beta, \gamma, \delta$ and $\mu : X \times X \times X \to [0, 1)$ such that for all $x, y \in X$, for a fixed $a \in X$:*

(i) $\alpha(TSx, y, a) \leq \alpha(x, y, a)$ and $\alpha(x, STy, a) \leq \alpha(x, y, a)$;

(ii) $\beta(TSx, y, a) \leq \beta(x, y, a)$ and $\beta(x, STy, a) \leq \beta(x, y, a)$;

(iii) $\gamma(TSx, y, a) \leq \gamma(x, y, a)$ and $\gamma(x, STy, a) \leq \gamma(x, y, a)$;

(iv) $\delta(TSx, y, a) \leq \delta(x, y, a)$ and $\delta(x, STy, a) \leq \delta(x, y, a)$;

(v) $\mu(TSx, y, a) \leq \mu(x, y, a)$ and $\mu(x, STy, a) \leq \mu(x, y, a)$. *Also, suppose that S and T satisfy either*

$$
\begin{aligned}
(vi) \quad d(Sx, Ty) \preceq\ & \alpha(x, y, a) d(x, y) \\
& + \beta(x, y, a) \frac{d(x, Sx) d(y, Ty)}{d(x, Ty) + d(y, Sx) + d(x, y)} \\
& + \gamma(x, y, a) \frac{d(x, Ty) d(y, Sx)}{d(x, Sx) + d(y, Ty) + d(x, y)} \\
& + \delta(x, y, a) \frac{[d(x, Sx) d(x, Ty) + d(y, Ty) d(y, Sx)]}{d(x, Sx) + d(y, Ty) + d(x, y)} \\
& + \mu(x, y, a) \frac{[d(x, Sx) d(y, Sx) + d(y, Ty) d(x, Ty)]}{d(x, Sx) + d(y, Ty) + d(x, y)},
\end{aligned}
\tag{2.1.1}
$$

where $|\alpha(x, y, a) + \beta(x, y, a) + \gamma(x, y, a) + \delta(x, y, a) + \mu(x, y, a)| < 1$, $d(x, Sx) + d(y, Ty) + d(x, y) \neq 0$ *and* $d(x, Ty) + d(y, Sx) + d(x, y) \neq 0$ *or*

(vii) $d(Sx, Ty) = 0$ *if* $d(x, Sx) + d(y, Ty) + d(x, y) = 0$ *or* $d(x, Ty) + d(y, Sx) + d(x, y) = 0$.

Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\} \in X$ by $x_{2n+1} = Sx_{2n}$ and

$$x_{2n+2} = Tx_{2n+1}. \tag{2.1.2}$$

First we suppose that

$$d(x_{2k}, x_{2k+1}) + d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k}) = 0, \quad \text{for some } k, \tag{2.1.3}$$

therefore we have

$$x_{2k} = Sx_{2k} = Tx_{2k+1} = x_{2k+1} \tag{2.1.4}$$

hence there exists k_1 and k_2 such that

$$k_1 = Sk_1 = k_2 = Tk_2. \tag{2.1.5}$$

Thus from (2.1.5) it follows that k_1 is a common fixed point of S and T.

Now, we will prove that k_1 is a unique common fixed point of S and T. Suppose k_1 and k_2 be two common fixed points of S and T which implies $k_1 = Sk_1 = Tk_1$ and $k_2 = Tk_2 = Sk_2$.

Hence from (2.1.3), we have

$$d(k_1, k_2) + d(k_1, Tk_2) + d(k_2, Sk_1) = 0,$$

this implies $k_1 = k_2$.

Similarly proceeding on the same lines of the above argument S and T have a unique fixed point if $d(x_{2k},x_{2k+1})+d(x_{2k},Sx_{2k})+d(x_{2k+1},Tx_{2k+1})=0$. Thus we suppose that $d(x,Sx)+d(y,Ty)+d(x,y)\neq 0$ and $d(x,Ty)+d(y,Sx)+d(x,y)\neq 0$. First we show that $\{x_n\}$ is a Cauchy sequence in X. Now on using the inequality (2.1.1) with $x=x_0$ $y=x_1$, we have

$$\begin{aligned}d(x_1,x_2)=d(Sx_0,Tx_1)&\leq \alpha(x_0,x_1,a)d(x_0,x_1)\\&+\beta(x_0,x_1,a)\frac{d(x_0,x_1)d(x_1,x_2)}{d(x_0,x_1)+d(x_0,x_2)}\\&+\delta(x_0,x_1,a)\frac{d(x_0,x_1)d(x_0,x_2)}{d(x_0,x_1)+d(x_1,x_2)+d(x_0,x_1)}\\&+\mu(x_0,x_1,a)\frac{d(x_1,x_2)d(x_0,x_2)}{d(x_0,x_1)+d(x_1,x_2)+d(x_0,x_1)}\end{aligned}$$

so that

$$\begin{aligned}|d(x_1,x_2)|\leq \alpha(x_0,x_1,a)|d(x_0,x_1)|&+\beta(x_0,x_1,a)\frac{|d(x_0,x_1)||d(x_1,x_2)|}{|d(x_0,x_1)+d(x_0,x_2)|}\\&+\delta(x_0,x_1,a)\frac{|d(x_0,x_1)||d(x_0,x_2)|}{|d(x_0,x_1)+d(x_1,x_2)|}\\&+\mu(x_0,x_1,a)\frac{|d(x_1,x_2)||d(x_0,x_2)|}{|d(x_0,x_1)+d(x_1,x_2)|}.\end{aligned}\tag{2.1.6}$$

On using (CM3), we have

$$|d(x_0,x_1)+d(x_0,x_2)|\geq |d(x_1,x_2)|\tag{2.1.7}$$

and

$$|d(x_0,x_1)+d(x_1,x_2)|\geq |d(x_0,x_2)|.\tag{2.1.8}$$

Thus, from (2.1.6), (2.1.7) and (2.1.8), it follows that

$$|d(x_1,x_2)|\leq [\frac{\alpha(x_0,x_1,a)+\beta(x_0,x_1,a)+\delta(x_0,x_1,a)}{1-\mu(x_0,x_1,a)}]|d(x_0,x_1)|\tag{2.1.9}$$

Similarly, once again on using the inequality with $x=x_2, y=x_1$, we have $d(x_2,x_3)=d(STx_1,Tx_1)\leq \alpha(Tx_1,x_1,a)d(x_2,x_1)+\beta(Tx_1,x_1,a)\frac{d(x_3,x_2)d(x_1,x_2)}{d(x_1,x_3)+d(x_1,x_2)}+\delta(Tx_1,x_1,a)\frac{d(x_1,x_2)d(x_1,x_3)}{d(x_2,x_3)+d(x_1,x_2)+d(x_1,x_2)}+\mu(Tx_1,x_1,a)\frac{d(x_2,x_3)d(x_1,x_3)}{d(x_2,x_3)+d(x_1,x_2)+d(x_1,x_2)}$. Thus

$$\begin{aligned}|d(x_2,x_3)|\leq \alpha(x_0,x_1,a)|d(x_2,x_1)|&+\beta(x_0,x_1,a)\frac{|d(x_3,x_2)||d(x_1,x_2)|}{|d(x_1,x_3)+d(x_1,x_2)|}\\&+\delta(x_0,x_1,a)\frac{|d(x_1,x_2)||d(x_1,x_3)|}{|d(x_2,x_3)+d(x_1,x_2)|}\\&+\mu(x_0,x_1,a)\frac{|d(x_2,x_3)||d(x_1,x_3)|}{|d(x_2,x_3)+d(x_1,x_2)|}.\end{aligned}\tag{2.1.10}$$

Again using triangle inequality, we have

$$|d(x_1,x_3)+d(x_1,x_2)| \geq |d(x_2,x_3)| \tag{2.1.11}$$

and

$$|d(x_2,x_3)+d(x_1,x_2)| \geq |d(x_1,x_3)|. \tag{2.1.12}$$

Thus, from (2.1.10), (2.1.11) and (2.1.12), it follows that

$$|d(x_3,x_2)| \leq [\frac{\alpha(x_0,x_1,a)+\beta(x_0,x_1,a)+\delta(x_0,x_1,a)}{1-\mu(x_0,x_1,a)}]|d(x_2,x_1)|. \tag{2.1.13}$$

Hence from (2.1.9) , (2.1.13) and Proposition 1.8, we can conclude that

$$|d(x_n,x_{n+1})| \leq p|d(x_n,x_{n-1})|,$$

for all $n=1,2,3,....$ and $p=\frac{\alpha(x_0,x_1,a)+\beta(x_0,x_1,a)+\delta(x_0,x_1,a)}{1-\mu(x_0,x_1,a)}<1$.

Hence

$$|d(x_n,x_{n+1})| \leq p^n|d(x_0,x_1)| \to 0 \text{ as } n\to\infty. \tag{2.1.14}$$

Thus, by Proposition 1.5, it follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exists $u\in X$ such that $\lim_{n\to\infty} x_n=u$. Therefore

$$\lim_{n\to\infty} x_{2n}=\lim_{n\to\infty} Sx_{2n}=\lim_{n\to\infty} Tx_{2n+1}=u. \tag{2.1.15}$$

We now show that u is a fixed point of S. On the contrary $u\neq Su$. In view of inequality (2.1.1) and proposition 1.8, we have

$$\begin{aligned}
d(u,Su) &\leq d(u,x_{2n+2})+d(x_{2n+1},Su)\leq d(u,x_{2n+2})+d(Tx_{2n+1},Su)\\
&\leq d(u,x_{2n+2})+\alpha(u,x_{2n+1},a)d(u,x_{2n+1})\\
&+\beta(u,x_{2n+1},a)\frac{d(u,Su)d(x_{2n+1},Tx_{2n+1})}{d(u,Tx_{2n+1})+d(x_{2n+1},Su)+d(x_{2n+1},u)}\\
&+\gamma(u,x_{2n+1},a)\frac{d(u,Tx_{2n+1})d(x_{2n+1},Su)}{d(x_{2n+1},Tx_{2n+1})+d(u,Su)+d(x_{2n+1},u)}\\
&+\delta(u,x_{2n+1},a)\frac{d(u,Tx_{2n+1})d(u,Su)+d(Tx_{2n+1},x_{2n+1})d(x_{2n+1},Su)}{d(x_{2n+1},Tx_{2n+1})+d(u,Su)+d(x_{2n+1},u)}\\
&+\mu(u,x_{2n+1},a)\frac{d(x_{2n+1},Su)d(u,Su)+d(Tx_{2n+1},x_{2n+1})d(u,Tx_{2n+1})}{d(x_{2n+1},Tx_{2n+1})+d(u,Su)+d(x_{2n+1},u)}.\\
&\leq d(u,x_{2n+2})+\alpha(u,x_1,a)d(u,x_{2n+1})\\
&+\beta(u,x_1,a)\frac{d(u,Su)d(x_{2n+1},Tx_{2n+1})}{d(u,Tx_{2n+1})+d(x_{2n+1},Su)+d(x_{2n+1},u)}\\
&+\gamma(u,x_1,a)\frac{d(u,Tx_{2n+1})d(x_{2n+1},Su)}{d(x_{2n+1},Tx_{2n+1})+d(u,Su)+d(x_{2n+1},u)}\\
&+\delta(u,x_1,a)\frac{d(u,Tx_{2n+1})d(u,Su)+d(Tx_{2n+1},x_{2n+1})d(x_{2n+1},Su)}{d(x_{2n+1},Tx_{2n+1})+d(u,Su)+d(x_{2n+1},u)}\\
&+\mu(u,x_1,a)\frac{d(x_{2n+1},Su)d(u,Su)+d(Tx_{2n+1},x_{2n+1})d(u,Tx_{2n+1})}{d(x_{2n+1},Tx_{2n+1})+d(u,Su)+d(x_{2n+1},u)}.
\end{aligned}$$

Letting $n \to \infty$ and from (2.1.15), gives rise to

$$|d(u, Su)| \leq |\mu(u, x_1, a)||d(u, Su)| < |d(u, Su)|.$$

Hence $u = Su$. We now show that u is a fixed point of T. Suppose that $u \neq Tu$. Now on utilizing the inequality (2.1.1) and Proposition (1.8), we have

$$\begin{aligned}
d(u, Tu) &\leq d(u, x_{2n+1}) + d(x_{2n+1}, Tu) \leq d(u, x_{2n+1}) + d(Sx_{2n}, Tu) \\
&\leq d(u, x_{2n+1}) + \alpha(x_{2n}, u, a)d(u, x_{2n}) \\
&+ \beta(x_{2n}, u, a)\frac{d(u, Tu)d(x_{2n}, Sx_{2n})}{d(u, Sx_{2n}) + d(x_{2n}, Tu) + d(x_{2n}, u)} \\
&+ \gamma(x_{2n}, u, a)\frac{d(u, Sx_{2n})d(x_{2n}, Tu)}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&+ \delta(x_{2n}, u, a)\frac{d(u, Sx_{2n})d(u, Tu) + d(x_{2n}, Tu)d(x_{2n}, Sx_{2n})}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&+ \mu(x_{2n}, u, a)\frac{d(x_{2n}, Sx_{2n})d(Sx_{2n}, u) + d(x_{2n}, Tu)d(u, Tu)}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&\leq d(u, x_{2n+1}) + \alpha(x_0, u, a)d(u, x_{2n}) \\
&+ \beta(x_0, u, a)\frac{d(u, Tu)d(x_{2n}, Sx_{2n})}{d(u, Sx_{2n}) + d(x_{2n}, Tu) + d(x_{2n}, u)} \\
&+ \gamma(x_0, u, a)\frac{d(u, Sx_{2n})d(x_{2n}, Tu)}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&+ \delta(x_0, u, a)\frac{d(u, Sx_{2n})d(u, Tu) + d(x_{2n}, Tu)d(Sx_{2n}, Sx_{2n})}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)} \\
&+ \mu(x_0, u, a)\frac{d(x_{2n}, Sx_{2n})d(Sx_{2n}, u) + d(x_{2n}, Tu)d(u, Tu)}{d(x_{2n}, Sx_{2n}) + d(u, Tu) + d(x_{2n}, u)},
\end{aligned}$$

which on making $n \to \infty$, using (2.1.15), we get

$$|d(u, Tu)| \leq |\mu(x_0, u, a)||d(u, Tu)|.$$

Hence, we find that $u = Tu$. Accordingly, we conclude that u is a common fixed point of S and T. To prove uniqueness let u and $u*$ be two distinct fixed points of S and T. This implies $d(Su*, u) + d(u*, Tu) + d(u, u*) \neq 0$ and $d(Su, u) + d(u*, Tu*) + d(u, u*) \neq 0$.

$$\begin{aligned}
d(u, u*) = d(Su, Tu*) &\leq \alpha(u, u*, a)d(u, u*) \\
&+ \beta(u, u*, a)\frac{d(u, Su)d(u*, Tu*)}{d(u, Tu*) + d(u*, Su) + d(u, u*)} \\
&+ \gamma(u, u*, a)\frac{d(u, Tu*)d(u*, Su)}{d(u, Su) + d(u*, Tu*) + d(u, u*)}
\end{aligned}$$

$$
\begin{aligned}
&+\delta(u,u*,a)\frac{[d(u,Su)d(u,Tu*)+d(u*,Tu*)d(u*,Su)]}{d(u,Su)+d(u*,Tu*)+d(u,u*)}\\
&+\mu(u,u*,a)\frac{[d(u,Su)d(u*,Su)+d(u*,Tu*)d(u,Tu*)]}{d(u,Su)+d(u*,Tu*)+d(u,u*)}\\
&\leq[\alpha(u,u*,a)+\gamma(u,u*,a)]d(u,u*),
\end{aligned}
$$

a contradiction. Thus u is a common fixed point of S and T. This concludes the proof of this theorem.

Theorem 2.2. *If $\{S_i\}_1^m$ and $\{T_i\}_1^n$ are two finite pairwise commuting finite families of self maps defined on a complete complex valued metric space such that the mappings S and T satisfy condition* (2.1.1)*, then two families of maps $\{S_i\}_1^m$ and $\{T_i\}_1^n$ have a unique common fixed point in X.*

Proof. In view of Theorem 2.1, we hypothesize that S and T have a unique common fixed point z in X *i.e.*, $Sz = Tz = z$. By pairwise commutative property of $\{S_i\}_1^m$ and $\{T_i\}_1^n$ for every $(1 \leq p \leq n)$, we have

$$S_pz = S_pSz = S(S_pz)$$

and

$$S_pz = S_pTz = T(S_pz)$$

which implies that S_pz is a common fixed point of S and T. On using uniqueness of fixed point of S and T, we have $S_pz = z$, for all $(1 \leq p \leq n)$. Hence it follows that z is a common fixed point of the family $\{S_i\}_1^m$.

Similar to the above argument we can show that z is a common fixed point of the family $\{T_i\}_1^n$.

Thus the conclusion of this theorem follows.

In the following we prove similar type of results for different rational expressions.

Theorem 2.3. *Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If there exist mappings $\alpha, \beta, \gamma, \delta$ and $\mu : X \times X \times X \to [0, 1)$ such that for all $x, y \in X$, for a fixed $a \in X$:*

(i) $\alpha(TSx, y, a) \leq \alpha(x, y, a)$ *and* $\alpha(x, STy, a) \leq \alpha(x, y, a)$*;*

(ii) $\beta(TSx, y, a) \leq \beta(x, y, a)$ *and* $\beta(x, STy, a) \leq \beta(x, y, a)$*;*

(iii) $\gamma(TSx, y, a) \leq \gamma(x, y, a)$ *and* $\gamma(x, STy, a) \leq \gamma(x, y, a)$.

Also, suppose that S and T satisfy either

(iv)

$$
\begin{aligned}
d(Sx,Ty) &\preceq \alpha(x,y,a)d(x,y)+\beta(x,y,a)\frac{d(x,Sx)d^2(x,Ty)+d(y,Ty)d^2(y,Sx)}{d^2(x,Ty)+d^2(y,Sx)}\\
&+\gamma(x,y,a)\frac{d(x,Sx)d(x,Ty)+d^2(x,y)+d(x,Sx)d(x,y)}{d(x,Sx)+d(x,Ty)+d(x,y)}, \qquad (2.3.1)
\end{aligned}
$$

where $|\alpha(x,y,a)+\beta(x,y,a)+\gamma(x,y,a)| < 1$ for all $x, y \in X$, $d(x,Sx)+d(x,Ty)+d(x,y) \neq 0$ and $d^2(x,Ty)+d^2(y,Sx) \neq 0$ or

(v) $d(Sx,Ty) = 0$ *if* $d(x,Sx)+d(x,Ty)+d(x,y) = 0$ or $d^2(x,Ty)+d^2(y,Sx) = 0$. Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. We define a sequence $\{x_n\} \in X$ by

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}. \tag{2.3.2}$$

First we suppose that

$$d^2(x_{2k}, Tx_{2k+1}) + d^2(x_{2k+1}, Sx_{2k}) = 0 \quad \text{for some} \quad k, \tag{2.3.3}$$

then we have

$$x_{2k} = Sx_{2k} = Tx_{2k+1} = x_{2k+1}, \tag{2.3.4}$$

therefore exists k_1 and k_2 such that

$$k_1 = Sk_1 = k_2 = Tk_2. \tag{2.3.5}$$

Thus from (2.3.5) it follows that k_1 is a common fixed point of S and T.

We now prove that k_1 is a unique common fixed point of S and T. Suppose k_1 and k_2 be two common fixed points of S and T. This implies $k_1 = Sk_1 = Tk_1$ and $k_2 = Tk_2 = Sk_2$. Hence from (2.3.3), we have

$$d^2(k_1, k_2) + d^2(k_1, k_2) = 0,$$

this implies $k_1 = k_2$. Similarly proceeding on the same lines of the above argument S and T have a unique fixed point if $d(x_{2k}, x_{2k+1}) + d(x_{2k}, Sx_{2k}) + d(x_{2k}, Tx_{2k+1}) = 0$. Thus we suppose that $d(x,Sx)+d(x,Ty)+d(x,y) \neq 0$ and $d^2(x,Ty)+d^2(y,Sx) \neq 0$.

First we show that $\{x_n\}$ is a Cauchy sequence in X. On using the inequality (2.3.1) with $x = x_0$ $y = x_1$, we have

$$\begin{aligned}
d(x_1,x_2) = d(Sx_0,Tx_1) &\leq \alpha(x_0,x_1,a)d(x_0,x_1) \\
&+ \beta(x_0,x_1,a)\frac{d(x_0,Sx_0)d^2d(x_0,Tx_1)+d(x_1,Tx_1)d^2d(x_1,Sx_0)}{d^2(x_0,Tx_1)+d^2(x_1,Sx_0)} \\
&+ \gamma(x_0,x_1,a)\frac{d(x_0,Sx_0)d(x_0,Tx_1)+d^2(x_0,x_1)+d(x_0,Sx_0)d(x_0,x_1)}{d(x_0,Sx_0)+d(x_0,Tx_1)+d(x_0,x_1)} \\
&\leq \alpha(x_0,x_1,a)d(x_0,x_1) + \beta(x_0,x_1,a)\frac{d(x_0,x_1)d^2d(x_0,x_2)}{d^2(x_0,x_2)} \\
&+ \gamma(x_0,x_1,a)\frac{d(x_0,x_1)d(x_0,x_2)+d^2(x_0,x_1)+d(x_0,x_1)d(x_0,x_1)}{d(x_0,x_1)+d(x_0,x_1)+d(x_0,x_2)} \\
&\leq [\alpha(x_0,x_1,a)+\beta(x_0,x_1,a)\gamma(x_0,x_1,a)]d(x_0,x_1). \qquad (2.3.6)
\end{aligned}$$

Again on using the inequality (2.3.1) with $x = x_1$ $y = x_2$, we have

$$
\begin{aligned}
& d(x_2, x_3) = d(Sx_2, Tx_1) \le \alpha(Tx_1, x_1, a)d(x_2, x_1) \\
& + \beta(Tx_1, x_1, a)\frac{d(x_2, Sx_2)d^2 d(x_2, Tx_1) + d(x_1, Tx_1)d^2(x_1, Sx_2)}{d^2 d(x_2, Tx_1) + d^2(x_1, Sx_2)} \\
& + \gamma(Tx_1, x_1, a)\frac{d(x_2, Sx_2)d(x_2, Tx_1) + d^2(x_2, x_1) + d(x_2, Sx_2)d(x_1, x_2)}{d(x_2, Sx_2) + d(x_2, Tx_1) + d(x_2, x_1)} \\
& \le [\alpha(x_0, x_1, a) + \beta(x_0, x_1, a) + \gamma(x_0, x_1, a)]d(x_2, x_1) \qquad (2.3.7)
\end{aligned}
$$

Thus, from (2.3.6), (2.3.7)and Proposition 1.8 , it follows that

$$|d(x_n, x_{n+1})| \le p^n |d(x_0, x_1)|, \quad \text{for} \quad \text{all } n \in N,$$

where $p = |\alpha(x_0, x_1, a) + \beta(x_0, x_1, a) + \gamma(x_0, x_1, a)| < 1$.

Thus, by Proposition 1.5, it follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Therefore

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = u. \tag{2.3.8}$$

We now show that u is a fixed point of S. On the contrary $u \neq Su$. In view of inequality (2.3.1) and Proposition (1.8), we have

$$
\begin{aligned}
& d(u, Su) \le d(u, x_{2n+2}) + d(x_{2n+2}, Su) \le d(u, x_{2n+2}) + d(Tx_{2n+1}, Su) \\
& \le d(u, x_{2n+2}) + \alpha(u, x_{2n+1}, a)d(u, x_{2n+1}) \\
& + \beta(u, x_{2n+1}, a)\frac{d(u, Su)d^2(u, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})d^2(x_{2n+1}, Su)}{d^2(u, Tx_{2n+1}) + d^2(x_{2n+1}, Su)} \\
& + \gamma(u, x_{2n+1}, a)\frac{d(u, Su)d(u, Tx_{2n+1}) + d^2(u, x_{2n+1}) + d(x_{2n+1}, u)d(u, Su)}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)} \\
& \le d(u, x_{2n+2}) + \alpha(u, x_1, a)d(u, x_{2n+1}) \\
& + \beta(u, x_1, a)\frac{d(u, Su)d^2(u, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})d^2(x_{2n+1}, Su)}{d^2(u, Tx_{2n+1}) + d^2(x_{2n+1}, Su)} \\
& + \gamma(u, x_1, a)\frac{d(u, Su)d(u, Tx_{2n+1}) + d^2(u, x_{2n+1}) + d(u, Su)d(u, x_{2n+1})}{d(x_{2n+1}, Tx_{2n+1}) + d(u, Su) + d(x_{2n+1}, u)}.
\end{aligned}
$$

Letting $n \to \infty$ and in view of condition (2.3.8), it follows that $|d(u, Su)| \le 0$. Hence $u = Su$. We now show that u is a fixed point of T. Suppose that $u \neq Tu$.

Now on utilizing the inequality (2.3.1) and Proposition (1.8), we have

$$
\begin{aligned}
& d(u,Tu) \leq d(u,x_{2n+1}) + d(x_{2n+1},Tu) \leq d(u,x_{2n+1}) + d(Sx_{2n},Tu) \\
& \leq d(u,x_{2n+1}) + \alpha(x_{2n},u,a)d(u,x_{2n}) \\
& + \beta(x_{2n},u,a)\frac{d(x_{2n},Sx_{2n})d^2(x_{2n},Tu) + d(u,Tu)d^2(u,Sx_{2n})}{d^2(x_{2n},Tu) + d^2(u,Sx_{2n})} \\
& + \gamma(x_{2n},u,a)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu) + d^2(u,x_{2n}) + d(x_{2n},Sx_{2n})d(x_{2n},u)}{d(x_{2n},Sx_{2n}) + d(x_{2n},Tu) + d(x_{2n},u)} \\
& \leq d(u,x_{2n+1}) + \alpha(x_0,u,a)d(u,x_{2n+1}) \\
& + \beta(x_0,u,a)\frac{d(x_{2n},Sx_{2n})d^2(x_{2n+1},Tu) + d(u,Tu)d^2(u,Sx_{2n+1})}{d^2(x_{2n},Tu) + d^2(u,Sx_{2n})} \\
& + \gamma(x_0,u,a)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu) + d^2(u,x_{2n}) + d(x_{2n},Sx_{2n})d(x_{2n},u)}{d(x_{2n},Sx_{2n}) + d(x_{2n},Tu) + d(x_{2n},u)}.
\end{aligned}
$$

On taking limits $n \to \infty$ and using (2.3.8), it follows that $|d(u,Tu)| \leq 0$. Hence u is a common fixed point of S and T. Uniqueness of the fixed point trivially follows from (2.3.1). This completes the proof of the theorem.

3. Corollaries and examples

Setting $S = T$ in Theorem 2.1, we have the following corollary.

Corollary 3.1. *Let (X,d) be a complete complex valued metric space and $T : X \to X$. If there exist mappings $\alpha, \beta, \gamma, \delta$ and $\mu : X \times X \times X \to [0,1)$ such that for all $x,y \in X$, for a fixed $a \in X$:*

(i) $\alpha(Tx,y,a) \leq \alpha(x,y,a)$ and $\alpha(x,Ty,a) \leq \alpha(x,y,a)$;
(ii) $\beta(Tx,y,a) \leq \beta(x,y,a)$ and $\beta(x,Ty,a) \leq \beta(x,y,a)$;
(iii) $\gamma(Tx,y,a) \leq \gamma(x,y,a)$ and $\gamma(x,Ty,a) \leq \gamma(x,y,a)$;
(iv) $\delta(Tx,y,a) \leq \delta(x,y,a)$ and $\delta(x,Ty,a) \leq \delta(x,y,a)$.
(v) $\mu(Tx,y,a) \leq \mu(x,y,a)$ and $\mu(x,Ty,a) \leq \mu(x,y,a)$.
Also, suppose that T satisfy either
(vi)

$$
\begin{aligned}
d(Tx,Ty) \preceq\ & \alpha(x,y,a)d(x,y) + \beta(x,y,a)\frac{d(x,Tx)d(y,Ty)}{d(x,Ty) + d(y,Tx) + d(x,y)} \\
& + \gamma(x,y,a)\frac{d(x,Ty)d(y,Tx)}{d(x,Tx) + d(y,Ty) + d(x,y)} \\
& + \delta(x,y,a)\frac{[d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)]}{d(x,Tx) + d(y,Ty) + d(x,y)} \\
& + \mu(x,y,a)\frac{[d(x,Tx)d(y,Tx) + d(y,Ty)d(x,Ty)]}{d(x,Tx) + d(y,Ty) + d(x,y)},
\end{aligned}
\tag{3.1.1}
$$

where $|\alpha(x,y,a) + \beta(x,y,a) + \gamma(x,y,a) + \delta(x,y,a) + \mu(x,y,a)| < 1$, $d(x,Tx) + d(y,Ty) + d(x,y) \neq 0$ and $d(x,Ty) + d(y,Tx) + d(x,y) \neq 0$ or *(vi)* $d(Tx, Ty) = 0$ *if* $d(x, Tx) + d(y, Ty) + d(x, y) = 0$ *or* $d(x, Ty) + d(y, Tx) + d(x, y) = 0$.

Then T *has a unique common fixed point in* X.

If $\mu = 0$ *in Theorem 2.1, we have the following corollary.*

Corollary 3.2. *Let* (X, d) *be a complete complex valued metric space and* $S, T : X \to X$. *If there exist mappings* α, β, γ *and* $\delta : X \times X \times X \to [0, 1)$ *such that for all* $x, y \in X$, *for a fixed* $a \in X$:

(i) $\alpha(TSx, y, a) \leq \alpha(x, y, a)$ *and* $\alpha(x, STy, a) \leq \alpha(x, y, a)$;

(ii) $\beta(TSx, y, a) \leq \beta(x, y, a)$ *and* $\beta(x, STy, a) \leq \beta(x, y, a)$;

(iii) $\gamma(TSx, y, a) \leq \gamma(x, y, a)$ *and* $\gamma(x, STy, a) \leq \gamma(x, y, a)$;

(iv) $\delta(TSx, y, a) \leq \delta(x, y, a)$ *and* $\delta(x, STy, a) \leq \delta(x, y, a)$.

Also, suppose that S *and* T *satisfy either*

(v)

$$\begin{aligned} d(Sx, Ty) &\preceq \alpha(x, y, a)d(x, y) \\ &+ \beta(x, y, a)\frac{d(x, Sx)d(y, Ty)}{d(x, Ty) + d(y, Sx) + d(x, y)} \\ &+ \gamma(x, y, a)\frac{d(x, Ty)d(y, Sx)}{d(x, Sx) + d(y, Ty) + d(x, y)} \\ &+ \delta(x, y, a)\frac{[d(x, Sx)d(x, Ty) + d(y, Ty)d(y, Sx)]}{d(x, Sx) + d(y, Ty) + d(x, y)}, \end{aligned} \tag{3.2.1}$$

where $|\alpha(x, y, a) + \beta(x, y, a) + \gamma(x, y, a) + \delta(x, y, a)| < 1$, $d(x, Sx) + d(y, Ty) + d(x, y) \neq 0$ *and* $d(x, Ty) + d(y, Sx) + d(x, y) \neq 0$ *or*

(vi) $d(Sx, Ty) = 0$ *if* $d(x, Sx) + d(y, Ty) + d(x, y) = 0$ *or* $d(x, Ty) + d(y, Sx) + d(x, y) = 0$.

Then S *and* T *have a unique common fixed point in* X.

If $\mu = \delta = 0$, *in Theorem 2.1, we have the following corollary.*

Corollary 3.3. *Let* (X, d) *be a complete complex valued metric space and* $S, T : X \to X$. *If there exist mappings* α, β *and* $\gamma : X \times X \times X \to [0, 1)$ *such that for all* $x, y \in X$, *for a fixed* $a \in X$:

(i) $\alpha(TSx, y, a) \leq \alpha(x, y, a)$ *and* $\alpha(x, STy, a) \leq \alpha(x, y, a)$;

(ii) $\beta(TSx, y, a) \leq \beta(x, y, a)$ *and* $\beta(x, STy, a) \leq \beta(x, y, a)$;

(iii) $\gamma(TSx, y, a) \leq \gamma(x, y, a)$ *and* $\gamma(x, STy, a) \leq \gamma(x, y, a)$.

Also, suppose that S *and* T *satisfy either*

(iv)

$$\begin{aligned} d(Sx, Ty) \preceq \alpha(x, y, a)d(x, y) + \beta(x, y, a)&\frac{d(x, Sx)d(y, Ty)}{d(x, Ty) + d(y, Sx) + d(x, y)} \\ + \gamma(x, y, a)&\frac{d(x, Ty)d(y, Sx)}{d(x, Sx) + d(y, Ty) + d(x, y)}. \end{aligned} \tag{3.3.1}$$

where $|\alpha(x,y,a)+\beta(x,y,a)+\gamma(x,y,a)|<1$, $d(x,Sx)+d(y,Ty)+d(x,y)\neq 0$ *and* $d(x,Ty)+d(y,Sx)+d(x,y)\neq 0$ *or*

(v) $d(Sx,Ty)=0$ *if* $d(x,Sx)+d(y,Ty)+d(x,y)=0$ *or* $d(x,Ty)+d(y,Sx)+d(x,y)=0$.

Then S and T have a unique common fixed point in X.

On choosing $\mu=\delta=\gamma=0$, in Theorem 2.1, we get the following corollary.

Corollary 3.4. *Let (X,d) be a complete complex valued metric space and $S,T: X\to X$. If there exist mappings α and $\beta: X\times X\times X\to[0,1)$ such that for all $x,y\in X$, for a fixed $a\in X$;*

(i) $\alpha(TSx,y,a)\leq\alpha(x,y,a)$ *and* $\alpha(x,STy,a)\leq\alpha(x,y,a)$;

(ii) $\beta(TSx,y,a)\leq\beta(x,y,a)$ *and* $\beta(x,STy,a)\leq\beta(x,y,a)$.

Also, suppose that S and T satisfy either (iii)

$$d(Sx,Ty)\preceq\alpha(x,y,a)d(x,y)+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} \tag{3.4.1}$$

where $|\alpha(x,y,a)+\beta(x,y,a)|<1$ *and* $d(x,Ty)+d(y,Sx)+d(x,y)\neq 0$ *or*

(iv) $d(Sx,Ty)=0$ *if* $d(x,Ty)+d(y,Sx)+d(x,y)=0$.

Then S and T have a unique common fixed point in X.

Remark 3.5. Theorem 1.7 follows as a corollary to Corollary 3.4 if we choose $\alpha(x,y,a)=\alpha$ and $\beta(x,y,a)=\beta$ in Corollary 3.4.

The following examples corroborates the validity of our main results.

Example 3.6. Let $X=[0,1]$ and define $d: X\times X\to C$ by $d(x,y)=i|x-y|$, we define $S,T: X\to X$ by $Sx=\frac{x}{6}$ and $Tx=\frac{y}{6}$. We define $\alpha,\beta,\ \gamma,\delta$ and μ : $X\times X\times X\to[0,1)$ by $\alpha(x,y,a)=\frac{x}{6}+\frac{y}{8}+\frac{1}{3}$, $\beta(x,y,a)=\frac{x^2y^2a}{16}$, $\gamma(x,y,a)=\frac{xya}{16}$ $\delta(x,y,a)=\frac{xya}{8}$ and $\mu(x,y,a)=\frac{x^8}{8}+\frac{y^8}{8}+\frac{1}{5}$.

Clearly, $|\alpha(x,y,a)+\beta(x,y,a)+\gamma(x,y,a)+\delta(x,y,a)+\mu(x,y,a)|<1$ with $a=\frac{1}{3}$. Now consider $\alpha(TSx,y,a)=\alpha(\frac{x}{36},y,a)=\frac{x}{216}+\frac{y}{8}+\frac{1}{3}<\frac{x}{6}+\frac{y}{8}+\frac{1}{3}$ and $\alpha(x,STy,a)=\alpha(x,y,a)=x+\frac{y}{216}+\frac{1}{3}$.

We now show that the inequality (2.1.1) holds for $x,y\in X$. Consider,

$$\begin{aligned}
d(Sx,Ty)=d(\frac{x}{6},\frac{y}{6})=i\frac{|x-y|}{6}\leq i\frac{|x-y|}{3}\\
&\leq(\frac{x}{6}+\frac{y}{8}+\frac{1}{3})i|x-y|\leq\alpha(x,y,a)d(x,y)\\
&+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}\\
&+\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}\\
&+\delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)}\\
&+\mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}.
\end{aligned}$$

Therefore all the conditions of Theorem 2.1 are satisfied 0 is a unique fixed point of S and T.

Example 3.7. Let $X=\{2,3,4,5\}$ and we write $A=\{(2,3),(3,2),(2,5),(5,2),(5,3),(3,5),(4,3),(3,4),(4,5),(5,4)\}$, $B=\{(2,4),(4,2)\}$ and $C=\{(2,2),(3,3),(4,4),(5,5)\}$.

We define $d: X\times X\to C$ by $d(x,y)=\begin{cases}2e^{i\theta}, & \text{if } (x,y)\in A\\ e^{\theta}, & \text{if } (x,y)\in B\\ 0, & \text{if } (x,y)\in C\end{cases}$

Then (X,d) is a complex metric space for $\theta\in[0,\frac{\pi}{2}]$.

We now define S and $T: X\to X$ by

$Sx=\begin{cases}2, & \text{if } x\in\{2,4,5\}\\ 4, & \text{if } x=3\end{cases}$ and $Tx=\begin{cases}2, & \text{if } x\in\{2,3,4\}\\ 4, & \text{if } x=5.\end{cases}$

Let $\alpha(x,y,a)=\frac{1}{4},\beta(x,y,a)=0,\gamma(x,y,a)=\frac{1}{21},\delta(x,y,a)=\frac{1}{2}$ and $\mu(x,y,a)=\frac{1}{5}$.

Then clearly, $|\alpha(x,y,a)+\beta(x,y,a)+\gamma(x,y,a)+\delta(x,y,a)+\mu(x,y,a)|=.99<1$. Now we verify the inequality (2.1.1).

Case (i). If $(x,y)=\{(2,3),(2,4),(2,2),(3,5),(4,2),(5,2),(4,3),(4,4),(5,3),(5,4)\}$ then we have $d(Sx,Ty)=0$ so that the inequality (2.1.1) holds.

Case (ii). If $x=2$, $y=5$ then $d(Sx,Ty)=d(2,4)=e^{i\theta}\leq 1.123e^{i\theta}=\alpha(x,y,a)d(x,y)+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}+\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}+\delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)}+\mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$, so that the inequality (2.1.1) holds.

Case (iii). If $(x,y)=(3,3)$ then $d(Sx,Ty)=d(2,4)=e^{i\theta}\leq 1.44e^{i\theta}=\alpha(x,y,a)d(x,y)+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}+\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}+\delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)}+\mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$, so that the inequality (2.1.1) holds.

Case (iv). If $(x,y)=(3,4)$ then $d(Sx,Ty)=d(2,4)=e^{i\theta}\leq 1.64e^{i\theta}=\alpha(x,y,a)d(x,y)+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}+\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}+\delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)}+\mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$ so that the inequality (2.1.1) holds.

Case (v). If $(x,y)=(3,2)$ then $d(Sx,Ty)=d(2,4)=e^{i\theta}\leq 1.64e^{i\theta}=\alpha(x,y,a)d(x,y)+\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}+\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}+\delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)}+\mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$ so that the inequality (2.1.1) holds.

Case (vi). If $(x,y)=(4,5)$ then $d(Sx,Ty)=d(2,4)=e^{i\theta}\leq 1.29e^{i\theta}=\alpha(x,y,a)d(x,y)\ +\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}\ +\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}\ +\delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)}\ +\mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$ so that the inequality (2.1.1) holds.

Case (vii). If $(x,y)=(5,5)$ then $d(Sx,Ty)=d(2,4)=e^{i\theta}\leq 1.44e^{i\theta}=\alpha(x,y,a)d(x,y)\ +\beta(x,y,a)\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}\ +\gamma(x,y,a)\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)+d(x,y)}\ +\delta(x,y,a)\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Sx)+d(y,Ty)+d(x,y)}\ +\mu(x,y,a)\frac{[d(x,Sx)d(y,Sx)+d(y,Ty)d(x,Ty)]}{d(x,Sx)+d(y,Ty)+d(x,y)}$ so that the inequality (2.1.1) holds.

Thus all the hypotheses of Theorem 2.1 are verified. Here, 2 is the unique common fixed point of S and T.

Here we note that inequality (1.7.1) fails to hold when $x=y=3$ and for any $\alpha,\beta\in[0,1)$. Indeed, when $(x,y)=(3,3)$, we have $d(Sx,Ty)=d(2,4)=e^{i\theta}\not\lesssim\beta\frac{d(3,4)d(3,2)}{d(3,2)+d(3,4)}=\beta e^{i\theta}$.

Hence Theorem 1.7 is not applicable.

Example 3.8. Let $X=\{(1,2),(3,4),(5,6),(7,8)\}$ and we define from $d:X\times X\to C$ by $d(z_1,z_2)=2|x_1-x_2|+3i|y_1-y_2|$, where $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$.

Then (X,d) is a complex valued metric space. We now define Sz and Tz with $z=x+iy$ on X by

$$Tz=Sz=\begin{cases}|x-y|+2i|x-y|, & \text{if } z\in\{(1,2),(3,4),(5,6)\}\\ 3|x-y|+4i|x-y|, & \text{if } z=(7,8)\end{cases}$$

We define $\alpha(x,y,a)=\frac{1}{3},\beta(x,y,a)=\frac{1}{43},\gamma(x,y,a)=0,\delta(x,y,a)=\frac{1}{2}$ and $\mu(x,y,a)=\frac{1}{7}$.

Then clearly, $|\alpha(x,y,a)+\beta(x,y,a)+\gamma(x,y,a)+\delta(x,y,a)+\mu(x,y,a)|=0.96<1$. Now we verify the inequality (2.1.1).

Case (i). If $(z_1,z_2)=\{((1,2),(1,2)),((1,2),(3,4)),((1,2),(5,6)),((3,4),(1,2)),((3,4),(3,4)),((3,4),(5,6)),((5,6),(1,2)),((5,6),(3,4)),((5,6),(5,6)),((7,8),(7,8))\}$ then $d(Sz_1,Tz_2)=0$, so that the inequality (2.1.1) holds.

Case (ii). If $z_1=(3,4)$ and $z_2=(7,8)$, then $d(Sz_1,Tz_2)=4+6i\leq\frac{1}{3}(8+12i)+\frac{2}{215}(4+6i)+\frac{3}{5}(4+6i)+\frac{3}{35}(4+6i)=1.29(4+6i)$, so that inequality (2.1.1) holds.

Case (iii). If $z_1=(5,6)$ and $z_2=(7,8)$, then $d(Sz_1,Tz_2)=4+6i\leq\frac{1}{3}(4+6i)+\frac{1}{43}(4+6i)+\frac{4}{5}(4+6i)+\frac{8}{35}(4+6i)=1.3(4+6i)$, so that inequality (2.1.1) holds.

Case (iv). If $z_1=(1,2)$ and $z_2=(7,8)$, then $d(Sz_1,Tz_2)=4+6i\leq\frac{1}{3}3(4+6i)+(4+6i)+\frac{2}{21}(4+6i)=2.09(4+6i)$, so that inequality (2.1.1) holds.

Case (v). If $z_1 = (7,8)$ and $z_2 = (3,4)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{2}{3}(4+6i) + \frac{2}{215}(4+6i) + \frac{3}{5}(4+6i) + \frac{3}{35}(4+6i) = 1.29(4+6i)$, so that inequality (2.1.1) holds.

Case (vi). If $z_1 = (7,8)$ and $z_2 = (1,2)$, then $d(Sz_1, Tz_2) = 4 + 6i \leq \frac{1}{3}3(4+6i) + \frac{2}{5}(4+6i) + \frac{2}{35}(4+6i) = 1.44(4+6i)$ so that inequality (2.1.1) holds.

Case (vii). If $z_1 = (7,8)$ and $z_2 = (5,6)$, then $d(Sz_1, Tz_2) = 4 + 6i \leq \frac{1}{3}(4+6i) + \frac{2}{215}(4+6i) + \frac{4}{5}(4+6i) + \frac{8}{35}(4+6i) = 1.3(4+6i)$.

Thus all the hypotheses of Theorem 2.1 are verified. Also $(1,2)$ is the unique common fixed point of S and T.

Here we note that inequality (1.7.1) fails to hold when $z_1 = (5,6)$ and $z_2 = (7,8)$, for any α and $\beta \in [0,1)$ with $\alpha + \beta < 1$. Indeed, we have

$$d(Sz_1, Tz_2) = 4 + 6i \leq \alpha(4+6i) + \beta\frac{4}{5}(4+6i) = (\alpha+\beta)(4+6i) < (4+6i).$$

Thus Theorem 1.7 is not applicable.

Example 3.9. Let $X = \{(1,2),(3,4),(5,6),(7,8)\}$ and we define $d : X \times X \to C$, $T : X \to X$ and $S : X \to X$ as in Example 3.8.

We define $\alpha(x,y,a) = \frac{1}{3}, \beta(x,y,a) = \frac{1}{2}, \gamma(x,y,a) = \frac{1}{7}$.

Then clearly, $|\alpha(x,y,a) + \beta(x,y,a) + \gamma(x,y,a)| = .97 < 1$. Now we verify the inequality (2.3.1).

Case (i). If $(z_1, z_2) = \{((1,2),(1,2)),((1,2),(3,4)),((1,2),(5,6)),((3,4),(1,2)),\ ((3,4),(3,4)),((3,4),(5,6)),((5,6),(1,2)),((5,6),(3,4)),((5,6),(5,6)),((7,8),(7,8))\}$, then $d(Sz_1, Tz_2) = 0$, so that the inequality (2.3.1) holds.

Case (ii). If $z_1 = (3,4)$ and $z_2 = (7,8)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}(8+12i) + \frac{1}{2}(8+12i) + \frac{8+12i}{7} = 1.864(4+6i)$, so that inequality (2.3.1) holds.

Case (iii). If $z_1 = (5,6)$ and $z_2 = (7,8)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}(4+6i) + \frac{1}{2}2(4+6i) + \frac{1}{7}\frac{5}{4}(4+6i) = 1.5(4+6i)$, so that inequality (2.3.1) holds.

Case (iv). If $z_1 = (1,2)$ and $z_2 = (7,8)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}3(4+6i) + \frac{1}{4}\frac{9}{5}(4+6i) + \frac{1}{7}\frac{9}{4}(4+6i) = 1.77(4+6i)$, so that inequality (2.3.1) holds.

Case (v). If $z_1 = (7,8)$ and $z_2 = (3,4)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}(8+12i) + \frac{1}{2}(8+6i) + \frac{1}{7}(8+6i) = 1.8(4+6i)$, so that inequality (2.3.1) holds.

Case (vi). If $z_1 = (7,8)$ and $z_2 = (1,2)$, then $d(Sz_1, Tz_1) = 4 + 6i \leq \frac{1}{3}3(4+6i) + \frac{1}{2}\ \frac{8}{5}(4+6i) + \frac{1}{7}\frac{19}{7}(4+6i) = 2.18(4+6i)$ so that inequality (2.3.1) holds.

Case (vii). If $z_1 = (7,8)$ and $z_2 = (5,6)$, then $d(Sz_1, Tz_1) = 4+6i \leq \frac{1}{3}(4+6i) + \frac{2}{5}(4+6i) + \frac{1}{7}\frac{6}{4}(4+6i) = 1.5(4+6i)$, so that inequality (2.3.1) holds.

Thus all the hypotheses of Theorem 2.3 are verified.

Here $(1,2)$ is a unique common fixed point of S and T.

References

[1] A. Azam , B. Fisher , M. Khan , *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim., 32 (2011), 243-253.

[2] S. Bhatt , S. Chaukiyal , R.C. Dimri, *A common fixed point theorem for weakly compatible maps in complex valued metric spaces*, Int. J. Math. Sci. Appl., 3 (2011), 1385-1389

[3] D.S. Jaggi, B.K. Dass, *An extension of Banach's fixed point through rational expression*, Bull. Cal. Math. Soc., 72 (1980), 261-262.

[4] A. Jamshaid , K. Chakkrid , A. Azam, *Common fixed points for multivalued mappings in complex valued metric spaces with applications*, Abstr. Appl. Anal., 12 (2013), (Article ID 854965).

[5] H. K. Nashine, M. Imdad, M. Hasan, *Common fixed point theorems under rational contractions in complex valued metric spaces*, J. Nonlinear Sci. Appl., 7 (2014), 42-50.

[6] N. Singh, Deepak Singh, Anil Badal and V. Joshi, *Fixed point theorems in complex valued metric spaces*, J. Egy. Math. Soc., 24 (2016), 402-409

[7] F. Rouzkard , M. Imdad , *Some common fixed point theorems on complex valued metric spaces*, Comput. Math. Appl., 64 (2012), 1866-1874.

[8] M. Sarwar, M. Bahadur Zada, S. Manro, *Common fixed point theorems for weakly compatible mappings in complex valued metric spaces*, Bull. Internat. Math. Virtual Inst., 7 (2017), 11-21.

[9] W. Sintunavarat, P. Kumam, *Generalized common fixed point theorems in complex valued metric spaces and applications*, J. Inequal. Appl., 2012 (2012), 84 .

[10] K. Sitthikul, S. Saejung, *Some fixed point theorems in complex valued metric space*, Fixed Point Theory Appl., 189 (2012).

[11] W. Sintunavarat , J.C. Yeol, P. Kumam, *Urysohn integral equa- tions approach by common fixed points in complex-valued metric spaces*, Adv. Differ. Eqn., 49 (2013).

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