The non-coprime graph of a finite group with respect to a subgroup

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Abstract. The non-coprime graph $\Pi_{G,H}$ of a finite group G with respect to a subgroup H is a simple graph with $G \setminus \{e\}$ as the vertex set. Two vertices u and v of this graph are adjacent if and only if $(|u|, |v|) \neq 1$ and $(u \in H$ or $v \in H)$. In this paper, the main properties of this graph are obtained. The isomorphism problem of this graph is also investigated for nilpotent groups.

Keywords: non-coprime graph, nilpotent group, Mathieu group.

1. Introduction

Throughout this paper, we will consider only simple graphs which are undirected, without loops and multiple edges. For any graph Π, the sets of all vertices and edges of Π are denoted by $V(\Pi)$ and $E(\Pi)$, respectively. We also denote the order and the length of a shortest cycle in Π by $|V(\Pi)|$ and girth(Π), respectively. The last quantity is called the girth of Π.

The study of an algebraic structures is a part of algebraic graph theory. In this topic, the researchers associate a graph structure to an algebraic system

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and then study the relationship between them. In this paper, a graph structure $\Pi_{G,H}$ will be associated to a pair (G,H) of a group G and a subgroup H of G. Then we will prove that some important algebraic properties of this pair has a graph theory meaning. In an exact phrase, the non-coprime graph $\Pi_{G,H}$ of G with respected to H is a simple graph with $G \setminus \{e\}$ as its vertex set and two distinct vertices x and y are adjacent if and only if $(|x|,|y|) \neq 1$ and $(x \in H$ or $y \in H)$. If $G = H$ then we use the notation Π_G as $\Pi_{G,G}$. The last graph first studied by Ma, Wei and Yang. [4]. In the mentioned paper, it was proved that this graph has diameter ≤ 2 and its automorphism group is studied in some special cases that G is cyclic or a $p-$group. In [2], Dorbidi proved that the chromatic and clique numbers of this graph are equal. He also classified all the groups for which this graph is complete $r-$partite or planar.

The non-coprime graph of a finite group G is a simple graph Π_G with vertex set $G\setminus\{e\}$ such that two distinct vertices are adjacent whenever their orders are relatively non-coprime. This graph was introduced by Mansoori, Erfanian and Tolue [5]. In the mentioned paper, the authors computed some graph invariants like diameter, girth, dominating number, independence and chromatic numbers. They also proved that the non-coprime graph of a group G is planar if and only if $|G| < 7$. The non-coprime graph of a group G with respect to a subgroup H, $\Pi_{G,H}$, is a generalization of the non-coprime graph Π_G presented in [5].

For a graph Π and a subset S of the vertex set $V(\Pi)$, we denote $N_\Pi[S]$ to be the set of all vertices in Π which are in S or adjacent to a vertex in S. If $N_\Pi[S] = V(\Pi)$, then S is said to be a dominating set in Π. The domination number of Π, $\gamma(\Pi)$, is the minimum size of a dominating set in Π. A planar graph is a graph that can be embedded in the plane so that two edges intersect geometrically except at a vertex which both are incident. The Kuratowski's theorem states that a graph is planar if and only if it contains no subgraph isomorphic with a subdivision of K_5 or $K_{3,3}$. A clique in the graph Π is a complete subgraph of Π and the number of vertices in a largest clique of Π is called the clique number of Π, denoted by $\omega(\Pi)$. An independence set of Π is a subset of the vertex set of Π such that no two vertices are adjacent in Π. The independence number of Π, denoted by $\alpha(\Pi)$, is the cardinality of a largest independent set in Π. The minimum number $\chi(\Pi)$ of colors which can be assigned to the vertices of Π in such a way that every two adjacent vertices have different colors is said to be the chromatic number of Π. The covering number of Π, denoted by $\theta(\Pi)$, is the minimum number of the cliques of Π that covering the vertex set of Π.

In this paper, our notations are almost standard and taken mainly from [1, 3] and our aim is to extend some results of [6, 7] to non-coprime graph of a group with respect to a subgroup.

2. Non-coprime graph of Mathieu groups

In this section the non-coprime graph of all Mathieu groups are obtained. The number of elements which is divisible by a prime divisor p of a group G is denoted by $m_p(G)$.

We start by the Mathieu group M_{11}. We first note that this is a group of order $8920 = 2^4 \times 3^2 \times 5 \times 11$. Apply Gap [8] to calculate the orders of elements and the number of elements of each order in M_{11}. These numbers are:

$$[[1,1],[2,165],[3,440],[4,990],[5,1584],[6,1320],[8,1980],[11,1440]].$$

Our calculations show that this graph has exactly three components. The first and second components are complete graph of orders 1584 and 1440, respectively. The orders of elements in the third component are $2, 3, 4, 6$ and 8. By our calculations, $m_2 = 4455$, $m_3 = 1760$, $m_5 = 1584$ and $m_{11} = 1440$. It can be calculated by Gap that the diameter of the third connected components is 2. We record our calculations in the following proposition:

Proposition 2.1. *The non-coprime graph of the Mathieu group M_{11} has three components K_{1440}, K_{1584} and a subgraph of diameter 2 with elements of orders $2, 3, 4, 6, 8$. Moreover, $\Pi_{M_{11}}$ has minimum degree 1439, maximum degree 4894, clique number 4895 and $\chi(\Pi_{M_{11}}) = \omega(\Pi_{M_{11}}) = 4895$.*

The Mathieu group M_{12} has order $95040 = 2^6 \times 3^2 \times 5 \times 11$. Furthermore, the orders of elements and the number of elements of each order in the Mathieu group M_{12} are as follows $[[1,1],[2,891],[3,4400],[4,5940],[5,9504],[6,23760],$ $[8,23760],[10,9504],[11,17280]]$. Thus, the number of elements that are divisible by $2, 3, 5, 11$ are

$$[[2,63855],[3,28160],[5,19008],[11,17280]].$$

Hence this graph has two connected components which one of them is isomorphic to K_{17280} and the second one is a subgraph with elements of orders $2, 3, 4, 5, 6, 8, 10$. The elements of order 10 have maximum degree and the elements of order 11 have minimum degree. It is obvious that the diameter of the second component is 3, the clique number of this component is 63855 and since $17280 < 63855$, $\omega(\Pi_{M_{12}}) = 63855$. We record our calculations in the following proposition:

Proposition 2.2. *The non-coprime graph $\Pi_{M_{12}}$ has two connected components, $\delta(M_{12}) = 17279$, $\Delta(M_{12}) = 73359$, one of the component is isomorphic to K_{17280} and another component has diameter 3. Moreover, $\chi(\Pi_{M_{12}}) = \omega(\Pi_{M_{12}})$ $= 63855$.*

We are now ready to calculate the graph invariants related to the non-coprime graph of the Mathieu group M_{22} of order $443520 = 2^7 \times 3^2 \times 5 \times 7 \times 11$. The order of elements and the number of each element in M_{22} are as follows:

$[[1,1],[2,1155],[3,12320],[4,41580],[5,88704],[6,36960],[7,126720],[8,55440],$ $[11,80640]]$.

To do this, we first notice that the number of elements that can be divided by $2,3,5,11$ are

$$[[2,135135],[3,49280],[5,88704],[7,126720],[11,80640]].$$

So, this graph has four connected components K_{88704} , K_{126720} , K_{80640} and a subgraph with elements of orders $2,3,4,6,8$. The maximum degree are attained by elements of order 6 and the minimum degree of vertices are related to elements of order 11. It is obvious that the diameter of non-complete component is 2 and by [1, Theorem 1.7.2], this component is a line graph. Since the clique number of non-complete component of M_{12} is 135135 and by a simple calculation $\omega(\Pi_{M_{22}}) = 135135$. We record our calculations in the following proposition:

Proposition 2.3. *The non-coprime graph* $\Pi_{M_{22}}$ *has four connected components which three of them are complete graphs of orders* $88704, 126720, 80640$*, respectively. The last component has diameter four,* $\delta(M_{22}) = 49279$*,* $\Delta(M_{22}) = 147454$*, the fourth component has diameter* 3*,* $\chi(\Pi_{M_{22}}) = \omega(\Pi_{M_{22}}) = 135135$*.*

By [8], $\mid M_{23} \mid = 10200960 = 2^7 \times 3^2 \times 5 \times 7 \times 11 \times 23$. The orders of elements and the number of elements of each orders in the Mathieu group M_{23} are:

$$[[1,1],[2,3795],[3,56672],[4,318780],[5,680064],[6,850080],$$
$$[7,1457280],[8,1275120],[11,1854720],[14,1457280],[15,1360128],[23,887040]]$$

and number of elements that their orders are divisible by $2,3,5,7,11,23$ are

$$[[2,3905055],[3,2266880],[5,2040192],[7,2914560],[11,1854720],[23,887040]].$$

Therefore, this graph has three connected components $K_{1854720}$, K_{887040} and a subgraph with elements of orders $2,3,4,5,6,7,8,14,15$. Hence we prove the following proposition:

Proposition 2.4. *The non-coprime graph* $\Pi_{M_{23}}$ *has three component with minimum degree* 887039 *and maximum degree* 5362334*. This graph has two complete components* $K_{1854720}$ *and* K_{887040} *and the non-complete component has diameter* 3*. Moreover,* $\chi(\Pi_{M_{23}}) = \omega(\Pi_{M_{23}}) = 3905055$*.*

By similar calculations by Gap such as our calculations in Propositions 2.1, 2.2, 2.3 and 2.4, one can easily prove the following result:

Proposition 2.5. *The non-coprime graph* $\Pi_{M_{24}}$ *has three connected components. Two complete of these components are isomorphic to the complete graphs* $K_{21288960}$ *and* $K_{22256640}$ *and the third one is a non-complete graph of diameter* 3*. The minimum and maximum degree of vertices in this graph are* 21288959 *and* 247003646*, respectively. Moreover,* $\chi(\Pi_{M_{24}}) = \omega(\Pi_{M_{24}}) = 128866815$*.*

3. Non-coprime graphs of nilpotent groups

Suppose G is a finite group of order $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, where $p_i, 1 \leq i \leq r$, are distinct primes. The number of elements of orders divisible by $p_{i_1}, p_{i_2}, \dots, p_{i_t}$ which is not divisible by p_j, $1 \leq j \leq r$ and $j \neq i_1, i_2, \dots, i_t$, is denoted by $\Delta_G(p_{i_1}, p_{i_2}, \dots, p_{i_t}; p_j)$. The aim of this section is to study the main properties of the non-coprime graph. We first prove a result for nilpotent groups which is crucial throughout this paper.

Theorem 3.1. *Let G be a nilpotent group of order $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, where $p_i, 1 \leq i \leq r$, be distinct primes. Then $\Delta_G(p_{i_1}, p_{i_2}, \dots, p_{i_t}; p_j) = \Delta_{Z_n}(p_{i_1}, p_{i_2}, \dots, p_{i_t}; p_j)$.*

Proof. Suppose G is a nilpotent group, P_{i_1}, P_{i_2}, $\dots$, P_{i_t} are Sylow subgroups of G and $H = P_{i_1} P_{i_2} \dots P_{i_t}$. It is enough to prove that $x \in H \setminus (P_{i_2} P_{i_3} \dots P_{i_t} \cup P_{i_1} P_{i_3} \dots P_{i_t} \cup \dots \cup P_{i_1} P_{i_2} \dots P_{i_{t-1}})$ if and only if $p_{i_1} p_{i_2} \dots p_{i_t} \mid o(x)$ and for each j, $j \neq i_1, i_2, \dots, i_t$ and $1 \leq j \leq r$, we have $p_j \nmid o(x)$. We first note that if $p_{i_1} p_{i_2} \dots p_{i_t} \mid o(x)$ and $p_j \nmid o(x)$ then $x \notin P_{i_2} P_{i_3} \dots P_{i_t}$, $x \notin P_{i_1} P_{i_3} \dots P_{i_t}$, $\dots$, $x \notin P_{i_1} P_{i_2} \dots P_{i_{t-1}}$ which shows that

$$x \in H \setminus (P_{i_2} P_{i_3} \dots P_{i_t} \cup P_{i_1} P_{i_3} \dots P_{i_t} \cup \dots \cup P_{i_1} P_{i_2} \dots P_{i_{t-1}}).$$

On the other hand, we assume that $x \in H \setminus (P_{i_2} P_{i_3} \dots P_{i_t} \cup P_{i_1} P_{i_3} \dots P_{i_t} \cup \dots \cup P_{i_1} P_{i_2} \dots P_{i_{t-1}})$. Then $x \in H$ and $x \notin P_{i_2} P_{i_3} \dots P_{i_t} \cup P_{i_1} P_{i_3} \dots P_{i_t} \cup \dots \cup P_{i_1} P_{i_2} \dots P_{i_{t-1}}$. This proves that $p_{i_1} p_{i_2} \dots p_{i_t} \mid o(x)$ and $p_j \nmid o(x)$.

On the other hand,

$$\begin{aligned}
t &= |H \setminus (P_{i_2} P_{i_3} \dots P_{i_t} \cup P_{i_1} P_{i_3} \dots P_{i_t} \cup \dots \cup P_{i_1} P_{i_2} \dots P_{i_{t-1}})| \\
&= |H| - (|P_{i_2} P_{i_3} \dots P_{i_t}| + |P_{i_1} P_{i_3} \dots P_{i_t}| + \dots + |P_{i_1} P_{i_2} \dots P_{i_{t-1}}| - |P_{i_3} P_{i_4} \dots P_{i_t}| \\
&\quad - |P_{i_1} P_{i_4} \dots P_{i_{t-1}}| - \dots - |P_{i_1} P_{i_2} \dots P_{i_{t-2}}| + 1) \\
&= p_{i_1}^{k_{i_1}} p_{i_2}^{k_{i_2}} \dots p_{i_t}^{k_{i_t}} - p_{i_2}^{k_{i_2}} p_{i_3}^{k_{i_3}} \dots p_{i_t}^{k_{i_t}} - p_{i_1}^{k_{i_1}} p_{i_3}^{k_{i_3}} \dots p_{i_t}^{k_{i_t}} - \dots - p_{i_1}^{k_{i_1}} p_{i_2}^{k_{i_2}} \dots p_{i_{t-1}}^{k_{i_{t-1}}} \\
&\quad + p_{i_3}^{k_{i_3}} p_{i_4}^{k_{i_4}} \dots p_{i_t}^{k_{i_t}} + p_{i_1}^{k_{i_1}} p_{i_4}^{k_{i_4}} \dots p_{i_t}^{k_{i_t}} + \dots + p_{i_1}^{k_{i_1}} p_{i_2}^{k_{i_2}} \dots p_{i_{t-2}}^{k_{i_{t-2}}} - 1 \\
&= (p_{i_1}^{k_{i_1}} - 1)(p_{i_2}^{k_{i_2}} - 1) \dots (p_{i_t}^{k_{i_t}} - 1)
\end{aligned}$$

For every j, $1 \leq j \leq t$, we define:

$$A_{p_{i_j}} = \left\{ \frac{n}{{p_{i_j}}^{k_{i_j}}}, \frac{2n}{{p_{i_j}}^{k_{i_j}}}, \frac{3n}{{p_{i_j}}^{k_{i_j}}}, \dots, \frac{({p_{i_j}}^{k_{i_j}} - 1)n}{{p_{i_j}}^{k_{i_j}}} \right\}.$$

Then $|A_{p_{i_1}} + A_{p_{i_2}} + \dots + A_{p_{i_t}}| = (p_{i_1}^{k_{i_1}} - 1)(p_{i_2}^{k_{i_2}} - 1) \dots (p_{i_t}^{k_{i_t}} - 1)$, proving the result. □

It is clear that if G_1 and G_2 are isomorphic then $\Pi_{G_1} \cong \Pi_{G_2}$. In what follows the converse of this result will considered into account.

Theorem 3.2. *Let G_1 and G_2 be finite nilpotent groups. Then $\Pi_{G_1} \cong \Pi_{G_2}$ if and only if $|G_1| = |G_2|$.*

Proof. If $\Pi_{G_1} \cong \Pi_{G_2}$ then by definition of non-coprime graph one can easily be seen that $|G_1| = |G_2|$. Suppose G is an arbitrary nilpotent group of order n and $n = |G| = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$, where p_i, $1 \leq i \leq r$, are distinct prime numbers. We will prove that $\Pi_G \cong \Pi_{Z_n}$. Choose $P_i, 1 \leq i \leq r$, to be the Sylow p_i-subgroup of G. Since G is nilpotent, $G \cong P_1 \times P_2 \times \ldots \times P_r$. Define:

$$\begin{aligned}
A_{P_i} &= \{(e, e, \ldots, a_i, e, \ldots, e) \ \mid \ a_i \in P_i, a_i \neq e\}, \\
B_{P_i} &= \left\{ \frac{n}{{p_i}^{k_i}}, \frac{2n}{{p_i}^{k_i}}, \frac{3n}{{p_i}^{k_i}}, \ldots, \frac{({p_i}^{k_i} - 1)n}{{p_i}^{k_i}} \right\}, \\
A_{P_i P_j} &= \{(e, e, \ldots, a_i, e, \ldots, e, a_j, e, \ldots, e) \ \mid \ a_i \in P_i, a_j \in P_j, a_i \neq e, a_j \neq e\}, \\
B_{P_i P_j} &= B_{p_i} + B_{p_j} = \{x + y \ \mid \ x \in B_{p_i}, y \in B_{p_j}\}, \\
\vdots &= \vdots \\
A_{P_1 P_2 \ldots P_r} &= \{(a_1, a_2, \ldots, a_r) \ \mid \ 1 \leq i \leq r-1; a_i \in P_i; a_i \neq e\}, \\
B_{P_1 P_2 \ldots P_r} &= B_{p_1} + B_{p_2} + \ldots + B_{p_r}.
\end{aligned}$$

Define $\varphi : v(\Pi_{P_1 \times P_2 \times \ldots \times P_r}) \longrightarrow v(\Pi_{Z_n})$ by $\varphi(a_{1_{j_1}}, a_{2_{j_2}}, \ldots, a_{r_{j_r}}) = j_1 \frac{n}{{p_1}^{k_1}} + j_2 \frac{n}{{p_2}^{k_2}} + \ldots j_r \frac{n}{{p_1}^{k_1}}$, where $a_{i_{j_i}} \in P_i$, $1 \leq j_i \leq {p_i}^{k_i}$ and if $a_{i_{j_i}} = 0$ then $j_i = 0$. A simple argument shows that φ is a graph isomorphism, as desired. □

Corollary 3.3. *Suppose G_1 and G_2 are nilpotent groups of the same order, $H_1 \leq G_1$ and $H_2 \leq G_2$. Then $\Pi_{G_1,H_1} \cong \Pi_{G_2,H_2}$ if and only if $|H_1| = |H_2|$.*

Theorem 3.4. *Suppose G is a finite $p-$group and $\{e\} \neq H \leq G$. Then,*

1) $\Pi_{G,H}$ *is connected,*

2) $\mathrm{diam}\Pi_{G,H} \leq 2$ *and* $\Pi_{G,H}$ *is a complete graph if and only* $H = G$.

Proof. Suppose G has order p^k, where $k \in \mathbb{N}$. We first notice that $V(\Pi_{G,H}) = V(\Pi_G)$. By definition of $\Pi_{G,H}$, Π_H is an induced subgraph of $\Pi_{G,H}$ and all vertices of $V(H \setminus \{e\})$ are adjacent with all vertices of $V(G \setminus H)$. Furthermore, there is no edge connecting vertices of $V(G \setminus H)$. If $x, y \in V(G)$, then we consider the following three cases:

a $x \in H$ *and* $y \in G \setminus H$. Since $|x|, |y| \mid p^k$, x and y are adjacent;

b $x \in H$ *and* $y \in H$. Since $H \cong K_{p^h - 1}$, $d(x, y) = 1$;

c $x \in G \setminus H$ *and* $y \in G \setminus H$. Then $x \nsim y$ and by definition of $\Pi_{G,H}$, for all $z \in H$, $x \sim z$, $y \sim z$ and so $d(x, y) = 2$.

Thus $\Pi_{G,H}$ is connected and $\mathrm{diam}\Pi_{G,H} \leq 2$. Now it is clear that $\Pi_{G,H}$ is complete if and only if $G = H$. □

Theorem 3.5. *Suppose G is a nilpotent group of order $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ and H is a subgroup of G of order $h = p_1^{h_1} p_2^{h_2} \dots p_r^{h_r}$, where $p_1, p_2, \dots, p_r$ are distinct prime numbers, $h_i, k_i \in \mathbb{N}$ with $1 \leq i \leq r$. Then,*

1) *$\Pi_{G,H}$ is connected and $\mathrm{diam}\Pi_{G,H} \leq 2$.*
2) $\gamma(\Pi_{G,H}) = 1$

Proof. We consider each part separately.

1) By Theorem 3.2, $\Pi_H \cong \Pi_{Z_n}$ and by [5, Theorem 2.4], Π_H is a connected graph with $\mathrm{diam}\Pi_H \leq 2$. We now consider the following three cases.

   a $x, y \in H$. If $x \nsim y$ then there is $z \in H$ such that $x \sim z \sim y$;

   b $x \in H$, $y \in G \setminus H$ and $x \nsim y$. Then there is an element $x_1 \in V(H)$ such that $p_1 p_2 \dots p_r$ divides $\mid x_1 \mid$ and so $x \sim x_1 \sim y$;

   c $x \in G \setminus H$ and $y \in G \setminus H$. So, there is an element $x_1 \in V(H)$ of order $p_1 p_2 \dots p_r$, where $x \sim x_1 \sim y$.

   This proves that $\mathrm{diam}\Pi_{G,H} \leq 2$.

2) Since there is an element of order $p_1 p_2 \dots p_r \in H$ as x_1 that is adjacent to all vertices of $\Pi_{G,H}$, $\gamma(\Pi_{G,H}) = 1$, as desired.

Hence the result. □

Theorem 3.6. *Let G be a finite group of order n, $n \geq 4$, and $H \leq G$ with $|H| > 2$. Then $\mathrm{girth}(\Pi_{G,H}) = 3$.*

Proof. If $|H| \geq 4$ then by [5, Theorem 2.2], $\mathrm{girth}\Pi_{G,H} = 3$. If $|H| = 3$ then $\Pi_H \cong K_2$, where each element of $V(H)$ has order 3. Since there exists at least an element a of order 3 in $G \setminus H$ such that $3 \big| \, |a|$, $a \sim x$, $a \sim y$ and $x \sim y$. Hence $\mathrm{girth}(\Pi_{G,H}) = 3$. □

Corollary 3.7. *Let G be a group and $H \leq G$ has order 2. Then $\Pi_{G,H}$ is acyclic and if $\mid G \mid = 2^k$ then $\Pi_{G,H}$ is tree.*

Theorem 3.8. *Let G be a nilpotent group of order ≥ 4, $\{e\} \neq H \leq G$ and $H \ncong Z_2$. If $\Pi_{G,H}$ is connected then it has Eulerian spanning subgraph.*

Proof. We first prove that if H is a subgroup of G of order $\neq 2$ then Π_H has Eulerian subgraph. It is a well-known fact in graph theory that if each edge of Π_H belong to a triangle then Π_H has Eulerian spanning subgraph. Let H be a subgroup of G of order $h = p_1^{h_1} p_2^{h_2} \dots p_r^{h_r}$, where p_i's are primes and r_i's are non-negative integers. It is enough to prove that Π_{Z_h} has Eulerian subgraph. If h is a prime power then by Theorem 3.4 Π_H is complete and so each edge of Π_G belong to a triangle. Suppose $r \geq 2$. If e is an edge of Π_G then we will consider the following three cases:

a) $e = \{x, y\}$ *and* $\langle x\rangle, \langle y\rangle \neq Z_h$. Then there exists an element $z \in Z_h$ such that $\langle z\rangle = H$. Thus $x \sim z$ and $y \sim z$, as desired. Choose generators $a, b \in Z_h$. Then a and b are adjacent to all vertices of Π_{Z_H} that leads to our result.

b) $e = \{x, y\}$, $\langle x\rangle = Z_h$ *and* $\langle y\rangle \neq Z_h$. Choose another generator z for Z_h. By definition, $z \sim x$ and $z \sim y$ and so the edge e belongs to a triangle.

c) $e = \{x, y\}$, $Z_h = \langle x\rangle\ \langle y\rangle$. Since each generator of Π_{Z_h} is adjacent to all vertices and $h \geq 4$, by choosing an arbitrary non-identity element $z \in Z_h$, we have $z \sim x$ and $z \sim y$. Hence e belongs to a triangle.

This proves that the graph Π_G has Eulerian spanning subgraph. Next we assume that $Z_2 \ncong H < G$ and choose an edge $e = \{x, y\}$ of $\Pi_{G,H}$. If $x, y \in V(H)$ then $e \in \Pi_H$ and so it is an edge of a triangle, and if $x \in V(H)$, $y \in V(G)\backslash V(H)$ then there exists an element $z \in H$ which is adjacent to all vertices of $V(G)$. This completes our argument. □

Theorem 3.9. *Let G be a group and $H \leqslant G$. $\Pi_{G,H}$ is planar if and only if $|G| < 7$ or $|G| \geq 7$ and $H \cong Z_2$.*

Proof. If $G = H$ Then by [5, Theorem 2.11], Π_G is planar if and only if $|G| < 7$. Suppose $H \neq G$. If $|G| < 7$ then $\Pi_{G,H}$ is a subgraph of a planar graph and so it is planar. In the case that $|G| \geq 7$ and $H \cong Z_2$, $\Pi_{G,H}$ is a subgraph of a star graph and so it is planar.

Next we assume that $\Pi_{G,H}$ is planar. Since Π_H is an induced subgraph of $\Pi_{G,H}$, $|H| < 7$. We now consider the following three cases for the subgroup H:

1) $|H| = 2$. In this case $\Pi_{G,H}$ is a subgraph of a star graph and the result is obvious.

2) $|H| = 3$. If $|G| = 3$ or 6 then by [5, Theorem 2.11], $\Pi_{G,H}$ is planar, and if $|G| > 6$ then we have at least three vertices in $G \setminus H$ adjacent to all vertices of H. Hence $K_{3,3} \leq \Pi_{G,H}$ which is not possible.

3) $|H| = r, r \geq 4$. Then $|G| > 6$ and there exist at least r vertices in $G \setminus H$ adjacent to all vertices in H. Again $\Pi_{G,H}$ has a subgraph isomorphic to $K_{3,3}$, that is impossible.

Hence the result. □

Theorem 3.10. *$\Pi_{G,H}$ has perfect matching if and only if $G = H$ or $|G|$ is odd.*

Proof. Suppose $H < G$. Since $|V(\Pi_H)| < |V(G \setminus H)|$, there is not perfect matching in $\Pi_{G,H}$. So $H = G$ and the proof follows from [5, Theorem 2.12]. □

Theorem 3.11. *If G is a non-trivial $p-$group and $H \leq G$ then $\Pi_{G,H}$ is not Eulerian.*

Proof. Suppose $|G| = p^n$, where n is a positive integer. If $p = 2$ then each vertex of $G \setminus H$ is adjacent to all vertices of $V(\Pi_H)$. Since the number of such vertices are $2^n - 1$, $\Pi_{G,H}$ is not Eulerian. Next we assume that p is odd. Then $|V(G \setminus H)| = p^n - p^k$, where $|H| = p^k$. Since Π_H is a complete graph isomorphic to K_{p^k-1}, each vertex in $V(H)$ is adjacent to $p^k - 2$ vertices in H and all vertices in $G \setminus H$. Hence each vertex in $V(H)$ is adjacent to $p^k - 2 + p^n - p^k = p^n - 2$ vertices in $\Pi_{G,H}$. Hence $\Pi_{G,H}$ cannot be Eulerian. □

Theorem 3.12. *Let G be a nilpotent group of order $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, where p_i's are distinct primes, r_i's are positive integers, $p_r^{k_r} < \dots < p_2^{k_2} < p_1^{k_1}$ and put*

$$a_1 = \sum_{\alpha_1 \neq 0 \text{ and } 0 \leq \alpha_i \leq k_i} \Phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}) = \frac{n(p_r^{k_r} - 1)}{p_r^{k_r}}$$

$$a_2 = \sum_{\forall i_1 \neq i_2 \in \{1,2,3\},\ \alpha_{i_1}\alpha_{i_2} \neq 0 \text{ and } 0 \leq \alpha_i \leq k_i} \Phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r})$$

$$a_3 = \sum_{\forall i_1 \neq i_2 \neq i_3 \in \{1,2,3,4,5\},\ \alpha_{i_1}\alpha_{i_2}\alpha_{i_3} \neq 0 \text{ and } 0 \leq \alpha_i \leq k_i} \Phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r})$$

$$\vdots\vdots$$

$$a_{[\frac{r}{2}]+1} = \sum_{\forall i_1 \neq i_2 \dots\dots \neq i_{[\frac{r}{2}]+1} \in \{1,2,\dots,r\},\ \alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_{[\frac{r}{2}]+1}} \neq 0 \text{ and } 0 \leq \alpha_i \leq k_i} \Phi(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}).$$

Then $w(\Pi_G) = max\{a_1, a_2, \dots, a_{[\frac{r}{2}]+1}\}$.

Proof. Let A_1 be the set of all elements of G that p_1 divides its order; A_2 be the set of all elements of G that p_1p_2, p_1p_3 or p_2p_3 divides its order; $\dots$; A_t be the set of all elements of G that at least a product of t distinct primes in $\{p_1, p_2, p_3, \dots p_{2t-1}\}$ divides its order, $1 \leq t \leq [\frac{r}{2}] + 1$. We also assume that $|A_i| = a_i$. It is now obvious that $w(\Pi_G) = max\{a_1, a_2, \dots, a_{[\frac{r}{2}]+1}\}$, as desired. □

Corollary 3.13. *Let G be a nilpotent group, $\{e\} \neq H < G$, $n_1 = |H| = p_1^{h_1} p_2^{h_2} \dots p_s^{h_s}$ be the prime factorization of n_1 and $p_1^{h_1} < p_2^{k_2} < \dots < p_s^{h_s}$. Then $\omega(\Pi_{G,H}) = \omega(\Pi_H) + 1$.*

Proof. Since $H \neq G$, there exists $g \in G \setminus H$ such that $p_1p_2 \dots p_r \big| |g|$. Thus, all vertices of $V(\Pi_H)$ are adjacent to g. Since the induced subgraph of $G \setminus H$ is empty, $\omega(\Pi_{G,H}) = \omega(H) + 1$ and the proof follows from Theorem 3.12. □

Theorem 3.14. *Let G be a group of order n containing a non-trivial subgroup H of order n_1. Then $\alpha(\Pi_G) = \pi(|G|)$ and if $G \neq H$, $\alpha(\Pi_{G,H}) = n - n_1$.*

Proof. The case of $H = G$ follows from [5, Theorem 2.10]. Suppose $H \neq G$. Then by definition of $\Pi_{G,H}$, there is no edge connecting two vertices of $G \setminus H$ and so $\alpha(\Pi_{G,H}) \leq n - n_1$. On the other hand, for each vertex h of H, there exists a vertex in $G \setminus H$ adjacent to h. Hence $\alpha(\Pi_{G,H}) = n - n_1$. □

Theorem 3.15. *Let G be a group and $\{e\} \neq H \leqslant G$. If $G = H$ then $\theta(\Pi_G) = \alpha(\Pi_G) = \pi(|G|)$ and if $G \neq H$ and $\Pi_{G,H}$ is connected then $\theta(\Pi_{G,H}) = \pi(|H|) + (|G| - |H|)$.*

Proof. Suppose $G = H$. Then the size of a minimum clique covering vertices of Π_G is equal to $\pi(|G|)$. Hence $\theta(\Pi_G) = \alpha(\Pi_G)$. If $G \neq H$ then the size of a minimum clique covering all vertices of $\Pi_{G,H}$ is the sum of the size of a minimum clique covering all vertices of $V(\Pi_H)$ and the size of a minimum clique that covers all vertices of $G \setminus H$. If $G \neq H$ and $\Pi_{G,H}$ is connected then each vertex of $G \setminus H$ is adjacent to at least a vertex of H and there is no edge connecting vertices of $V(G \setminus H)$. Hence $\theta(\Pi_{G,H}) = \pi(H) + (|G| - |H|)$. □

Theorem 3.16. *Let G be a nilpotent group of order $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$. Then $\chi(\Pi_G) = \omega(\Pi_G)$.*

Proof. It is well-known that $\chi(\Pi_G) \geq \omega(\Pi_G)$ and so it is enough to prove that $\chi(\Pi_G) \leq \omega(\Pi_G)$. Let $p_r^{k_r} < \ldots < p_2^{k_2} < p_1^{k_1}$ and for $i \in \{1, 2, r\}$ define a subset of G that p_i divides its order by B_i and let A_i, $i \in \{1, 2, \ldots [\frac{r}{2}] + 1\}$, are subset of G that are defined in Theorem 3.12. If $\omega(\Pi_G) = |B_1| = |A_1|$, since $|B_1| > |B_2| > \ldots |B_r|$ by definition of Π_G vertices of Π_G are colorable with $|B_1|$ colors. If $\omega(\Pi_G) = |A|$ and $|A| \neq |A_1|$, then $|A| > |B_1| > |B_2| > \ldots |B_r|$ and so vertices of Π_G are colorable with $|A|$ colors. □

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