On the sum of element orders of $\mathrm{PSL}(2,p)$ for some p

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Abstract. Let G be a finite group and $\psi(G) = \sum_{g\in G} o(g)$, where $o(g)$ denotes the order of $g \in G$. In this paper, we show that if G is a group of order $|\mathrm{PSL}(2,p)|$, then $\psi(G) \geq \psi(\mathrm{PSL}(2,p))$, where $p \in \{11, 13, 19, 23, 29, 37, 61\}$. Also we prove that $|G| = |\mathrm{PSL}(2,p)|$ and $\psi(G) = \psi(\mathrm{PSL}(2,p))$ if and only if $G \cong \mathrm{PSL}(2,p)$, where $p \in \{11, 13, 19, 23, 29, 37, 61\}$. Furthermore, we prove that $\mathrm{PSL}(2,17)$ is determined by its order and the sum of element orders.
Keywords: finite groups, simple group, element orders, sum of element orders.

1. Introduction

In this paper all groups are finite. Let $\psi(G) = \sum_{g\in G} o(g)$, the sum of element orders in a group G. The function $\psi(G)$ was introduced by Amiri, Jafarian and Isaacs [2]. They proved that if G is a non-cyclic group of order n, then $\psi(G) < \psi(\mathbb{Z}_n)$. In fact $\mathbb{Z}_n$ is characterized by $\psi(\mathbb{Z}_n)$ and $|\mathbb{Z}_n|$. In [7], Jafarian proved that $\mathrm{PSL}(2,5)$ and $\mathrm{PSL}(2,7)$ are uniquely determined by their orders and the sum of the element orders.

In [1, 8, 9, 12], finite groups G of order n with the second largest value of $\psi(G)$ were studied. Recently, in [6], Marcel Herzog et al. proved that $\psi(G) \leq 7/11\psi(\mathbb{Z}_n)$ and $\psi(G) < \psi(\mathbb{Z}_n)/(q-1)$, where G is a non-cyclic finite group of order n and q is the least prime divisor of n.

In [1], the following conjecture was stated: *Let S be a simple group. If G is a non-simple group of order $|S|$, then $\psi(S) < \psi(G)$.* In fact, in [7], Jafarian proved that this conjecture holds for $\mathrm{PSL}(2,5)$ and $\mathrm{PSL}(2,7)$. In [11], Marefat et al. by providing groups $\mathrm{PSL}(2,64)$ and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathrm{Sz}(8)$ showed that the conjecture of minimality of $\psi(G)$ in simple groups is incorrect.

In [7, 10], the following general question was proposed: *What information about a group G can be obtained from $\psi(G)$ and $|G|$?* In general, the invariants $\psi(G)$ and $|G|$ do not characterize G. For example, there exist two non-

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isomorphic p-groups of order p^3 and exponent p with sum of element orders $p^4 - p + 1$, where p is an odd prime number.

In this paper, we prove that if $p \in \{11, 13, 17, 19, 23, 29, 37, 61\}$, then $|\mathrm{PSL}(2, p)|$ is uniquely determined by its order and the sum of element orders.

2. Preliminary results

Let s_n be the number of elements of order n and $\omega(G)$ denotes the set of element orders of G. We note that

$$\psi(G) = \sum_{i \in \omega(G)} i s_i, \qquad |G| = \sum_{i \in \omega(G)} s_i. \tag{1}$$

Lemma 2.1 ([3])**.** *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

We know that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \in \omega(G)$, then by the above lemma we have

$$\phi(n) \mid s_n, \qquad n \mid \sum_{d|n} s_d. \tag{2}$$

Lemma 2.2 ([5])**.** *An integer $n = {p_1}^{\alpha_1} \cdots {p_k}^{\alpha_k}$ is the number of Sylow p-subgroups of a finite solvable group G if and only if ${p_i}^{\alpha_i} \equiv 1 \pmod p$ for $i = 1, \cdots, k$.*

Lemma 2.3 ([13, Lemma 1])**.** *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \bigm| |\mathrm{Out}(K/H)|$.*

Lemma 2.4 ([2, Corollary B])**.** *Let $P \in \mathrm{Syl}_p(G)$, and assume that $P \trianglelefteq G$ and that P is cyclic. Then $\psi(G) \leq \psi(P)\psi(G/P)$, with equality if and only if P is central in G.*

Lemma 2.5 ([1, Lemma 2.1])**.** *If G and H are finite groups, then $\psi(G \times H) \leq \psi(G)\psi(H)$. Also $\psi(G \times H) = \psi(G)\psi(H)$ if and only if $\gcd(|G|, |H|) = 1$.*

3. Main results

Theorem 3.1. *Let G be a finite group of order $|\mathrm{PSL}(2, p)|$, where*

$$p \in \{11, 13, 19, 23, 29, 37, 61\}.$$

Then $\psi(\mathrm{PSL}(2, p)) \leq \psi(G)$. Also $|G| = |\mathrm{PSL}(2, p)|$ and $\psi(G) = \psi(\mathrm{PSL}(2, p))$ if and only if $G \cong \mathrm{PSL}(2, p)$.

Proof. We prove the theorem in the following steps.

Step 1. Let $p = 11$.

Let G be a group such that $\psi(G) = 3741$ and $|G| = 660 = 2^2 \cdot 3 \cdot 5 \cdot 11$. First, we show that if G is a solvable group, then $\psi(G) > 3741$. By Lemma 2.2, we have $n_5 \in \{1, 11\}$ and $n_{11} = 1$. By NC-theorem, we have $s_{22} \neq 0$ and $s_{33} \neq 0$. Let Q_{22} and Q_{33} be cyclic subgroups of orders 22 and 33, respectively. If $n_5 = 1$, then $s_{55} \neq 0$ and we have

$$(3) \qquad \psi(G) \geq \psi(\mathbb{Z}_{55}) + \phi(33) \cdot 33 + \phi(22) \cdot 22 + (|G| - 85) \cdot 2 = 4361,$$

and we get the result. If $n_5 = 11$, then $|N_G(P_5)| = 2^2 \cdot 3 \cdot 5$, where $P_5 \in \mathrm{Syl}_5(G)$. By NC-theorem, $s_{15} \neq 0$. Let Q_{15} be a cyclic subgroup of order 15.

- If $11 \mid |N_G(Q_{15})|$, then $11 \cdot 15 \in \omega(G)$ and similarly to (3), we get the result.
- If $11 \nmid |N_G(P_{15})|$, then $s_{15} = \phi(15) \cdot 11 \cdot k_{15}$, where $k_{15} \in \mathbb{N}$.

If $3 \mid |N_G(Q_{22})|$, then $3 \cdot 22 \in \omega(G)$. Hence $\psi(G) \geq \psi(\mathbb{Z}_{66}) + 88 \cdot 15 + (|G| - (66+88)) \cdot 2 = 4663$, as required. If $3 \nmid |N_G(Q_{22})|$, then $s_{22} = \phi(22) \cdot 3 \cdot k_{22}$, where $k_{22} \in \mathbb{N}$. Hence $\psi(G) \geq \psi(\mathbb{Z}_{33}) + 30 \cdot 22 + 88 \cdot 15 + (|G| - (33+30+88)) \cdot 2 = 3775$, as wanted.

If G is non-solvable, then by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong \mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,11)$ and $|G/K| \mid |\mathrm{Out}(K/H)|$. If $K/H \cong \mathrm{PSL}(2,5)$, then $|H| = 11$ and $G = K$. Therefore G is an extension of $\mathbb{Z}_{11}$ by $\mathrm{PSL}(2,5)$. We know that $G/C_G(H) \hookrightarrow \mathrm{Aut}(H)$ and $(G/H)/(C_G(H)/H) \cong G/C_G(H)$. So G is a central extension of H by $\mathrm{PSL}(2,5)$. Since the Shur multiplier of $\mathrm{PSL}(2,5)$ is 2, we get that $G \cong \mathbb{Z}_{11} \times \mathrm{PSL}(2,5)$. Using Lemma 2.5, $\psi(G) = \psi(\mathbb{Z}_{11})\psi(\mathrm{PSL}(2,5)) = 111 \cdot 211 = 23421$ and the result holds. If $K/H \cong \mathrm{PSL}(2,11)$, then $G \cong \mathrm{PSL}(2,11)$ and the proof is complete.

Step 2. Let $p = 13$.

In this case, $\psi(G) = 7281$ and $|G| = 1092 = 2^2 \cdot 3 \cdot 7 \cdot 13$. If G is a solvable group, then by Lemma 2.2, $n_7 = 1$ and $n_{13} = 1$. Therefore $s_{7 \cdot 13} \neq 0$ and we have

$$\psi(G) \geq \psi(\mathbb{Z}_{91}) + (|G| - 91) \cdot 2 = 6751 + (1092 - 91) \cdot 2 = 8753,$$

as required.

If G is non-solvable, then $G \cong \mathrm{PSL}(2,13)$. This completes the proof.

Step 3. Let $p = 19$.

We know that $\psi(G) = 28843$ and $|G| = 3420 = 2^2 \cdot 3^2 \cdot 5 \cdot 19$. First, we show that if G is a solvable group, then $\psi(G) > 28843$.

If G is a solvable group, then there exists a Hall subgroup H of order $3^2 \cdot 5 \cdot 19$. By Lemma 2.2, we get that $n_5(H) = 1$. NC-theorem implies that P_5 is central in H, where $P_5 \in \mathrm{Syl}_5(H)$. Using Lemma 2.4, we have $\psi(H) = \psi(P_5)\psi(A)$, where A is a group of order $3^2 \cdot 19$. By [4], we get that $\psi(A) \geq 1483$. Hence

$$\psi(G) > \psi(H) \geq \psi(P_5)\psi(A) \geq 21 \cdot 1483 = 31143,$$

and we get the result. If G is non-solvable, then by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong \mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,19)$ and $|G/K| \mid |\mathrm{Out}(K/H)|$.

If $K/H \cong \mathrm{PSL}(2,5)$, then $|H| = 3\cdot 19$ and $G = K$. We know that $H \cong \mathbb{Z}_{3\cdot 19}$ or $19 : 3$. If $H \cong \mathbb{Z}_{3\cdot 19}$, then G is a central extension of $\mathbb{Z}_{3\cdot 19}$ by $\mathrm{PSL}(2,5)$. Since the Shur multiplier of $\mathrm{PSL}(2,5)$ is 2, we get that $G \cong \mathbb{Z}_{3\cdot 19} \times \mathrm{PSL}(2,5)$. If $H \cong 19 : 3$, then by [4], $|\mathrm{Out}(19 : 3)| = 6$ and $Z(19 : 3) = 1$. NC-theorem implies that $C_G(H) \neq 1$. Therefore $H \cap C_G(H) = 1$ and it follows that $HC_G(H) \cong H \times C_G(H)$. Also $C_G(H) \cong HC_G(H)/H \trianglelefteq G/H \cong \mathrm{PSL}(2,5)$ which implies that $G \cong (19 : 3) \times \mathrm{PSL}(2,5)$. In both cases, G has a subgroup isomorphic to $\mathbb{Z}_{19} \times \mathrm{PSL}(2,5)$. Hence $\psi(G) > \psi(\mathbb{Z}_{19} \times \mathrm{PSL}(2,5)) = \psi(\mathbb{Z}_{19})\psi(\mathrm{PSL}(2,5)) = 72373$, and we get the result.

If $K/H \cong \mathrm{PSL}(2,19)$, then $G \cong \mathrm{PSL}(2,19)$ and the proof is complete.

Step 4. Let $p = 23$.

Let G be a group such that $\psi(G) = 61733$. By assumption $|G| = 6072 = 2^3\cdot 3\cdot 11\cdot 23$. We claim that $\psi(G) > 61733$, when G is a solvable group. Let G be a solvable group. Then by Lemma 2.2, $n_3 \in \{1,4\}$, $n_{11} \in \{1,23\}$ and $n_{23} = 1$. By NC-theorem, we have $s_{69} \neq 0$ and $s_{46} \neq 0$. If $n_{11} = 1$, then $s_{11\cdot 23} \neq 0$ and we have

$$\psi(G) \geq \psi(\mathbb{Z}_{11\cdot 23}) + (|G| - 11\cdot 23)\cdot 2 = 67915, \tag{4}$$

as required. If $n_{11} = 23$, then $s_{11} = 230$ and $|N_G(P_{11})| = 2^3\cdot 3\cdot 11$, where $P_{11} \in \mathrm{Syl}_{11}(G)$. By NC-theorem, we get that $s_{22} \neq 0$ and $s_{33} \neq 0$. Let Q_{22} and Q_{33} be cyclic subgroups of orders 22 and 33, respectively.

If $23 \mid |N_G(Q_{33})|$, then $23\cdot 33 \in \omega(G)$. Hence $\psi(G) \geq \psi(\mathbb{Z}_{23\cdot 33}) = 393939$. If $23 \nmid |N_G(Q_{33})|$, then $s_{33} = \phi(33)\cdot 23\cdot k_{33}$, where $k_{33} \in \mathbb{N}$, and similarly $s_{22} = \phi(22)\cdot 23\cdot k_{22}$, where $k_{22} \in \mathbb{N}$. If $k_{33} \geq 3$, then $\psi(G) \geq \psi(\mathbb{Z}_{69}) + \phi(33)\cdot 23\cdot 3\cdot 33 + \phi(22)\cdot 23\cdot 22 + s_{11}\cdot 11 + (|G| - (69 + 20\cdot 23\cdot 3 + 10\cdot 23 + 230))\cdot 2 = 65005$, as required. We assume that $k_{33} \in \{1,2\}$, therefore $s_{33} \in \{460, 920\}$. Since $n_3 \in \{1,4\}$, therefore $s_3 \in \{2,8\}$. On the other hand, by (2), we have $33 \mid (1 + s_3 + s_{11} + s_{33})$. Hence $s_3 = 2$ and $s_{33} = 460$. Therefore $|G : N_G(Q_{33})| = 23$ and $|N_G(Q_{33})| = 2^3\cdot 3\cdot 11$. By NC-theorem for Q_{33}, we get that $s_{66} \neq 0$. Let Q_{66} be a cyclic subgroup of order 66. If $23 \mid |N_G(Q_{66})|$, then $23\cdot 66 \in \omega(G)$ and the result holds. If $23 \nmid |N_G(Q_{66})|$, then $s_{66} = \phi(66)\cdot 23\cdot k_{66}$, where $k_{66} \in \mathbb{N}$. Hence

$$\begin{aligned}\psi(G) &\geq \psi(\mathbb{Z}_{69}) + s_{66}66 + s_{33}33 + s_{22}22 + s_{11}11 \\ &\quad + 2(|G| - (69 + s_{66} + s_{33} + s_{22} + s_{11})) \\ &\geq 3549 + 20\cdot 23\cdot 66 + 460\cdot 33 + 230\cdot 22 + 230\cdot 11 + 2(6072 - 1449) \\ &= 65925,\end{aligned}$$

as required. If G is non-solvable, then by Lemma 2.3, $G \cong \mathrm{PSL}(2,23)$ and the proof is complete.

Step 5. Let $p = 29$.

We have $\psi(G) = 139317$ and $|G| = 12180 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$. If G is a solvable group, then by Lemma 2.2, $n_5 = 1$ and $n_{29} = 1$. By NC-theorem, $s_{3\cdot 5\cdot 29} \neq 0$ and so $\psi(G) \geq \psi(\mathbb{Z}_{3\cdot 5\cdot 29}) + (|G| - 3 \cdot 5 \cdot 29) \cdot 2 = 143001$, as required. If G is non-solvable, then by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong \mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,29)$ and $|G/K| \mid |\mathrm{Out}(K/H)|$.

If $K/H \cong \mathrm{PSL}(2,5)$, then $|H| = 7 \cdot 29$ and $G = K$. We know that $H \cong \mathbb{Z}_{7\cdot 29}$ or $29 : 7$. If $H \cong \mathbb{Z}_{7\cdot 29}$, then G is a central extension of $\mathbb{Z}_{7\cdot 29}$ by $\mathrm{PSL}(2,5)$. Since the Shur multiplier of $\mathrm{PSL}(2,5)$ is 2, we get that $G \cong \mathbb{Z}_{7\cdot 29} \times \mathrm{PSL}(2,5)$. Using Lemma 2.5, $\psi(G) = \psi(\mathbb{Z}_{7\cdot 29})\psi(\mathrm{PSL}(2,5)) = 34959 \cdot 211$, as required. If $H \cong 29 : 7$, then by [4], we get that $|\mathrm{Out}(H)| = 4$ and $Z(H) = 1$. NC-theorem implies that $C_G(H) \neq 1$. Therefore $H \cap C_G(H) = 1$ and it follows that $HC_G(H) \cong H \times C_G(H)$. Also $C_G(H) \cong HC_G(H)/H \trianglelefteq G/H \cong \mathrm{PSL}(2,5)$ which implies that $G \cong (29 : 7) \times \mathrm{PSL}(2,5)$. Using Lemma 2.5, $\psi(G) = \psi(29 : 7)\psi(\mathrm{PSL}(2,5)) = 2031 \cdot 211$, as required. If $K/H \cong \mathrm{PSL}(2,29)$, then $G \cong \mathrm{PSL}(2,29)$ and the proof is complete.

Step 6. Let $p = 37$.

Note that $\psi(G) = 406335$ and $|G| = 25308 = 2^2 \cdot 3^2 \cdot 19 \cdot 37$. If G is a solvable group, then by Lemma 2.2, $n_{19} = 1$ and $n_{37} = 1$. By NC-theorem, we have $s_{19\cdot 37} \neq 0$ and so $\psi(G) \geq \psi(\mathbb{Z}_{19\cdot 37}) = 457219$, as required. If G is non-solvable, then by Lemma 2.3, $G \cong \mathrm{PSL}(2,37)$ and the proof is complete.

Step 7. Let $p = 61$.

Let G be a group such that $\psi(G) = 2760861$ and $|G| = 2^2 \cdot 3 \cdot 5 \cdot 31 \cdot 61 = 113460$. If G is a solvable group, then by Lemma 2.2, $n_{31} = 1$ and $n_{61} = 1$. By NC-theorem, we have $s_{31\cdot 61} \neq 0$ and so

$$\psi(G) \geq \psi(\mathbb{Z}_{31\cdot 61}) = 3408391, \tag{5}$$

as required. If G is non-solvable, then by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong \mathrm{PSL}(2,5)$ or $\mathrm{PSL}(2,61)$. If $K/H \cong \mathrm{PSL}(2,5)$, then $|H| = 31 \cdot 61$ and similarly to (5) we obtain the result. If $K/H \cong \mathrm{PSL}(2,61)$, then $G \cong \mathrm{PSL}(2,61)$ and the proof is complete. □

Lemma 3.2. *If G is a finite group with a normal subgroup N, then*

$$|L_n(G)| \leq |N||L_n(G/N)|. \tag{6}$$

Proof. The proof is straightforward. □

Theorem 3.3. *Let G be a finite group such that $|G| = 2448 = 2^4 \cdot 3^2 \cdot 17$ and $\psi(G) = 19483$. Then $G \cong \mathrm{PSL}(2,17)$.*

Proof. If G is a solvable group, then $C_G(\mathrm{Fit}(G)) \leq \mathrm{Fit}(G)$. By NC-theorem, we have

$$|G| \mid |\mathrm{Fit}(G)| \cdot |\mathrm{Aut}(\mathrm{Fit}(G))| \tag{7}$$

and by Lemma 2.2, we get that $n_{17} = 1$. Obviously,

$$\psi(\mathrm{Fit}(G)) + 2 \cdot (|G| - |\mathrm{Fit}(G)|) \leq \psi(G). \tag{8}$$

As $n_{17} = 1$, using (7) and (8), we have the following cases: $\mathrm{Fit}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{17}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$. Now we consider each possibility:

• Let $\mathrm{Fit}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{17}$. As $n_3 \in \{1, 4, 16\}$, therefore $s_3 \leq 16 \cdot 8 = 128$. Also, since $O_2(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Sylow 2-subgroup of G, we have $s_2 = 15$ and $s_4 = 0$. Hence $\psi(G) \geq \psi(\mathrm{Fit}(G)) + 3 \cdot 128 + 6 \cdot (|G| - (|\mathrm{Fit}(G)| + 128)) = 21135$, which is a contradiction.

• Let $\mathrm{Fit}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$. In Lemma 3.2, if we consider $N \cong O_2(G) \times O_{17}(G)$, then by [4], $|L_2(G)| \leq |N||L_2(G/N)| = 68 \cdot 19 = 1292$ and so $s_2 \leq 1291$. Similarly, for $N \cong O_2(G) \times O_3(G)$, we get that $s_3 \leq 107$. Therefore $\psi(G) \geq \psi(\mathrm{Fit}(G)) + 2 \cdot 1291 + 3 \cdot 107 + 4 \cdot (|G| - (|\mathrm{Fit}(G)| + 1291 + 107)) = 19664$, and we get a contradiction.

• Let $\mathrm{Fit}(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$. Let c be a number such that $c \mid |G|$, $s_{17c} \neq 0$ and $\gcd(c, 17) = 1$. If $c \geq 9$, then $\psi(G) \geq \psi(\mathbb{Z}_{17c}) + 2 \cdot (|G| - 17c)$, a contradiction. If $c = 8$, then $s_{8 \cdot 17} = s_{136} = \phi(136) k_{136}$, where $k_{136} \in \mathbb{N}$. If $k_{136} > 2$, then we get a contradiction. If $k_{136} \leq 2$, then $s_{136 \cdot 3} \neq 0$, a contradiction. Similarly, we get a contradiction for $c = 4$. If $c = 6$, then $s_{6 \cdot 17} = s_{102} = \phi(102) k_{102}$, where $k_{102} \in \mathbb{N}$. We assume that Q_{102} is a cyclic subgroup of order 102. If $k_{102} > 2$, then $\psi(G) \leq \psi(\mathrm{Fit}(G)) + 3 \cdot \phi(102) \cdot 102 + (|G| - (|\mathrm{Fit}(G)| + 3 \cdot \phi(102))) \cdot 2 = 21015$, which is a contradiction. If $k_{102} \leq 2$, then $\mathrm{Fit}(N_G(Q_{102})) \cong O_2(N_G(Q_{102})) \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{17}$, where $O_2(N_G(Q_{102})) \neq 1$. Therefore $\psi(G) \geq \psi(N_G(Q_{102})) \geq \psi(\mathrm{Fit}(N_G(Q_{102}))) \geq 3 \cdot 25 \cdot 273 = 20475$, which is a contradiction.

Now if $c = 3$, then by Lemma 3.2, $|L_{51}(G)| \leq |\mathrm{Fit}(G)||L_{51}(G/\mathrm{Fit}(G))| = 153$ and since $s_{51}(G) = 32 \cdot k_{51} \geq s_{51}(\mathrm{Fit}(G)) = 128$, we get that $s_{51}(G) = 128$. Also, we know that $s_3 = 8, s_{17} = 16$.

For $c = 2$, we claim that $s_{34} = 0$ or 144. Let $s_{34} \neq 0$. We assume that Q_{34} is a cyclic subgroup of order 34. Since we proved that $s_{102} = 0$, so we get that $3 \nmid |N_G(Q_{34})|$, and so $s_{34} = \phi(34) \cdot 9 \cdot k_{34}$, where $k_{34} \in \mathbb{N}$. If $k_{34} > 1$, then we get a contradiction. Therefore $s_{34} = 0$ or $s_{34} = 144$. Similarly we get that $s_d = \phi(d) \cdot 17 \cdot k_d$, where $d \in \omega(G)$, $\gcd(d, 17) = 1$ and $d \notin \{1, 2, 3\}$.

By (1) and the above restrictions, we have the following system of equations:

$$\begin{cases} 1 + s_2 + 8 + 16 + s_{34} + 128 + 34 \cdot t_1 = 2448, \\ 1 + 2 \cdot s_2 + 3 \cdot 8 + 17 \cdot 16 + 34 \cdot s_{34} + 51 \cdot 128 + 2 \cdot 34 \cdot t_2 = 19483. \end{cases}$$

The solution of these equations shows that $t_2 - t_1 \notin \mathbb{Z}$.

Hence, we showed that if G is solvable, then $\psi(G) \neq 19483$. If G is a non-solvable group, then by Lemma 2.3, $G \cong \mathrm{PSL}(2, 17)$. This completes the proof. □

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