# $C^*$-ALGEBRA-VALUED $M$-METRIC SPACES AND SOME RELATED FIXED POINT RESULTS

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**Abstract.** In this paper, the concept of $C^*$-algebra-valued $M$-metric spaces is initiated, generalizing the $M$-metric spaces introduced by Asadi *et al.* [8]. Some fixed point theorems are also established via $C_*$-class functions in such spaces. Moreover, some illustrative examples are given. The obtained results generalize and improve some fixed point results in the literature.

**Keywords:** fixed point, $C^*$-algebra-valued $M$-metric space, $C_*$-class function.

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## 1. Introduction

The fixed point theorem, generally known as the Banach contraction mapping principle [11], appeared in explicit form in Banach thesis in 1922. Fixed point theory is also very famous due to its variety of applications in numerous areas such as engineering, computer sciences, economics, etc. The contractive type conditions play an important role in the fixed point theory. Many researchers have extended and generalized Banach contraction principle because it is the heart of this theory.

In 1994, Matthews [20] introduced the notion of a partial metric space and proved the contraction principle of Banach in this new framework. Next, many fixed point theorems in partial metric spaces have been given by several mathematicians. Recently, Haghi *et al.* published [17] a paper which stated that we should be careful on partial metric fixed point results along with giving some results. They showed that some fixed point results in partial metric spaces can be obtained from the corresponding results in metric spaces. Going in same direction, see [7, 26].

In 2014, Asadi *et al.* [8] extended the partial metric space to a $M$-metric space, and proved some of the main theorems by generalizing contractions to get existence of (common) fixed points. For more information on $M$-metric spaces, see also [1, 9, 10, 22, 23, 27].

Consider the operator equation

$$X - \sum_{n=1}^{\infty} L_n^* X L_n = Q, \tag{1.1}$$

where $\{L_1, L_2, \cdots, L_n\}$ is a subset of the set of linear bounded operators on an Hilbert space $H$, $X \in L(H)$ and $Q \in L(H)_+$ is a positive linear bounded operator on the Hilbert space $H$. Then we convert the operator equation to the mapping $F : L(H) \to L(H)$ which is defined by

$$F(X) = \sum_{n=1}^{\infty} L_n^* X L_n + Q. \tag{1.2}$$

Observe that the range of mapping $F$ is not a real number, but it is a linear bounded operator on the Hilbert space $H$. Therefore, the Banach contraction principle cannot be applied with this problem. Afterward, does such mapping have a fixed point which is equivalent to the solution of operator equation?

Recently in 2014, Ma et al. in [19] introduced $C^*$-algebra-valued metric spaces as a new concept which is more general than metric spaces, replacing the set of real numbers by $C^*$-algebras, and established a fixed point theorem for self-maps involving contractive or expansive conditions on such spaces, analogous to the Banach contraction principle. As applications, the existence and uniqueness results for an integral type equation and operator type equation were given and were able to solve the above problem if the elements $L_1, L_2, \cdots, L_n \in L(H)$ satisfy $\sum_{n=1}^{\infty} \|L_n\|^2 < 1$.

In 2014, Ansari [2] introduced the concept of $C$-class functions covering a large class of contractive conditions. For more details, see also [4, 5, 6, 12, 16, 21].

This paper is organized as follows: In section 2, we give the required information, notions and definitions about $M$-metric spaces and $C^*$-algebras. In section 3, we introduce the concept of $C^*$-algebra-valued $M$-metric spaces. Some properties and examples of such spaces are given and several essential lemmas are proved. Finally in section 4, our main results are established and by applying $C_*$-class functions, some fixed point results are proved in $C^*$-algebra-valued $M$-metric spaces.

## 2. Preliminaries

To begin with, we give some basic definitions, notations and theorems which will be used later.

**Definition 2.1** ([8]). *Let $X$ be a non empty set. A function $m : X \times X \to \mathbb{R}_+$ is called a $M$-metric if the following conditions are satisfied:*

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$;

(m2) $m_{xy} \leq m(x,y)$ *where* $m_{xy} := \min\{m(x,x), m(y,y)\}$;

(m3) $m(x,y) = m(y,x)$;

(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

*Then the pair $(X, m)$ is called a $M$-metric space.*

We begin with the basic concept of $C^*$-algebras. A real or a complex linear space $\mathbb{A}$ is an algebra if the vector multiplication is defined for every pair of elements of $\mathbb{A}$ satisfying two conditions such that $\mathbb{A}$ is a ring with respect to vector addition and vector multiplication and for every scalar $\alpha$ and every pair of elements $x, y \in \mathbb{A}$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$. A norm $\|.\|$ on $\mathbb{A}$ is said to be sub-multiplicative if $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathbb{A}$. In this case $(\mathbb{A}, \|.\|)$ is called normed algebra. A complete normed algebra is called Banach algebra. An involution on the algebra $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$ on $\mathbb{A}$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. $(\mathbb{A}, *)$ is called an $*$-algebra. A Banach $*$-algebra $\mathbb{A}$ is an $*$-algebra $\mathbb{A}$ with a complete sub-multiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$. A $C^*$-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$. There are many examples of $C^*$-algebras, such as the set of complex numbers, the set of all bounded linear operators on a Hilbert space $H$, $L(H)$, and the set of $n \times n$-matrices, $M_n(\mathbb{C})$. If a normed algebra $\mathbb{A}$ admits a unit $I$, $Ia = aI = a$ for all $a \in \mathbb{A}$ and $\|I\| = 1$, we say that $\mathbb{A}$ is a unital normed algebra. A complete unital normed algebra $\mathbb{A}$ is called a unital Banach algebra.

For properties on $C^*$-algebras, we refer to [13, 14, 24] and the references therein. A positive element of $\mathbb{A}$ is an element $a \in \mathbb{A}$ such that $a^* = a$ and its spectrum $\sigma(a) \subset \mathbb{R}_+$, where $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda I - a \text{ is noninvertible}\}$. The set of all positive elements will be denoted by $\mathbb{A}_+$. Such elements allow us to define a partial ordering '$\succeq$' on the elements of $\mathbb{A}$. That is,

$$b \succeq a \text{ if and only if } b - a \in \mathbb{A}_+.$$

If $a \in \mathbb{A}$ is positive, then we write $a \succeq \theta$, where $\theta$ is the zero element of $\mathbb{A}$ ($\theta = 0_{\mathbb{A}}$). Each positive element $a$ of a $C^*$-algebra $\mathbb{A}$ has a unique positive square root. From now on, by $\mathbb{A}$, we mean a unital $C^*$-algebra with identity element $I$. Further, $\mathbb{A}_+ = \{a \in \mathbb{A} : a \succeq \theta\}$ and $(a^*a)^{\frac{1}{2}} = |a|$.

**Lemma 2.2** ([15])**.** *Suppose that $\mathbb{A}$ is a unital $C^*$-algebra with a unit $I$.*

(1) *For any $x \in \mathbb{A}_+$, we have $x \preceq I \Leftrightarrow \|x\| \leq 1$;*

(2) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I-a)^{-1}\| < 1$;*

(3) *Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$;*

(4) *By $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$ if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$, and $I - a \in \mathbb{A}'$ is an invertible operator, then*

$$(I-a)^{-1}b \succeq (I-a)^{-1}c.$$

Notice that in a $C^*$-algebra, if $\theta \preceq a, b$, one cannot conclude that $\theta \preceq ab$. For example, consider the $C^*$-algebra $\mathbb{M}_2(\mathbb{C})$ and set $a = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, then $ab = \begin{bmatrix} -1 & 2 \\ -4 & 8 \end{bmatrix}$. Clearly $a, b \in \mathbb{M}_2(\mathbb{C})_+$, while $ab$ is not.

**Definition 2.3** ([19])**.** *Let $X$ be a non empty set. A function $d : X \times X \to \mathbb{A}$ is called a $C^*$-algebra-valued metric on $X$ if the following conditions are satisfied:*

($c1$) *$\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;*

($c2$) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*

($c3$) *$d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

*Then the pair $(X, \mathbb{A}, d)$ is called a $C^*$-algebra-valued metric space.*

**Definition 2.4** ([19])**.** *Suppose that $(X, \mathbb{A}, d)$ is a $C^*$-algebra-valued metric space, $x \in X$ and $\{x_n\}$ is a sequence in $(X, \mathbb{A}, d)$. Then*

(1) *$\{x_n\}$ converges to $x$ with respect to $\mathbb{A}$, if for any $\epsilon > 0$, there is a positive integer $N$ such that $\|d(x_n, x)\| \leq \epsilon$ for all $n \geq N$. We denote it by $\lim_{n\to\infty} x_n = x$;*

(2) $\{x_n\}$ *is Cauchy with respect to* $\mathbb{A}$, *if for any* $\epsilon > 0$ *there is a positive integer* $N$ *such that* $\|d(x_n, x_m)\| \leq \epsilon$ *for all* $n, m \geq N$*;*

(3) $(X, \mathbb{A}, d)$ *is complete if every Cauchy sequence with respect to* $\mathbb{A}$ *in* $X$ *converges to a point in* $X$*.*

In 2017, Ansari *et al.* [3] introduced the concept of complex $C$-class functions as follows:

**Definition 2.5.** *Let* $S = \{z \in \mathbb{C} : 0 \precsim z\}$*, then a continuous function* $F : S^2 \to \mathbb{C}$ *is called a complex* $C$*-class function if for any* $s, t \in S$*, the following conditions hold:*
(1) $F(s, t) \precsim s$*;*
(2) $F(s, t) = s$ *implies that either* $s = 0$ *or* $t = 0$*.*

The same letter $\mathcal{C}$ will denote the class of all complex $C$*-class* functions. For some examples of these functions, see [3].

## 3. $C^*$-algebra-valued $M$-metric spaces

In this section, let $\mathbb{A}$ be a unital $C^*$-algebra with unit $I$. We introduce the concept of $C^*$-algebra-valued $M$-metric spaces, which is more general than $M$-metric spaces.

Define a partial ordering $\succeq$ on the elements of $\mathbb{A}$ as

$$B \succeq A \iff B - A \in \mathbb{A}_+ \iff B - A \succeq \theta.$$

$A, B \in \mathbb{A}$ are comparable if and only if

$$A \succeq B \quad \text{or} \quad B \succeq A.$$

So if all elements of a set $\mathbb{D} \subseteq \mathbb{A}$ are comparable pairwise, then we can define "min" and "max" for $\mathbb{D}$ as follows:

$$\begin{aligned} &\max\{A_i : A_i \in \mathbb{D}, i = 1, 2, \cdots, n\} = A_k \\ &\iff A_k \succeq A_i, \quad \forall i = 1, 2, \cdots, n \\ &\iff A_k - A_i \in \mathbb{A}_+, \quad \forall i = 1, 2, \cdots, n \\ &\iff A_k - A_i \succeq \theta, \quad \forall i = 1, 2, \cdots, n, \end{aligned}$$

and

$$\begin{aligned} &\min\{A_i : A_i \in \mathbb{D}, i = 1, 2, \cdots, n\} = A_k \\ &\iff A_i \succeq A_k, \quad \forall i = 1, 2, \cdots, n \\ &\iff A_i - A_k \in \mathbb{A}_+, \quad \forall i = 1, 2, \cdots, n \\ &\iff A_i - A_k \succeq \theta, \quad \forall i = 1, 2, \cdots, n, \end{aligned}$$

for some $k \in \{1, 2, 3, \cdots, n\}$.

**Definition 3.1.** *Let $X$ be a non empty set. A function $m : X \times X \to \mathbb{A}$ is called a $C^*$-algebra-valued $M$-metric if the following conditions are satisfied:*

(cm1) $\theta \preceq m(x, y)$ *for all* $x, y \in X$ *and* $m(x, x) = m(y, y) = m(x, y) \iff x = y$,

(cm2) $m(x, x)$ *and* $m(y, y)$ *be comparable for all* $x, y \in X$;

(cm3) $m_{xy} \preceq m(x, y)$ *for all* $x, y \in X$, *where* $m_{xy} = \min\{m(x, x), m(y, y)\}$;

(cm4) $m(x, y) = m(y, x)$ *for all* $x, y \in X$;

(cm5) $(m(x, y) - m_{xy}) \preceq (m(x, z) - m_{xz}) + (m(z, y) - m_{zy})$ *for all* $x, y, z \in X$.

*Then the pair* $(X, \mathbb{A}, m)$ *is called a* $C^*$*-algebra-valued* $M$*-metric space.*

**Remark 3.2.** Note that if we take $\mathbb{A} = \mathbb{R}$, then the new notion of $C^*$-algebra-valued $M$-metric space becomes equivalent to Definition 2.1 of the real $M$-metric space.

Let $(X, \mathbb{A}, m)$ be a $C^*$-algebra-valued $M$-metric space. Define $M_{xy}$ by

$$M_{xy} = \max\{m(x, x), m(y, y)\}.$$

**Remark 3.3.** For every $x, y, z \in X$, we have

1. $\theta \preceq M_{xy} + m_{xy} = m(x, x) + m(y, y)$;
2. $\theta \preceq M_{xy} - m_{xy} = (m(x, x) - m(y, y)) \vee (m(y, y) - m(x, x))$;
3. $M_{xy} - m_{xy} \preceq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

It is clear that each $C^*$-algebra-valued $M$-metric $p$ on $X$ generates a $T_0$ topology $\tau_m$ on $X$. Let $\theta \prec \varepsilon \in \mathbb{A}$. The set

$$\{B_m(x, \varepsilon) : x \in X, \varepsilon \succ \theta\},$$

where

$$B_m(x, \varepsilon) = \{y \in X : m(x, y) \prec m_{xy} + \varepsilon\},$$

for all $x \in X$ and $\varepsilon \succ \theta$, forms the base of $\tau_m$.

**Definition 3.4.** *Let* $(X, \mathbb{A}, m)$ *be a* $C^*$*-algebra-valued* $M$*-metric space,* $x \in X$ *and* $\{x_n\}$ *be a sequence in* $X$. *Then*

1. $\{x_n\}$ *converges to* $x$ *with respect to* $\mathbb{A}$, *whenever for every* $\epsilon > 0$ *there is a natural number* $N$ *such that* $\|m(x_n, x) - m_{x_n x}\| \leq \epsilon$ *for all* $n \geq N$. *We denote this by*

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n x}) = \theta; \tag{3.1}$$

2. *$\{x_n\}$ is a m-Cauchy sequence with respect to $\mathbb{A}$, whenever for every $\epsilon > 0$ there is a natural number $N$ such that $\|m(x_n, x_m) - 2m_{x_n x_m} + M_{x_n x_m}\| \leq \epsilon$, for all $m, n \geq N$;*

3. *$(X, \mathbb{A}, m)$ is complete if every m-Cauchy sequence with respect to $\mathbb{A}$, converges to a point $X$ such that*

$$\lim_{n\to\infty} (m(x_n, x) - 2m_{x_n x} + M_{x_n x}) = \theta.$$

The next example states that $m^s$ and $m^w$ are $C^*$-algebra-valued metrics.

**Example 3.5.** Let $m$ be a $C^*$-algebra-valued $M$-metric. Put

1. $m^w(x, y) = m(x, y) - 2m_{xy} + M_{xy}$;
2. $m^s(x, y) = m(x, y) - m_{xy}$ if $x \neq y$ and $m^s(x, y) = \theta$ if $x = y$.

Then $m^w$ and $m^s$ are $C^*$-algebra-valued metrics.

**Proof.** We have $m^w(x, y) \succeq \theta$ and if $m^w(x, y) = \theta$, then

$$m(x, y) = 2m_{xy} - M_{xy}. \tag{3.2}$$

From (3.2) and the fact that $m_{xy} \preceq m(x, y)$, we get $m_{xy} = M_{xy} = m(x, x) = m(y, y)$, so by (3.2), we obtain $m(x, y) = m(x, x) = m(y, y)$. Therefore, $x = y$. For the triangle inequality, it is enough that we consider $(cm5)$ together with Remark 3.3. Similarly, we can show that $m^s$ is a $C^*$-algebra-valued metric. □

**Remark 3.6.** For every $x, y \in X$, we have

1. $m(x, y) - M_{xy} \preceq m^w(x, y) \preceq m(x, y) + M_{xy}$;
2. $(m(x, y) - M_{xy}) \preceq m^s(x, y) \preceq m(x, y)$.

**Lemma 3.7.** *Let $(X, \mathbb{A}, m)$ be a $C^*$-algebra-valued $M$-metric space. Then*

1. *$\{x_n\}$ is a m-Cauchy sequence in $(X, \mathbb{A}, m)$ if and only if it is Cauchy in the $C^*$-algebra-valued metric space $(X, \mathbb{A}, m^w)$;*
2. *A $C^*$-algebra-valued $M$-metric space $(X, \mathbb{A}, m)$ is complete if and only if the $C^*$-algebra-valued metric space $(X, \mathbb{A}, m^w)$ is complete. Furthermore,*

$$\lim_{n\to\infty} m^w(x_n, x) = \theta \iff (\lim_{n\to\infty} (m(x_n, x) - m_{x_n x}) = \theta \quad \& \quad \lim_{n\to\infty} (M_{x_n x} - m_{x_n x}) = \theta).$$

**Proof.** It suffices to use Definition 2.3, Definition 3.4 and Example 3.5. □

Likewise, above lemma also holds for $m^s$.

**Lemma 3.8.** *Assume that* $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$ *in a* $C^*$*-algebra-valued* $M$*-metric space* $(X, \mathbb{A}, m)$*. Then*

$$\lim_{n\to\infty} (m(x_n, y_n) - m_{x_n y_n}) = m(x, y) - m_{xy}.$$

**Proof.** It suffices to write that

$$(m(x_n, y_n) - m_{x_n y_n}) - (m(x, y) - m_{xy}) \preceq (m(x_n, x) - m_{x_n x}) + (m(y, y_n) - m_{y y_n}),$$

and

$$(m(x, y) - m_{xy}) - (m(x_n, y_n) - m_{x_n y_n}) \preceq (m(x_n, x) - m_{x_n x}) + (m(y, y_n) - m_{y y_n}).$$

□

From Lemma 3.8, we can deduce the following.

**Lemma 3.9.** *Assume that* $x_n \to x$ *as* $n \to \infty$ *in a* $C^*$*-algebra-valued* $M$*-metric space* $(X, \mathbb{A}, m)$*. Then*

$$\lim_{n\to\infty} (m(x_n, y) - m_{x_n y}) = m(x, y) - m_{xy},$$

*for all* $y \in X$*.*

Similarly, we may state

**Lemma 3.10.** *Assume that* $x_n \to x$ *and* $x_n \to y$ *as* $n \to \infty$ *in a* $C^*$*-algebra-valued* $M$*-metric space* $(X, \mathbb{A}, m)$*. Then* $m(x, y) = m_{xy}$*. Further, if* $m(x, x) = m(y, y)$*, then* $x = y$*.*

**Proof.** By Lemma 3.8, we have $\theta = \lim_{n\to\infty}(m(x_n, x_n) - m_{x_n x_n}) = m(x, y) - m_{xy}$. □

We present the following examples.

**Example 3.11.** Let $X = [0, \infty)$ and $\mathbb{A} = M_n(\mathbb{C})$. An element $A = (a_{ij})_{n\times n} \in \mathbb{A} = M_n(\mathbb{C})$ is a positive element ( written as $A \succeq \theta$) means that

$$a_{ij} \succeq 0 \quad \forall i, j \in \{1, 2, \cdots, n\},$$

where $\theta$ is the zero matrix in $M_n(\mathbb{C})$. We define a partial ordering $\preceq$ on $\mathbb{A}$ as follows

$$A \preceq B \text{ iff } \theta \preceq B - A.$$

It is clear that $\preceq$ is a partial order relation. Define

$$m(x, y) = \begin{bmatrix} \frac{x+y}{2} + i\frac{x+y}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{x+y}{2} + i\frac{x+y}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{x+y}{2} + i\frac{x+y}{2} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{x+y}{2} + i\frac{x+y}{2} \end{bmatrix}, \tag{3.3}$$

where $x, y \in X$. A norm $\|.\|$ on $\mathbb{A}$ is defined by

$$\|A\| = \max_{i,j} |a_{ij}|^2,$$

where $A = (a_{ij})_{n\times n} \in \mathbb{A}$. The involution is given by $A^* = (\overline{A})^T$, the conjugate transpose of matrix $A$:

$$A^* = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{n2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{nn}} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

It is easy to verify that $m$ is a $C^*$-algebra valued $M$-metric and $(X, M_n(\mathbb{C}), m)$ is a complete $C^*$-algebra valued $M$-metric space of $\mathbb{C}$.

**Example 3.12.** Let $X = \{1, 2, 3\}$ and $\mathbb{A} = M_2(\mathbb{R})$. Define

$$m(1,2) = m(2,1) = m(1,1) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$m(1,3) = m(3,1) = m(3,2) = m(2,3) = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$m(2,2) = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \quad m(3,3) = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix},$$

so $m$ is a $C^*$-algebra-valued $M$-metric. Consider $D(x, y) = m(x, y) - m_{xy}$. We have $m(1,2) = m_{12} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$. Also, $D(1,2) = \theta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since $1 \neq 2$, this means that $D$ is not a $C^*$-algebra-valued metric.

**Example 3.13.** Let $(X, \mathbb{A}, d)$ be a $C^*$-algebra-valued metric space. Take $\phi : \mathbb{A}_+ \to \{A \in \mathbb{A} : \phi(\theta) \preceq A\}$ a one to one, nondecreasing or strictly increasing mapping where $\phi(\theta) \succeq \theta$ is defined, such that

$$\phi(x + y) \preceq \phi(x) + \phi(y) - \phi(\theta), \quad \forall x, y \succeq \mathbb{A}_+.$$

Then $m$ defined by $m(x, y) = \phi(d(x, y))$, is a complex valued $M$-metric.

**Proof.** (cm1), (cm2), (cm3) and (cm4) are clear. For (cm5), we have

$$\begin{aligned}\phi(d(x,y)) &\preceq \phi(d(x,z) + d(z,y)) \\ &\preceq \phi(d(x,z)) + \phi(d(z,y)) - \phi(\theta),\end{aligned}$$

then

$$(\phi(d(x,y)) - \phi(\theta)) \preceq (\phi(d(x,z)) - \phi(\theta)) + (\phi(d(z,y)) - \phi(\theta)).$$

This means that $(m(x,y) - m_{xy}) \preceq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$. □

**Example 3.14.** Let $(X, \mathbb{A}, d)$ be a $C^*$-algebra-valued metric space. Then $m(x,y) = ad(x,y) + bI$ where $a, b > 0$, is a $C^*$-algebra-valued $M$-metric. It suffices to take $\phi(t) = at + bI$, for all $t \in \mathbb{A}_+$.

## 4. Main results

In this section, first we introduce the concept of a $C_*$-class function. The main idea consists in using the set of elements of a unital $C^*$-algebra instead of the set of complex numbers.

**Definition 4.1** ($C_*$-class function)**.** *Suppose that $\mathbb{A}$ is a unital $C^*$-algebra, then a continuous function $F : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ is called a $C_*$-class function if for any $A, B \in \mathbb{A}_+$, the following conditions hold*

(1) *$F(A,B) \preceq A$;*

(2) *$F(A,B) = A$ implies that either $A = \theta$ or $B = \theta$.*

The letter $\mathcal{C}_*$ will denote the class of all $C_*$-class functions.

**Remark 4.2.** The class $\mathcal{C}_*$ includes the set of complex $C$-class functions introduced in [3]. It is sufficient to take $\mathbb{A} = \mathbb{C}$ in Definition 4.1.

The following examples show that the class $\mathcal{C}_*$ is nonempty.

**Example 4.3.** Let $\mathbb{A} = M_2(\mathbb{R})$, of all $2 \times 2$ matrices with the usual operation of addition, scalar multiplication, and matrix multiplication. Define a norm on $\mathbb{A}$ by $\|A\| = \left( \sum_{i,j=1}^{2} |a_{ij}|^2 \right)^{\frac{1}{2}}$. Consider $* : \mathbb{A} \to \mathbb{A}$, given by $A^* = A$, for all $A \in \mathbb{A}$. It defines a convolution on $\mathbb{A}$. Thus $\mathbb{A}$ becomes a $C^*$-algebra. For

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{A} = M_2(\mathbb{R}),$$

we denote $A \preceq B$ if and only if $(a_{ij} - b_{ij}) \leq 0$, for all $i, j = 1, 2$.

(1) Define $F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ by

$$F_*(A,B) = A - B, \text{ i.e.,}$$

$$F_*\Big( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Big) = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix}$$

for all $a_{i,j}, b_{i,j} \in \mathbb{R}_+$ and $i, j \in \{1, 2\}$. Then $F_*$ is a $C_*$-class function.

(2) Define $F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ by

$$F_*\Big( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \Big) = m \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

for all $a_{i,j}, b_{i,j} \in \mathbb{R}_+$, $(i, j \in \{1, 2\})$, where $m \in (0, 1)$. Then $F_*$ is a $C_*$-class function.

**Example 4.4.** Let $X = L^\infty(E)$ and $H = L^2(E)$, where $E$ is a Lebesgue measurable set. Denote $B(H)$ the set of bounded linear operators on the Hilbert space $H$. Clearly, $B(H)$ is a $C^*$-algebra with the usual operator norm.
Define $F_* : B(H)_+ \times B(H)_+ \to B(H)$ by

$$F_*(U, V) = U - \varphi(U),$$

where $\varphi : B(H)_+ \to B(H)_+$ is a continuous function such that $\varphi(U) = \theta$ if and only if $U = \theta$ $(\theta = 0_{B(H)})$. Then $F_*$ is a $C_*$-class function.

Let $\Psi$ be the set of all continuous functions $\psi : \mathbb{A}_+ \to \mathbb{A}_+$ satisfying the following conditions:

$(\psi_1)$ $\psi$ is continuous and non-decreasing;

$(\psi_2)$ $\psi(T) = \theta$ if and only if $T = \theta$.

Our essential main result is

**Theorem 4.5.** *Let $(X, \mathbb{A}, m)$ be a $C^*$-algebra-valued $M$-metric space and $T : X \to X$ be a self-mapping satisfying*

$$\psi(m(Tx, Ty)) \preceq F_*\Big(\psi(m(x, y)), \phi(m(x, y))\Big) \quad \text{for all} \quad x, y \in X, \tag{4.1}$$

*where $\psi, \phi \in \Psi$ and $F_* \in \mathcal{C}_*$. Then $T$ has a unique fixed point.*

**Proof.** Fix $x_0 \in X$. Define $x_n = T^n x_0$ for every $n = 1, 2, 3, \cdots$. We shall prove that

$$m(x_n, x_{n+1}) \to \theta \quad \text{as} \quad n \to \infty.$$

We have

$$\begin{aligned} \psi(m(x_n, x_{n+1})) &= \psi(m(Tx_{n-1}, Tx_n)) \\ &\preceq F_*\Big(\psi(m(x_{n-1}, x_n)), \phi(m(x_{n-1}, x_n))\Big) \\ &\preceq \psi(m(x_{n-1}, x_n)). \end{aligned} \tag{4.2}$$

So we get

$$\psi(m(x_n, x_{n+1})) \preceq \psi(m(x_{n-1}, x_n)).$$

$\psi$ is nondecreasing, so the sequence $\{m(x_n, x_{n+1})\}$ is monotone decreasing in $\mathbb{A}_+$ and hence there exists $\theta \preceq t \in \mathbb{A}_+$ such that

$$m(x_n, x_{n+1}) \to t \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$ in (4.2), by definition of $F_*$ and continuity of $\psi$ and $\phi$, we obtain $\psi(t) \preceq F_*(\psi(t), \phi(t)) \preceq \psi(t)$. Thus $F_*(\psi(t), \phi(t)) = \psi(t)$, so $\psi(t) = \theta$ or $\phi(t) = \theta$, hence $t = \theta$. That is

$$m(x_n, x_{n+1}) \to \theta \quad \text{as} \quad n \to \infty. \tag{4.3}$$

Now, we want to show that $\{x_n\}$ is a $m$-Cauchy sequence in $(X, \mathbb{A}, m)$. By Lemma 3.7, it suffices to prove that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, m^w)$. We obtained $\lim_{n\to\infty} m(x_n, x_{n+1}) = \theta$. Having in mind that $\theta \preceq m_{x_n x_{n+1}} \preceq m(x_n, x_{n+1})$, so

$$\lim_{n\to\infty} m_{x_n x_{n+1}} = \theta. \tag{4.4}$$

Also, $m_{x_n x_{n+1}} = \min\{m(x_n, x_n), m(x_{n+1}, x_{n+1})\}$. In view of the above, one can write

$$\lim_{n\to\infty} m(x_n, x_n) = \theta.$$

Recall that $m_{x_n x_m} = \min\{m(x_n, x_n), m(x_m, x_m)\}$ and $M_{x_n x_m} = \max\{m(x_n, x_n), m(x_m, x_m)\}$. We deduce that

$$\lim_{n,m\to\infty} m_{x_n x_m} = \lim_{n,m\to\infty} M_{x_n x_m} = \theta. \tag{4.5}$$

Assume that $\{x_n\}$ is not Cauchy in $(X, \mathbb{A}, m^w)$. Then there exist $\epsilon > 0$ and subsequences $\{x_{l_k}\}$, $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > l_k > k$ such that $\|m^w(x_{l_k}, x_{n_k})\| > \epsilon$. Now, corresponding to $l_k$, we can choose $n_k$ such that it is the smallest integer with $n_k > l_k$ and satisfying above inequality. Hence $\|m^w(x_{l_k}, x_{n_k-1})\| \leq \epsilon$. So, we have

$$\begin{aligned} \epsilon < \|m^w(x_{l_k}, x_{n_k})\| &\leq \|m^w(x_{l_k}, x_{n_k-1})\| + \|m^w(x_{n_k-1}, x_{n_k})\| \\ &\leq \epsilon + \|m^w(x_{n_k-1}, x_{n_k})\|. \end{aligned} \tag{4.6}$$

We know that

$$m^w(x_{n_k-1}, x_{n_k}) = m(x_{n_k-1}, x_{n_k}) - 2m_{x_{n_k-1}x_{n_k}} + M_{x_{n_k-1}x_{n_k}}. \tag{4.7}$$

Clearly, by (4.3) and (4.5),

$$\lim_{k\to\infty} \|m^w(x_{n_k-1}, x_{n_k})\| = 0. \tag{4.8}$$

Using (4.8) in (4.6), we have

$$\lim_{k\to\infty} \|m^w(x_{l_k}, x_{n_k})\| = \epsilon. \tag{4.9}$$

Again,

$$\begin{aligned} \|m^w(x_{n_k}, x_{l_k})\| \leq \|m^w(x_{n_k}, x_{n_k-1})\| &+ \|m^w(x_{n_k-1}, x_{l_k-1})\| \\ &+ \|m^w(x_{l_k-1}, x_{l_k})\|, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\|m^w(x_{n_k-1}, x_{l_k-1})\| &\le \|m^w(x_{n_k-1}, x_{n_k})\| + \|m^w(x_{n_k}, x_{l_k})\| \\ &\quad + \|m^w(x_{l_k}, x_{l_k-1})\|.\end{aligned} \tag{4.11}$$

Letting $k \to \infty$ in (4.10) and (4.11) and using (4.8) and (4.9), we have

$$\lim_{k\to\infty} \|m^w(x_{n_k-1}, x_{l_k-1})\| = \epsilon.$$

Thus

$$\begin{aligned}\lim_{k\to\infty} \|m(x_{n_k-1}, x_{l_k-1})\| &= \lim_{k\to\infty} \|(m(x_{n_k-1}, x_{l_k-1}) - 2m_{x_{n_k-1}x_{l_k-1}} + M_{x_{n_k-1}x_{l_k-1}})\| \\ &= \lim_{k\to\infty} \|m^w(x_{n_k-1}, x_{l_k-1})\| = \epsilon.\end{aligned}$$

Since $m(x_{n_k-1}, x_{l_k-1}), m(x_{n_k}, x_{l_k}) \in \mathbb{A}_+$ and

$$\lim_{k\to\infty} \|m(x_{n_k-1}, x_{l_k-1})\| = \lim_{k\to\infty} \|m(x_{n_k}, x_{l_k})\| = \epsilon,$$

so there exists $a \in \mathbb{A}_+$ with $\|a\| = \epsilon$ such that

$$\lim_{k\to\infty} m(x_{n_k-1}, x_{l_k-1}) = \lim_{k\to\infty} m(x_{n_k}, x_{l_k}) = a.$$

Now, by (4.1), we have

$$\begin{aligned}\psi(a) &= \lim_{k\to\infty} \psi(m(x_{n_k}, x_{l_k})) \\ &\preceq \lim_{k\to\infty} F_*\Big(\psi(m(x_{n_k-1}, x_{l_k-1})), \phi(m(x_{n_k-1}, x_{l_k-1}))\Big).\end{aligned}$$

Therefore,

$$\psi(a) \preceq F_*\Big(\psi(a), \phi(a)\Big) \preceq \psi(a).$$

Hence $\psi(a) = \theta$ or $\phi(a) = \theta$, so $a = \theta$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in the complete $C^*$-algebra-valued metric space $(X, \mathbb{A}, m^w)$, and so $\{x_n\}$ is $m$-Cauchy in the complete $C^*$-algebra-valued $M$-metric space $(X, \mathbb{A}, m)$. Hence there exists some $v \in X$ such that

$$\lim_{n\to\infty} (m(x_n, v) - m_{x_n v}) = \theta.$$

Due to (4.4), we have $\lim_{n\to\infty} m_{x_n v} = \theta$, hence $\lim_{n\to\infty} m(x_n, v) = \theta$. By Remark 3.3, $m(v, v) = \theta$. Now, we want to show that $v$ is the fixed point of $T$. By (4.1), we have $\theta \preceq \psi(m(Tv, Tv)) \preceq F_*(\psi(m(v, v)), \phi(m(v, v))) = F_*(\psi(\theta), \phi(\theta)) = \theta$. Thus $\psi(m(Tv, Tv)) = \theta \Rightarrow m(Tv, Tv) = \theta$. On the other hand

$$\psi(m(x_n, Tv)) \preceq F_*\Big(\psi(m(x_{n-1}, v)), \phi(m(x_{n-1}, v))\Big).$$

Then letting $n \to \infty$, making use of Lemma 3.9 and continuity of functions $F_*$, $\psi$ and $\phi$, we obtain that $m(v, Tv) = \theta$. Hence we have

$$m(v,v) = m(Tv,Tv) = m(v,Tv) = \theta, \tag{4.12}$$

so by (cm2), we have $Tv = v$. Now, let $u, v \in X$ be two fixed points of $T$ . From (4.1),

$$\psi(m(v,v)) = \psi(m(Tv,Tv)) \preceq F_*\Big(\psi(m(v,v)), \phi(m(v,v))\Big) \preceq \psi(m(v,v)),$$

so $\psi(m(v,v)) = \theta$ or $\phi(m(v,v)) = \theta$. Thus $m(v,v) = \theta$. Similarly, we obtain $m(u,u) = \theta$. Again, by (4.1), we have

$$\psi(m(v,u)) = \psi(m(Tv,Tu)) \preceq F_*\Big(\psi(m(v,u)), \phi(m(v,u))\Big) \preceq \psi(m(v,u)).$$

Hence $\psi(m(v,u)) = \theta$ or $\phi(m(v,u)) = \theta$, so $m(v,u) = \theta$. we obtained that $m(v,v) = m(u,u) = m(v,u) = \theta$. By (cm2), $u = v$. □

If in Theorem 4.5, we take $F_*(s,t) = s - t$, where $s, t \in \mathbb{A}_+$, then we get the following.

**Corollary 4.6.** *Let $(X, \mathbb{A}, m)$ be a complete $C^*$-algebra-valued $M$-metric space and $T : X \to X$ be a self-mapping satisfying*

$$\psi(m(Tx,Ty)) \preceq \psi(m(x,y)) - \phi(m(x,y)) \quad \text{for all} \quad x, y \in X, \tag{4.13}$$

*where $\psi, \phi \in \Psi$. Then $T$ has a unique fixed point.*

**Remark 4.7.** If in Corollary 4.6, we take $\mathbb{A} = \mathbb{R}$, then we obtain Theorem 3.1 of [23].

**Remark 4.8.** If in Corollary 4.6, we take $\mathbb{A} = \mathbb{R}$ and $\phi(t) = (1-k)\psi(t)$ with $0 < k < 1$, then we obtain the $M$-metric generalization of the result in [18].

**Remark 4.9.** If in Corollary 4.6, we take $\mathbb{A} = \mathbb{R}$ and $\psi(t) = t$, then we obtain the $M$-metric generalization for the weakly contractive fixed point theorem in [25].

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