ON TWO NEW APPROACHES IN MODULAR SPACES

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Abstract. In this paper, we prove a Reich-type fixed point theorem in modular spaces. Also, we introduce the concept of h-convex modular spaces and we get the related Banach-type theorem. Our results generalize several ones in the existing literature. Moreover, some examples are given supporting theoretical approaches.

Keywords: fixed point, metric modular, h-convex modular.

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1. Introduction and preliminaries

Two very important kinds of the notions in mathematical analysis are well known: modular function spaces and modular metric spaces. For details on modular function spaces, readers can see [22], while for modular metric spaces see [9].

Both the kinds of modular concept are in fact generalizations of the standard metric spaces, was introduced by Nakano [27] and was intensively developed by several authors. Example 2.1 presented by Abdon and Khamsi [1] is an important motivation for developing the theory of modular metric spaces. Also, see the introduction section in [3].

Otherwise, modular function spaces, that is, modular metric spaces theory has many applications for example in physical interpretations of the modular, the electrorheological fluids, economy, engineering, further in applications to integral operators, approximation and fixed point results. For more details, see ([1]-[7], [9]-[12], [14]-[28]).

Now, we will begin with a brief recollection of basic notions and the facts in modular (metric) spaces. Let X be a nonempty set and $w : (0,\infty) \times X \times X \to [0,\infty]$ be a function. For simplicity, it is denoted by

$$w_\lambda(x,y) = w(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1 ([9])**.** *Let X be a nonempty set. Assume that the map $\omega : (0,\infty) \times X \times X \to [0,\infty]$ satisfies the following conditions for all $x, y, z \in X$:*

(1) $\omega_\lambda(x,y) = 0$ *for all* $\lambda > 0$ *if and only if* $x = y$*;*

(2) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ *for all* $\lambda > 0$*;*

(3) $\omega_{\lambda+\mu}(x,z) \leq \omega_\lambda(x,y) + \omega_\mu(y,z)$ *for all* $\lambda, \mu > 0$*.*

In this case, ω is a said a (metric) modular on X.

A (metric) modular w is said to be strict if it has the following property: given $x, y \in X$ with $x \neq y$, we have $\omega(\lambda, x, y) > 0$ for all $\lambda > 0$.

A (metric) modular w on X is said to be convex [10] if, instead of (3), it satisfies the stronger inequality:

$$\omega_{\lambda+\mu}(x,z) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,y) + \frac{\mu}{\mu+\lambda}\omega_\mu(y,z) \tag{1.1}$$

for all $\lambda, \mu > 0$.

Let (X, d) be a metric space such that X has at least two elements. Then we obtain the following examples of a metric modular w.

Example 1.1 ([3])**.** Let

$$\omega(\lambda, x, y) = w_\lambda(x, y) = d(x, y),$$

for all $\lambda > 0$. This modular is not convex. Indeed, putting $z = y$ and $\mu = \lambda$ in (1.1), the result follows.

Example 1.2 ([3])**.** Let

$$\omega(\lambda, x, y) = w_\lambda(x, y) = \frac{d(x, y)}{\lambda},$$

for all $\lambda > 0$. This modular is convex.

Example 1.3 ([3])**.** Let

$$w_\lambda(x, y) = \begin{cases} \infty \text{ if } \lambda < d(x, y), \\ 0 \text{ if } \lambda \geq d(x, y). \end{cases}$$

This modular w is also convex.

Example 1.4. Let $X = \mathbb{R}$ be endowed with the mapping $w_\lambda : (0, \infty) \times X \times X$ defined as

$$w_\lambda(x, y) = \begin{cases} \frac{|x|+|y|}{\lambda}, \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$$

for all $x, y \in X$ and $\lambda > 0$. It is clear that this mapping is a modular.

Definition 1.2 ([9, 11])**.** *Given a modular ω on X, the two sets*

$$X_\omega \equiv X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\}$$

are said to be modular spaces (around x_0).

It is not hard to see that $X_w \equiv X_w(x_0) = \{x_0\}$ in Examples 1.1 and 1.2, while $X^*_w \equiv X^*_w(x_0) = X$ in both cases.

Definition 1.3 ([10])**.** *Let ω be a modular on X. A sequence $\{x_n\}$ in X is said to be ω-convergent (or modular convergent) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x, such that $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$. A sequence $\{x_n\}$ in X is said to be ω-Cauchy if there exists $\lambda > 0$, possibly depending on the sequence, such that $\omega_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$. X is said to be $\omega-$complete if every $\omega-$Cauchy sequence is $\omega-$convergent.*

Definition 1.4 ([10]). *Let w be a modular on a set X and let X_ω^* be a modular set. A mapping $T : X_\omega^* \to X_\omega^*$ is said to be modular contractive (or an $\omega-$contraction) if there exist numbers $k \in (0,1)$ and $\lambda_0 > 0$ such that*

$$\omega_{k\lambda}(Tx, Ty) \leq \omega_\lambda(x, y) \tag{1.2}$$

for all $0 < \lambda \leq \lambda_0$ and for all $x, y \in X_\omega^*$.

Definition 1.5 ([10]). *Let w be a modular on a set X and let X_w^* be a modular set. A mapping $T : X_w^* \to X_w^*$ is said to be strongly modular contractive (or a strongly ω-contraction) if there exist $k \in (0,1)$ and $\lambda_0 > 0$ such that*

$$w_{k\lambda}(Tx, Ty) \leq kw_\lambda(x, y) \tag{1.3}$$

for all $0 < \lambda \leq \lambda_0$ and all $x, y \in X_w^*$.

The following proposition is the key in enough proofs in the context of modular metric spaces.

Proposition 1.1 ([3], Proposition 2.4). *Let w be a modular on the set X.*

(a) *For every $x, y \in X$, the function $\lambda \mapsto w_\lambda(x, y)$ is non-increasing;*

(b) *Let w be a convex modular. For $x, y \in X$, if $w_\lambda(x, y)$ is finite for at least one value of λ, then $w_\lambda(x, y) \to 0$ as $\lambda \to \infty$ and $w_\lambda(x, y) \to \infty$ as $\lambda \to 0^+$;*

(c) *If w is a convex modular, then the function $v_\lambda(x, y) = \frac{w_\lambda(x,y)}{\lambda}$ is a modular on X.*

2. Reich-type theorem in modular spaces

Definition 2.1. *Let w be a modular on a set X and let X_ω^* be a modular set. A mapping $T : X_\omega^* \to X_\omega^*$ is said to be a Reich $\omega-$contraction if there exist $a, b, c \in (0,1)$ with $a + b + c < 1$ and $\lambda_0 > 0$ such that*

$$\omega_\lambda(Tx, Ty) \leq \omega_{\frac{\lambda}{a}}(x, y) + \omega_{\frac{\lambda}{b}}(x, Tx) + \omega_{\frac{\lambda}{c}}(y, Ty), \tag{2.1}$$

for all $0 < \lambda \leq \lambda_0$ and for all $x, y \in X_\omega^*$.

The following theorem is an analog of the fixed point theorem by Reich [29] in the framework of modular spaces.

Theorem 2.1. *Let ω be a strict convex modular on X such that the modular space X_ω^* is ω-complete and let $T : X_\omega^* \to X_\omega^*$ be a Reich ω-contractive map such that for each $\lambda > 0$, there exists $x = x(\lambda) \in X_\omega^*$ such that $\omega_\lambda(x, Tx) < \infty$. Then T has a fixed point x^* in X_ω^*. If the modular w assumes only finite values on X_ω^*, then the condition $\omega_\lambda(x, Tx) < \infty$ is redundant, and so the fixed point x^* of T is unique and for each $x_0 \in X_\omega^*$, the sequence of iterates $T^n x_0$ is modular convergent to x^*.*

Proof. Let $x_0 \in X^*_\omega$ and $x_n = T^n x_0, n = 0, 1, 2, \ldots$ Putting $x = x_n$ and $y = x_{n-1}$ in inequality (2.1), we obtain

$$\omega_\lambda(x_{n+1}, x_n) \leq \omega_{\frac{\lambda}{a}}(x_n, x_{n-1}) + \omega_{\frac{\lambda}{b}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{c}}(x_{n-1}, x_n). \tag{2.2}$$

Next, we have

$$\begin{aligned}
\omega_{\frac{\lambda}{a}}(x_n, x_{n-1}) &= \omega_{\lambda+\lambda\frac{1-a}{a}}(x_n, x_{n-1}) \\
&\leq \frac{\lambda}{\frac{\lambda}{a}}\omega_\lambda(x_n, x_{n-1}) + \frac{\lambda\frac{1-a}{a}}{\frac{\lambda}{a}}\omega_{\lambda\frac{1-a}{a}}(x_{n-1}, x_{n-1}) \\
&= a\omega_\lambda(x_n, x_{n-1}).
\end{aligned}$$

So,

$$\omega_{\frac{\lambda}{a}}(x_n, x_{n-1}) \leq a\omega_\lambda(x_n, x_{n-1}). \tag{2.3}$$

Similarly,

$$\omega_{\frac{\lambda}{b}}(x_n, x_{n+1}) \leq b\omega_\lambda(x_n, x_{n+1}) \tag{2.4}$$

and

$$\omega_{\frac{\lambda}{c}}(x_n, x_{n-1}) \leq c\omega_\lambda(x_n, x_{n-1}). \tag{2.5}$$

Using (2.2), (2.3), (2.4) and (2.5), we obtain that

$$\omega_\lambda(x_{n+1}, x_n) \leq \frac{a+c}{1-b}\omega_\lambda(x_n, x_{n-1}). \tag{2.6}$$

From (2.6), by induction we have

$$\omega_\lambda(x_{n+1}, x_n) \leq \left(\frac{a+c}{1-b}\right)^n \omega_\lambda(x_1, x_0). \tag{2.7}$$

If we put $= x_{m-1}$ and $y = x_{n-1}$ in the inequality (2.1), we get that

$$\omega_\lambda(x_m, x_n) \leq \omega_{\frac{\lambda}{a}}(x_{m-1}, x_{n-1}) + \omega_{\frac{\lambda}{b}}(x_{m-1}, x_m) + \omega_{\frac{\lambda}{c}}(x_{n-1}, x_n). \tag{2.8}$$

On the other hand, we have

$$\begin{aligned}
\omega_{\frac{\lambda}{a}}(x_{m-1}, x_{n-1}) &= \omega_{\frac{\lambda(1-a)}{2a}+\lambda+\frac{\lambda(1-a)}{2a}}(x_{m-1}, x_{n-1}) \\
&\leq \frac{1-a}{2}\omega_{\frac{\lambda(1-a)}{2a}}(x_{m-1}, x_m) + a\omega_\lambda(x_m, x_n) \\
&+ \frac{1-a}{2}\omega_{\frac{\lambda(1-a)}{2a}}(x_n, x_{n-1}).
\end{aligned}$$

Using (2.8), we get

$$\begin{aligned}(1-a)\omega_\lambda(x_m,x_n) &\le \omega_{\frac{\lambda}{b}}(x_{m-1},x_m)+\frac{1-a}{2}\omega_{\frac{\lambda(1-a)}{2a}}(x_{m-1},x_m)\\ &+ \omega_{\frac{\lambda}{c}}(x_{n-1},x_n)+\frac{1-a}{2}\omega_{\frac{\lambda(1-a)}{2a}}(x_n,x_{n-1}).\end{aligned}$$

Now, by (2.7), we conclude that $\{x_n\}$ is a Cauchy sequence. Since ω is strict, the modular limit X^* of the sequence $\{x_n\}$ is determined uniquely. Let us show that x^* is a fixed point of T. We have the following

$$\begin{aligned}\omega_{\frac{\lambda}{c}}(x^*,Tx^*) &= \omega_{\frac{\lambda(1-c)}{c}+\lambda}(x^*,Tx^*)\\ &\le (1-c)\omega_{\frac{\lambda(1-c)}{c}}(x^*,x_{n+1})+c\omega_\lambda(x_{n+1},Tx^*)\\ &= (1-c)\omega_{\frac{\lambda(1-c)}{c}}(x^*,x_{n+1})+c\omega_\lambda(Tx_n,Tx^*)\\ &\le (1-c)\omega_{\frac{\lambda(1-c)}{c}}(x^*,x_{n+1})+c[\omega_{\frac{\lambda}{a}}(x_n,x^*)\\ &+ \omega_{\frac{\lambda}{b}}(x_n,x_{n+1})+\omega_{\frac{\lambda}{c}}(x^*,Tx^*)].\end{aligned}$$

If we let $n\to\infty$ in the last inequality, we get

$$\omega_{\frac{\lambda}{c}}(x^*,Tx^*)\le c\omega_{\frac{\lambda}{c}}(x^*,Tx^*),$$

so, $Tx^*=x^*$.

Uniqueness. Let x^* and y^* be two fixed points. Then we have

$$\begin{aligned}\omega_\lambda(x^*,y^*) &= \omega_\lambda(Tx^*,Ty^*)\\ &\le \omega_{\frac{\lambda}{a}}(x^*,y^*)+\omega_{\frac{\lambda}{b}}(x^*,Tx^*)\\ &+ \omega_{\frac{\lambda}{c}}(y^*,Ty^*)\\ &\le a\omega_\lambda(x^*,y^*),\end{aligned}$$

so, $x^*=y^*$. □

Corollary 2.1. *If in the condition* (2.1)*, b and c tend to 0^+, then we obtain the Banach contraction principle in the context of strict convex modular metric spaces.*

Corollary 2.2. *If in the condition* (2.1)*, a tends to 0^+, then we obtain the Kannan-type contraction result in the context of strict convex modular metric spaces.*

Now, we give an example supporting Theorem 2.1.

Example 2.1. Let $X_w=[0,1]$ be endowed with the modular metric

$$w_\lambda(x,y)=\begin{cases}\frac{x+y}{\lambda}, & \text{if } x\ne y\\ 0, & \text{if } x=y\end{cases}$$

for all $x, y \in X_w$ and $\lambda > 0$. Clearly, (X_w, w_λ) is a $w-$complete modular metric space. Define the self-mapping $T : X_w \to X_w$ by $Tx = \frac{x}{2}$ and put $a = b = c = \frac{1}{4}$. It is easy to check that all the conditions of Theorem 2.1 hold and T has a fixed point, which is $x = 0$.

Definition 2.2. *Let w be a modular on a set X and let X_w^* be a modular set. A mapping $T : X_w^* \to X_w^*$ is said to be a Hardy-Rogers ([28], (18)) $\omega-$ contraction if there exist $a, b, c, d, e \in (0,1)$ with $a+b+c+d+e < 1$ and $\lambda_0 > 0$ such that*

(2.9)
$$w_\lambda (Tx, Ty) \leq w_{\frac{\lambda}{a}} (x, y) + w_{\frac{\lambda}{b}} (x, Tx) + w_{\frac{\lambda}{c}} (y, Ty) + w_{\frac{\lambda}{d}} (x, Ty) + w_{\frac{\lambda}{e}} (y, Tx),$$

for all $0 < \lambda \leq \lambda_0$ and all $x, y \in X_w^*$.

Now, we present the following open question (Hardy-Rogers $\omega-$ contraction): Prove or disprove the following statement:

• Let w be a strict convex modular on X such that the modular space X_w^* is w-complete and let $T : X_w^* \to X_w^*$ be a Hardy-Rogers ω-contractive map such that for each $\lambda > 0$, there exists $x = x(\lambda) \in X_w^*$ such that $w_\lambda (x, Tx) < \infty$. Then T has a fixed point x^* in X_w^*. If the modular w assumes only finite values on X_w^*, then the condition $w_\lambda (x, Tx) < \infty$ is redundant, and so the fixed point x^* of T is unique and for each $x_0 \in X_w^*$, the sequence of iterates $T^n x_0$ is modular convergent to x^*.

3. Contractions on h-convex modular spaces

We introduce the concept of $C-$type functions.

Definition 3.1. *A function $h : (0,1) \to [0, \infty)$ is said to be $C-$type if it has the following properties:*

(C_1) *there exists $C_h > 0$ such that $h(x)h(y) \leq C_h h(xy)$ for all $x, y \in (0,1)$;*

(C_2) $\lim_{t \to 0+} h(t) = 0$.

Definition 3.2. *Let X be a nonempty set and h be a $C-$type function. A map $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a h-convex modular if for all $x, y, z \in X$, the following conditions are satisfied:*

(1_h) *$\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;*

(2_h) *$\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$;*

(3_h) *$\omega_{\lambda+\mu}(x, z) \leq h\left(\frac{\lambda}{\lambda+\mu}\right) \omega_\lambda(x, y) + h\left(\frac{\mu}{\mu+\lambda}\right) \omega_\mu(y, z)$ for all $\lambda, \mu > 0$.*

A triplet (X, ω, h) is called a h-convex modular space.

Note that, if $h(x) = x$, then we obtain a convex modular. A convex modular is also a h-convex modular, but the converse is not true in general.

Example 3.1. 1. Let $h_1 : (0,1) \to [0,\infty)$ be a function defined by $h_1(x) = x^s$ with $s \in (0,1]$. Then h_1 is a $C-$type with $C_{h_1} = 1$ and we obtain

$$\omega_{\lambda+\mu}(x,z) \leq \left(\frac{\lambda}{\lambda+\mu}\right)^s \omega_\lambda(x,y) + \left(\frac{\mu}{\mu+\lambda}\right)^s \omega_\mu(y,z)$$

for all $\lambda, \mu > 0$.

2. Similarly, let $h_2 : (0,1) \to [0,\infty)$ be a function defined by $h_2(x) = sx$ with $s \geq 1$. Then h_2 is a $C-$type with $C_{h_2} = s$ and we obtain

$$\omega_{\lambda+\mu}(x,z) \leq s\left[\frac{\lambda}{\lambda+\mu}\omega_\lambda(x,y) + \frac{\mu}{\mu+\lambda}\omega_\mu(y,z)\right]$$

for all $\lambda, \mu > 0$.

Lemma 3.1. *Let ω be an $h-$convex modular. If for $x, y \in X$, $\omega_\lambda(x,y)$ is finite for at least one value of λ, then $\omega_\alpha(x,y) \to 0$ as $\alpha \to \infty$.*

Proof. If we set $z = y$ in condition (3_h), then

$$\omega_{\lambda+\mu}(x,y) \leq h\left(\frac{\lambda}{\lambda+\mu}\right)\omega_\lambda(x,y).$$

Now, from condition (C_2), it follows the assertion (by taking $\alpha = \lambda+\mu$ and $\mu \to \infty$). □

The following theorem is the analogue of Banach contraction principle in h-convex modular spaces.

Theorem 3.1. *Let ω be a strict h-convex modular on X such that the modular space X^*_ω is ω-complete and let $T : X^*_\omega \to X^*_\omega$ be a ω-contractive map such that for each $\lambda > 0$, there exists $x = x(\lambda) \in X^*_\omega$ such that $\omega_\lambda(x,Tx) < \infty$. Then T has a fixed point x^* in X^*_ω. If the modular w assumes only finite values on X^*_ω, then the condition $\omega_\lambda(x,Tx) < \infty$ is redundant, and so the fixed point x^* of T is unique and for each $x_0 \in X^*_\omega$ the sequence of iterates $T^n x_0$ is modular convergent to x^*.*

Proof. From the condition $\lim\limits_{t\to 0+} h(t) = 0$, we obtain that there exists $n_0 \in \mathbb{N}$ such that

$$h(k^{n_0}) < \min\left\{1, \frac{1}{C_h}\right\}. \tag{3.1}$$

Further, let $\lambda_1, \lambda_2 > 0$ be such that $\lambda_1 + \lambda_2 + k^{n_0}\lambda = \lambda$. Since ω is $h-$convex modular, we obtain that

$$\begin{aligned}
\omega_{\lambda_1+k^{n_0}\lambda+\lambda_2}(x,y) &\leq h(\frac{\lambda_1+k^{n_0}\lambda}{\lambda})\omega_{\lambda_1+k^{n_0}\lambda}w(x,T^{n_0}y)\\
&+ h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(T^{n_0}y,y)\\
&\leq h(\frac{\lambda_1+k^{n_0}\lambda}{\lambda})[h(\frac{\lambda_1}{\lambda_1+k^{n_0}\lambda})\omega_{\lambda_1}(x,T^{n_0}x)\\
&+ h(\frac{k^{n_0}\lambda}{\lambda_1+k^{n_0}\lambda})\omega_{k^{n_0}\lambda}(T^{n_0}x,T^{n_0}y)]\\
&+ h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(T^{n_0}y,y)\\
&\leq C_h[h(\frac{\lambda_1}{\lambda}t)\omega_{\lambda_1}(x,T^{n_0}x)+h(k^{n_0})\omega_{k^{n_0}\lambda}(T^{n_0}x,T^{n_0}y)]\\
&+ h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(T^{n_0}y,y).
\end{aligned}$$

The condition (1.2) yields that

$$\omega_{k^{n_0}\lambda}(T^{n_0}x,T^{n_0}y) \leq \omega_\lambda(x,y),$$

so we obtain

$$\omega_\lambda(x,y) \leq \frac{C_h h(\frac{\lambda_1}{\lambda})\omega_{\lambda_1}(x,T^{n_0}x)+h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(T^{n_0}y,y)}{1-C_h h(k^{n_0})}. \tag{3.2}$$

For $x_0 \in X^*_\omega$, let $x_n = T^n x_0$. By (3.2), we obtain

$$\begin{aligned}
\omega_\lambda(x_m,x_n) &\leq \frac{C_h h(\frac{\lambda_1}{\lambda})\omega_{\lambda_1}(x_m,x_{m+n_0})+h(\frac{\lambda_2}{\lambda})\omega_{\lambda_2}(x_{n+n_0},x_n)}{1-C_h h(k^{n_0})}\\
&\leq \frac{C_h h(\frac{\lambda_1}{\lambda})\omega_{k^{-m}\lambda_1}(x_0,x_{n_0})+h(\frac{\lambda_2}{\lambda})\omega_{k^{-n}\lambda_2}(x_{n_0},x_0)}{1-C_h h(k^{n_0})}\\
&\to 0 \text{ as } m,n\to\infty,
\end{aligned}$$

that is, $\{x_n\}$ is Cauchy. Let $\lim_{n\to\infty} x_n = x^*$. Then we have

$$\begin{aligned}
\omega_{(k+1)\lambda}(Tx^*,x^*) &\leq h\left(\frac{k\lambda}{(k+1)\lambda}\right)\omega_{k\lambda}(Tx^*,x_n)+h\left(\frac{\lambda}{(k+1)\lambda}\right)\omega_\lambda(x_n,x^*)\\
&\leq h\left(\frac{k}{k+1}\right)\omega_\lambda(x^*,x_{n-1})+h\left(\frac{1}{k+1}\right)\omega_\lambda(x_n,x^*)\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}$$

By the strictness of ω, $Tx^* = x^*$. Suppose $x^*, y^* \in X^*_\omega$ are such that $Tx^* = x^*$ and $Ty^* = y^*$. Then we have

$$\begin{aligned}
\omega_\lambda(x^*,y^*) &\leq h(k^{n_0})\omega_{k^{n_0}\lambda}(x^*,y^*)+h(1-k^{n_0})\omega_{(1-k^{n_0})\lambda}(y^*,y^*)\\
&= h(k^{n_0})\omega_{k^{n_0}\lambda}(x^*,y^*)\\
&= h(k^{n_0})\omega_{k^{n_0}\lambda}(T^{n_0}x^*,T^{n_0}y^*)\\
&\leq h(k^{n_0})\omega_\lambda(x^*,y^*).
\end{aligned}$$

Since $\omega_\lambda(x^*, y^*)$ is finite, by inequality (3.1), we obtain $x^* = y^*$. □

Remark 3.1. For some details on so-called h-convexity, see [31].

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