A NOTE ON GROUPS WHOSE ALL NON-LINEAR IRREDUCIBLE CHARACTERS ARE DEFECT ZERO

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Abstract. Let G be a finite group, χ a non-linear irreducible character of G. $\mathrm{dz}(\chi)$ denotes the set of all prime divisors p of $|G|$ such that χ is a $p-$defect zero character, $\mathrm{dz}(G)$ denotes the union of $\mathrm{dz}(\chi)$ for all non-linear irreducible characters χ of G, i.e., $\chi \in \mathrm{Irr}(G|G')$. A finite group G such that $\bigcap_{\chi\in \mathrm{Irr}(G|G')} \mathrm{dz}(\chi) \neq \emptyset$ was studied in 1996. Finite groups G satisfying $\mathrm{dz}(\chi) = \pi(\chi(1))$ for all $\chi \in \mathrm{Irr}(G|G')$ were classified in 2007. We are motivated to study more general case, i.e., a finite group G satisfying $\mathrm{dz}(\chi) \neq \emptyset$ for every $\chi \in \mathrm{Irr}(G|G')$. At first, we study a solvable group G and come to a necessary and sufficient condition. Secondly, for a non-solvable group, we prove that K_3-simple groups can be uniquely determined by $\mathrm{dz}(G)$ and the order of G.
Keywords: finite group, irreducible character, p-defect zero, structure of a group.

1. Introduction

All groups considered are finite, all characters considered are afforded by ordinary representations. Let n be a positive integer, define $\pi(n)$ to be the set of prime divisors of n. For a set of some primes π, we define n_π to be the positive divisor of n such that $n = n_\pi \times k$, where $\pi(n_\pi) \subseteq \pi$ and $\pi(k) \cap \pi = \emptyset$, especially while $\pi = \{p\}$, we write n_p instead of n_π. Let G be a finite group and N a normal subgroup of G, $\pi(G)$ denotes the set of prime divisors of $|G|$, $\mathrm{Irr}(G)$ the set of irreducible characters of G, and $\mathrm{Irr}(G|N) = \{\chi \big| \chi \in \mathrm{Irr}(G) \text{ and } N \nsubseteq Ker(\chi)\}$, $\mathrm{Irr}(G/N) = \{\chi \big| \chi \in \mathrm{Irr}(G) \text{ and } N \subseteq Ker(\chi)\}$. For a prime $p \in \pi(G)$, an irreducible character $\chi \in \mathrm{Irr}(G)$ is called p-defect zero if $\chi(1)_p = |G|_p$. Set

$$\mathrm{dz}(\chi) = \{\ p \in \pi(G) \mid \chi \text{ is } p\text{-defect zero }\} \text{ and } \mathrm{dz}(G) = \bigcup_{\chi\in \mathrm{Irr}(G|G')} \mathrm{dz}(\chi).$$

All other notations are referred to [2] and [4].

In 1996, Ren studied finite groups G such that $\bigcap_{\chi\in \mathrm{Irr}(G|G')} \mathrm{dz}(\chi) \neq \emptyset$ in [7]. In 2007, Liang, Qian and Shi classified finite groups G satisfying $\mathrm{dz}(\chi) = \pi(\chi(1))$ for all $\chi \in \mathrm{Irr}(G|G')$ in [5].

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We are motivated to study more general case, that is, a finite group G satisfying $\mathrm{dz}(\chi) \neq \emptyset$ for every $\chi \in \mathrm{Irr}(G|G')$. We first focus on a finite solvable group, and come to the following theorem in Section 2:

Theorem A. *Let G be a non-abelian solvable group. Then $\mathrm{dz}(\chi) \neq \emptyset$ holds for every $\chi \in \mathrm{Irr}(G|G')$ if and only if there exists a series of normal Hall-subgroups*

$$G = M_0 > M_1 > \cdots > M_r > 1$$

with $r \geqslant 1$, such that following statements hold.

(a) *For $0 \leq i \leq r-1$, M_i are non-abelian groups;*

(b) *For $1 \leqslant i \leqslant r$, M_{i-1}/M_i is a cyclic Sylow subgroup of G;*

(c) *For $1 \leqslant i \leqslant r$ and every $\lambda \in \mathrm{Irr}(M_i/M_i')$, it follows $\mathrm{I}_{M_{i-1}}(\lambda) = M_i$ or M_{i-1}. Moreover for each non-linear irreducible character χ of M_r with $\mathrm{dz}(\lambda) = \emptyset$, if there exists, it follows $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$.*

By Theorem A, we have the following two corollaries.

Corollary A. *Let G be a non-abelian solvable group. Then $\mathrm{dz}(\chi) = \pi(\chi(1))$ holds for every $\chi \in \mathrm{Irr}(G|G')$ if and only if there exists a series of normal Hall subgroups*

$$G = M_0 > M_1 > \cdots > M_r > 1$$

with $r \geqslant 1$, such that following statements hold.

(a) *For $0 \leq i \leq r-1$, M_i are non-abelian groups, but M_r is an abelian group;*

(b) *For $1 \leqslant i \leqslant r$, M_{i-1}/M_i is a cyclic Sylow subgroup of G;*

(c) *For $1 \leqslant i \leqslant r$ and every $\lambda \in \mathrm{Irr}(M_i)$, it follows $\mathrm{I}_{M_{i-1}}(\lambda) = M_i$ or M_{i-1}.*

Corollary B. *Let G be a group and π be a nonempty proper subset of $\pi(G)$. Suppose $\pi \subseteq \mathrm{dz}(\chi)$ for every $\chi \in \mathrm{Irr}(G|G')$. Then $\mathrm{cd}(G) = \{1,\ f\}$ if and only if G has an abelian normal subgroup N of index f such that*

(a) *G/N is cyclic;*

(b) *for every $\lambda \in \mathrm{Irr}(N)$, $\mathrm{I}_G(\lambda) = G$ or N.*

For a non-solvable group, we cannot get a beautiful result as Theorem A. It is worth to mention a result about simple group, which is in [3, Corollary 2], Granville and Ono proved that every finite simple group M satisfies $\mathrm{dz}(M) = \pi(M)$ with the following exceptions:

(a) M has no character of 2-defect zero if it is isomorphic to M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, BM, or A_n, where $n \neq 2m^2+m$ nor $2m^2+m+2$ for any integer m;

(b) M has no character of 3-defect zero if it is isomorphic to Suz, Co_3, or A_n with $3n+1 = m^2r$, where r is square free and divisible by some prime $q \equiv 2 \mod 3$.

Above result means that most simple groups M satisfies $\mathrm{dz}(M) = \pi(M)$. But if a finite group G satisfying $\mathrm{dz}(G) = \mathrm{dz}(M)$, can we get $G \cong M$? Surely we cannot. For example, let r be a positive integer such that $(r,\ |M|) = 1$ and C_r a cyclic group of order r, then $\mathrm{dz}(M) = \mathrm{dz}(M \times C_r)$, but $M \ncong M \times C_r$. So it is a meaningful topic to study 'adding what kind of condition to $\mathrm{dz}(G) = \mathrm{dz}(M)$, we can get $G \cong M$'.

Here we try to study K_3-simple groups. A finite group M is called a K_n-group if the order of M has exactly n distinct prime divisors. We shall prove in Section 3 that a K_3-simple group M can be uniquely determined by $\mathrm{dz}(M)$ and $|M|$, that is the following theorem.

Theorem B. *Let G be a finite group, M a K_3-simple group. Then $G \cong M$ if and only if*

(*a*) $|G| = |M|$,

(*b*) $\mathrm{dz}(G) = \mathrm{dz}(M)$.

2. Proof of Theorem A

At first we introduce a result in [6].

Lemma 2.1 ([6], Theorem A)**.** *Let G be a group, $1 < N \lhd G$, P a Sylow p-subgroup of G, where $p \in \pi(G)$. Then $p \notin V(G|N)$ if and only if N is a p'-group and P acts frobeniusly on N, where $V(G|N) = \bigcup_{\chi \in \mathrm{Irr}(G|N)} \pi\big(\frac{|G|}{|\ker \chi|\chi(1)}\big)$.*

Lemma 2.2. *Let G be a group, K a normal Hall subgroup of G. If G/K is solvable and $\xi \in \mathrm{Irr}(K)$ is invariant in G. Then ξ has a unique extension $\chi \in \mathrm{Irr}(G)$ with $o(\chi) = o(\xi)$.*

Proof. The lemma follows straight forward from [4, Corollary 6.28]. □

Lemma 2.3. *Let G be a group, π a nonempty proper subset of $\pi(G)$. Suppose $\pi \subseteq \mathrm{dz}(\chi)$ for every $\chi \in \mathrm{Irr}(G|G')$. Then the following statements hold.*

(*a*) *For every $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$, P acts frobeniusly on G'. In particular, G is solvable;*

(*b*) *$G = H \ltimes K$, where H is a cyclic Hall π-subgroup of G, K is a normal Hall π'-subgroup of G;*

(*c*) *For every $\xi \in \mathrm{Irr}(K)$, $\mathrm{I}_G(\xi) = G$ or K. Especially while $\mathrm{I}_G(\xi) = G$, ξ is linear.*

Proof. By formula

$$|G| = |G/G'| + \sum_{\chi \in \mathrm{Irr}(G|G')} \chi(1)^2,$$

we have $|G|_\pi \big| |G/G'|$. Hence $1 < G' \lneqq G$. For every $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$, it follows by Lemma 2.1 that P acts frobeniusly on G'. Then G' is a nilpotent π'-group. Thus G is solvable.

Since G/G' is abelian, we have $G/G' = HG'/G' \times K/G'$, where H is a Hall π-subgroup of G, K a normal Hall π'-subgroup of G. Note that $HG'/G' \cong H$ is abelian. Then H is cyclic by argument in preceding paragraph.

Let $\xi \in \mathrm{Irr}(K)$. If some constituents of ξ^G are non-linear, say, one is ϕ, then $\phi(1)/\xi(1)\big||H|$. Since $\pi(H) = \pi$, we have $|H|\big|\phi(1)$ by the hypothesis. Note that $\xi(1)$ is a π'-number. We get $\phi(1) = |H|\xi(1) = \xi^G(1)$. Then $\xi^G = \phi$, i.e., $\mathrm{I}_G(\xi) = K$. If all irreducible constituents of ξ^G are linear, then ξ is extendible to G. Therefore $\mathrm{I}_G(\xi) = G$ and $\xi(1) = 1$. □

Remark 2.1. The statements (a) and (b) have been given in [7, Theorem 1]. However, the approach of proof in [7] is different from ours.

Conversely, assume that G is solvable and satisfies statements (b) and (c) of Lemma 2.3. Let $\chi \in \mathrm{Irr}(G|G')$ and ξ be an irreducible constituent of χ_K. If $\mathrm{I}_G(\xi) = K$, then $\chi = \xi^G$ and $\chi(1) = |H|\xi(1)$. So $\pi \subseteq \mathrm{dz}(\chi)$.

If $\mathrm{I}_G(\xi) = G$, then ξ is linear by (c). By Lemma 2.2, there exists $\phi \in \mathrm{Irr}(G)$ such that $\phi_K = \xi$. Hence characters of form like $\beta\phi$ for $\beta \in \mathrm{Irr}(G/K)$ are all irreducible constituents of ξ^G by [4, Corollary 6.17]. Therefore all irreducible constituents of ξ^G are linear, so χ is linear, a contradiction. Thus we come to the following lemma.

Lemma 2.4. *Let G be a group and π a nonempty proper subset of $\pi(G)$. Then $\pi \subseteq \mathrm{dz}(\chi)$ holds for every $\chi \in \mathrm{Irr}(G|G')$ if and only if G is solvable and satisfies*

(*a*) *$G = H \ltimes K$ where H is a cyclic Hall π-subgroup of G, K is a normal Hall π'-subgroup of G;*

(*b*) *for every $\xi \in \mathrm{Irr}(K)$, $\mathrm{I}_G(\xi) = G$ or K. Especially while $\mathrm{I}_G(\xi) = G$, ξ is linear.*

In order to write the proof of Theorem A to be readable, we make following hypothesis for brevity.

Hypothesis 2.1. *Let X be a non-abelian solvable group. Suppose $\mathrm{dz}(\chi) \neq \emptyset$ holds for every $\chi \in \mathrm{Irr}(X|X')$.*

The Proof of Theorem A. Firstly, we show the necessity. Now G satisfies Hypothesis 2.1. Let K be a maximal normal subgroup of G such that G/K is non-abelian. By [4, Lemma 12.3], it follows that $|\mathrm{cd}(G/K)| = 2$. Let χ be a non-linear irreducible character of G/K, then $\mathrm{cd}(G/K) = \{1,\ \chi(1)\}$, and there exists a prime $p \in \mathrm{dz}(\chi)$. By setting $\pi = \{p\}$ in Lemma 2.4, we see that the Sylow p-subgroup of G/K is cyclic and G/K has a normal Hall p'-subgroup M/K. Since K is a p'-group, we conclude that M is a normal Hall p'-subgroup of G. By Schur-Zassenhaus Theorem the Sylow p-subgroup of G is isomorphic to G/M and is cyclic.

We denote above M as M_1, p as p_1. If M_1 does not satisfy Hypothesis 2.1, we stop and let $r = 1$. If M_1 satisfies Hypothesis 2.1, by the same argument in the preceding paragraph, we get that M_1 has a cyclic Sylow p_2-subgroup and has a normal p_2-complement M_2. If M_2 does not satisfy Hypothesis 2.1, we stop and let r=2. Repeating this process, we obtain a series of normal Hall subgroups

$$G = M_0 > M_1 > \cdots > M_r > 1$$

such that $M_1, M_2, \cdots, M_{r-1}$ satisfy Hypothesis 2.1 and M_r does not. This means the statements (a) and (b) are proved.

Let $|M_{i-1}/M_i| = p_i^{\alpha_i}$, where $p_i^{\alpha_i}\big|\big|\ |G|$, $1 \leqslant i \leqslant r$. For every $\lambda \in \mathrm{Irr}(M_i/M_i')$, if all irreducible constituents of $\lambda^{M_{i-1}}$ are linear, then λ is extendible to M_{i-1}, moreover $\mathrm{I}_{M_{i-1}}(\lambda) = M_{i-1}$. Otherwise there exists $\phi \in \mathrm{Irr}(M_{i-1}|M_{i-1}')$ to be an irreducible constituent of $\lambda^{M_{i-1}}$. Then $\phi(1)/\lambda(1)\big||M_{i-1}/M_i|$, i.e., $\phi(1)\big|p_i^{\alpha_i}$. Since M_{i-1} satisfies Hypothesis 2.1, it follows $\mathrm{dz}(\phi) \neq \emptyset$. So $\phi(1) = p_i^{\alpha_i} = p_i^{\alpha_i}\lambda(1)$, and consequently $\phi = \lambda^{M_{i-1}}$. This implies $\mathrm{I}_{M_{i-1}}(\lambda) = M_i$. We have proved the first part of (c).

Now assume M_r is non-abelian and $\xi \in \mathrm{Irr}(M_r|M_r')$ with $\mathrm{dz}(\xi) = \emptyset$. Let $\varphi \in \mathrm{Irr}(M_{r-1})$ be an irreducible constituent of $\xi^{M_{r-1}}$. Then $\varphi(1)/\xi(1)\big|p_r^{\alpha_r}$. Since M_{r-1} satisfies Hypothesis 2.1 and $\xi(1)$ is a p_r'-number and $\mathrm{dz}(\xi) = \emptyset$, it follows that φ has to be p_r-defect zero, and $\varphi(1) = p_r^{\alpha_r}\xi(1)$, further $\varphi = \xi^{M_{r-1}}$ and $\mathrm{I}_{M_{r-1}}(\xi) = M_r$. The second part of (c) is proved.

Now we prove the sufficiency. Assume $\chi \in \mathrm{Irr}(M_{r-1}|M_{r-1}')$. Let λ be an irreducible constituent of χ_{M_r}.

If λ is linear, then $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$ or M_{r-1} by (c). If $\mathrm{I}_{M_{r-1}}(\lambda) = M_{r-1}$, λ is extendible to M_{r-1} by Lemma 2.2. Hence all irreducible constituents of $\lambda^{M_{r-1}}$ are linear by [4, Corollary 6.17] and (b). This contradicts that χ is a non-linear irreducible constituent of $\lambda^{M_{r-1}}$. Hence $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$ and $\chi(1) = p_r^{\alpha_r}\lambda(1) = p_r^{\alpha_r}$. So $p_r \in \mathrm{dz}(\chi)$.

If λ is non-linear. Then M_r must be non-ableian. If $\mathrm{dz}(\lambda) \neq \emptyset$, then $\mathrm{dz}(\lambda) \subseteq \mathrm{dz}(\chi) \neq \emptyset$. Otherwise $\mathrm{dz}(\lambda) = \emptyset$, and it follows by (c) that $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$, which implies that $\chi(1) = p_r^{\alpha_r}\lambda(1)$, $p_r \in \mathrm{dz}(\lambda) \subseteq \mathrm{dz}(\chi)$. Hence it always follows that $\mathrm{dz}(\chi) \neq \emptyset$. Therefore M_{r-1} satisfies Hypothesis 2.1.

If $r = 1$, then sufficiency follows. Now assume $r > 1$. For any $\chi \in \mathrm{Irr}(M_{r-2}|M_{r-2}')$, let λ be an irreducible constituent of $\chi_{M_{r-1}}$. If λ is linear, then $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-1}$ or M_{r-2} by (c). By the same arguments as in preceding paragraph, we have $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-1}$, and $p_{r-1} \in \mathrm{dz}(\chi)$. If λ is non-linear. Since M_{r-1} satisfies Hypothesis 2.1 , we have $\mathrm{dz}(\lambda) \neq \emptyset$. By $\mathrm{dz}(\lambda) \subseteq \mathrm{dz}(\chi)$, we have $\mathrm{dz}(\chi) \neq \emptyset$. So anyhow $\mathrm{dz}(\chi) \neq \emptyset$ always follows. Thus M_{r-2} satisfies Hypothesis 2.1. Repeating above process, we get at last that G satisfies Hypothesis 2.1. □

Remark 2.2. By the proof of Theorem A, one can see to that M_i satisfies Hypothesis 2.1, $0 \leqslant i \leqslant r-1$. But M_r does not.

In order to write the proof readable, we make a another hypothesis.

Hypothesis 2.2. Let X be a finite non-abelian solvable group. Suppose $\mathrm{dz}(\chi) = \pi(\chi(1))$ for every $\chi \in \mathrm{Irr}(X|X')$.

The Proof of Corollary A. Firstly, by assumption G satisfies Hypothesis 2.2. By Theorem A, in order to prove statement (a), it is enough to prove M_r is abelian.

By Remark 2.2, we take r as large as possible, and M_r does not satisfy Hypothesis 2.1. By [5, Lemma 2.2], every non-abelian normal Hall subgroup of G satisfies Hypothesis 2.2. Therefore M_r is abelian. The statement (a) follows.

The statement (b) follows trivially from Theorem A.

Now we prove statement (c). For $1 \leqslant i \leqslant r$, let $\lambda \in \mathrm{Irr}(M_i)$ and χ an irreducible constituent of $\lambda^{M_{i-1}}$. Let $|M_{i-1}/M_i| = p_i^{\alpha_i}$, $1 \leqslant i \leqslant r$, where $p_i^{\alpha_i} \| \, |G|$. If $\mathrm{dz}(\chi) = \mathrm{dz}(\lambda)$, then $\chi(1) = \lambda(1)$. It follows that χ is an extension of λ, so $\mathrm{I}_{M_{i-1}}(\lambda) = M_{i-1}$. If $\mathrm{dz}(\chi) \neq \mathrm{dz}(\lambda)$. Then $\mathrm{dz}(\chi) \setminus \mathrm{dz}(\lambda) = \{p_i\}$, so $\chi(1) = p_i^{\alpha_i}\lambda(1) = \lambda^{M_{i-1}}(1)$. Hence $\chi = \lambda^{M_{i-1}}$ and $\mathrm{I}_{M_{i-1}}(\lambda) = M_i$.

Conversely, suppose $\chi \in \mathrm{Irr}(M_{r-1}|M'_{r-1})$ and λ is an irreducible constituent of χ_{M_r}. By statement (c), we see that $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$ or M_{r-1}. Assume $\mathrm{I}_{M_{r-1}}(\lambda) = M_r$. Then $\chi = \lambda^{M_{r-1}}$ and $\chi(1) = p_r^{\alpha_r} \cdot \lambda(1)$. Notice that λ is linear. Then $\mathrm{dz}(\chi) = \{p_r\} = \pi(\chi(1))$. Now assume $\mathrm{I}_{M_{r-1}}(\lambda) = M_{r-1}$. By Lemma 2.2, λ is extendible to M_{r-1}. By (b) and [4, Corollary 6.17], we have $\chi(1) = \lambda(1) = 1$. This contradicts $\chi \in \mathrm{Irr}(M_{r-1}|M'_{r-1})$. Therefore M_{r-1} satisfies Hypothesis 2.2.

If $r = 1$, then proof is finished. Now assume $r > 1$.

For any $\chi \in \mathrm{Irr}(M_{r-2}|M'_{r-2})$, let λ be an irreducible constituent of $\chi_{M_{r-1}}$. By (c), $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-1}$ or M_{r-2}. Firstly, assume $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-1}$. Then $\chi = \lambda^{M_{r-2}}$ and $\chi(1) = p_{r-1}^{\alpha_{r-1}} \cdot \lambda(1)$. If λ is linear, then $\mathrm{dz}(\chi) = \pi(\chi(1))$. Otherwise λ is non-linear. Since M_{r-1} satisfies Hypothesis 2.2, we have $\mathrm{dz}(\lambda) = \pi(\lambda(1))$. It follows that $\mathrm{dz}(\chi) = \pi(\lambda(1)) \cup \{p_r\} = \pi(\chi(1))$. Secondly, assume $\mathrm{I}_{M_{r-2}}(\lambda) = M_{r-2}$. By the same argument as in the preceding paragraph, we get $\chi(1) = \lambda(1)$. It follows that λ is non-linear, which implies $\mathrm{dz}(\lambda) = \pi(\lambda(1))$ since M_{r-1} satisfies Hypothesis 2.2. Therefore it always follows $\mathrm{dz}(\chi) = \pi(\chi(1))$, so M_{r-2} also satisfies Hypothesis 2.2. Repeating this process, we get that G satisfies Hypothesis 2.2. □

The Proof of Corollary B. Suppose $\mathrm{cd}(G) = \{1, f\}$. Let K be a maximal normal subgroup such that G/K is non-abelian. By Lemma 2.4, G is solvable. Thus G/K satisfies the assumption of Lemma 12.3 in [4]. By the proof of Theorem 12.5 in [4], G has an abelian normal subgroup N of index f such that G/N is cyclic.

Assume $\lambda \in \mathrm{Irr}(N)$ and ϕ is a non-linear irreducible constituent of λ^G. Notice that $\lambda^G(1) = |G : N| = f = \phi(1)$, we have that λ^G is irreducible. Hence $\mathrm{I}_G(\lambda) = N$.

Conversely, let $\chi \in \mathrm{Irr}(G|G')$ and λ be an irreducible constituent of χ_N. If $\mathrm{I}_G(\lambda) = N$, then $\lambda^G = \chi$ and $\chi(1) = |G : N| = f$.

Assume $\mathrm{I}_G(\lambda) = G$. We assert that λ^G has no linear irreducible constituent. Otherwise, let ϕ be a linear irreducible constituent of λ^G, then $\phi_N = \lambda$. By [4, Corollary 6.17], all irreducible constituents of λ^G are linear, a contradiction to χ is a non-linear irreducible constituent of λ^G.

Let $\mathrm{Irr}(\lambda^G) = \{\chi_1 = \chi,\ \chi_2,\ \cdots,\ \chi_t\}$, where $\chi_i \in \mathrm{Irr}(G|G')$, $i = 1,\ 2,\ \cdots,\ t$. By Frobenius reciprocity, we have

$$[\chi_i, \lambda^G] = [(\chi_i)_N, \lambda] = \chi_i(1), \quad i = 1, 2, \cdots, t.$$

This implies $\lambda^G = \sum_{i=1}^t \chi_i(1)\chi_i$, so $\lambda^G(1) = \sum_{i=1}^t \chi_i(1)^2$. Thus $\sum_{i=1}^t \chi_i(1)^2 = f$.

For $p \in \pi$, let $p^r \big|\big| |G|$. By the assumption, we have $p^r | \chi_i(1)$, so $p^{2r} | f$. But this contradicts $f \big| |G|$ and $p^r \big|\big| |G|$. The proof is finished. □

3. Proof of Theorem B

Lemma 3.1. *Let G be a group. If $\mathrm{dz}(G) = \pi(G)$, then G is a non-solvable group. Moreover, the minimal subnormal subgroup of G is a non-abelian simple group.*

Proof. We assert that every nontrivial normal subgroup N of G is non-abelian. Otherwise, let N be an abelian non-trivial subgroup of G, then $\chi(1) \big| |G : N|$ for every $\chi \in \mathrm{Irr}(G)$. But by assumption, for every $p \in \pi(N)$, there exists $\xi \in \mathrm{Irr}(G)$ such that $\xi(1)_p = |G|_p$, a contradiction.

By the fact that every finite solvable group contains an abelian minimal normal subgroup, we get that G is non-solvable.

By Clifford Theorem, it is easy to see that the condition $\mathrm{dz}(G) = \pi(G)$ is inherited by a normal subgroup, thus a minimal subnormal subgroup of G must be a non-abelian simple group. □

Since the group of order $p^a q^b$ is solvable, a finite group G satisfying $\mathrm{dz}(G) = \pi(G)$ has at least three prime factors by Lemma 3.1. That is to say, when we investigate a finite group G satisfying $\mathrm{dz}(G) = \pi(G)$, we must start from investigating a finite group with $|\pi(G)| = 3$, i.e., a K_3-simple group, they are A_5, A_6, $L_2(7)(= L_3(2))$, $L_2(17)$, $L_3(3)$, $U_4(2)(= S_4(3))$, $L_2(8)(= R(3)')$ and $U_3(3)$. Now we set up following lemma.

Lemma 3.2. *Let G be a group and $|G| = p^a q^b r$, where p, q, r are distinct primes with $p < q < r$. Suppose that G satisfies $\mathrm{dz}(G) = \pi(G)$. Then G is one of K_3-simple groups, $Aut(L_2(8))$, $Aut(L_3(3))$, $Aut(U_3(3))$ and subgroups of $Aut(A_6)$.*

Proof. Since every minimal normal subgroup N of G is a direct product of isomorphic non-abelian simple group by Lemma 3.1, we have $pqr \big| |N|$. But $r \big|\big| |G|$, so N is a non-abelian simple group and is a unique minimal normal subgroup of G. Consequently $C_G(N) = 1$, thus $N \trianglelefteq G \lesssim \mathrm{Aut}(N)$, and N is a K_3-simple group. For $N = A_5,\ A_6,\ L_2(7),\ L_2(17),\ L_3(3),\ U_4(2),\ L_2(8)$ and $U_3(3)$, checking character tables of group G such that $N \trianglelefteq G \lesssim \mathrm{Aut}(N)$ in [1], we found that $\mathrm{dz}(G) = \pi(G)$ holds only if $G = N$ is a K_3-group, $Aut(L_2(8))$, $Aut(L_3(3))$, $Aut(U_3(3))$ or a subgroup of $Aut(A_6)$. This completes the proof. □

The Proof of Theorem B. It is enough to prove the sufficiency. For any K_3-simple group M, it always follows $\mathrm{dz}(M) = \pi(M)$. By assumption $|G| = |M|$, one has that $\pi(G) = \pi(M)$, and $\mathrm{dz}(G) = \mathrm{dz}(M) = \pi(M) = \pi(G)$. Hence G satisfies conditions of Lemma 3.2, G is a K_3-simple group. The sufficiency follows from $|G| = |M|$. Thus Theorem B follows. □

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