NONHOLONOMIC FRAMES FOR FINSLER SPACE WITH DEFORMED (α,β)-METRIC

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Abstract. The purpose of present paper is to determine the Finsler spaces due to deformation of special (α,β)-metrics. Consequently, we determine the non-holonomic frames for Finsler space with help of Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$,one form metric $\beta = b_i(x)y^i$ and some special Finsler (α,β)- metric.
Keywords: (α,β)-metric, Riemannian metric, One form metric, Nonholonomic Finsler frame.

1. Introduction

In 1982, P.R. Holland [5, 6] studies about the nonholonomic frame on space time which was based on the consideration of a charged particle moving in an external electromagnetic field . Further in 1995, R.G. Beil [2] have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries.

In the present paper we have used the common Finsler idea to study the existence of a nonholonomic frame on the vertical sub bundle VTM of the tangent bundle of a base manifold M. In this case we have considered that the fundamental tensor field might be the deformation of four different special Finsler spaces from the (α, β)- metrics. First we consider a nonholonomic frame for a Finsler space with (α, β)- metrics such as:

I. $L(\alpha, \beta) = ((\alpha + \beta) + \frac{\alpha^2}{\beta})\alpha = \alpha^2 + \alpha\beta + \frac{\alpha^3}{\beta}$ i.e. product of (Randers + Kropina metric) and Riemannian metric.

II. $L(\alpha, \beta) = ((\alpha + \beta) + \frac{\alpha^2}{\beta})\beta = \alpha^2 + \alpha\beta + \beta^2$, i.e. product of (Randers + Kropina metric) and one form metric.

III. $L(\alpha, \beta) = (\alpha + \beta + \frac{\beta^2}{\alpha})\alpha = \alpha^2 + \alpha\beta + \beta^2$, i.e. product of first approximate Matsumoto Metric and Riemannian metric.

IV. $L(\alpha, \beta) = (\alpha + \beta + \frac{\beta^2}{\alpha})\beta = \beta^2 + \alpha\beta + \frac{\beta^3}{\alpha}$, i.e. product of first approximate Matsumoto Metric and one form metric.

Further we obtain a corresponding frame for each of these four Finsler deformations. This is an extension work of Ioan Bucataru and Radu Miron [4], Tripathi [12, 13] and Narasimhamurthy [10].

2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces (α, β)-metrics [8]. The first Finsler spaces with (α, β)-metrics were introduced by physicist G.Randers in 1940,are called Randers spaces [4].Recently, R.G. Beil suggested to consider a more general case, the class of Lagrange spaces with (α, β)-metric,which was discussed in [2]. A unified formalism which uses a nonholonomic frame on a space time,a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P. R. Holland [5, 6].If we do not ask for the function L to be homogeneous of order two with respect to the (α, β) variables,then we have Lagrange space with (α, β)-metric. Next we look for some different Finsler space with (α, β)-metrics.

Definition 2.1. *Let U be an open set of TM and $V_i : u \in U \longmapsto V_i(u) \in V_uTM, i \in \{1, 2.....n\}$ be a vertical frame over U. If $V_i(u) = V_i^j(u)\frac{\partial}{\partial y^j}|_u$,then $V_i^j(u)$ are the entries of invertible matrix for all $u \in U$.Denote by $\check{V}_k^j(u)$ the inverse of this matrix. This means that : $V_j^i\check{V}_k^j = \delta_k^i$, $\check{V}_j^i V_k^j = \delta_k^i$. We call V_j^i a nonholonomic Finsler Frame.*

Definition 2.2. *A Finsler space $F^n = \{M, F(x, y)\}$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by*

$$F^2(x, y) = L\{\alpha(x, y), \beta(x, y)\}, \tag{2.1}$$

where $\alpha^2(x, y) = a_{ij}(x)y^iy^j, \alpha$ is a Riemannian metric on the manifold M, and $\beta(x, y) = b_i(x)y^i$ is a 1-form on M.

Further consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space (M, F). Taking into account the homogeneity of a and F we have the following formulae:

$$\begin{aligned}
&p^i = \frac{1}{a}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j}; \quad P_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i};\\
&l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial l}{\partial y^j}; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = p_i + b_i\\
&l^i = \frac{1}{L}p^i; l^il_i = p^ip_i = 1; l^ip_i = \frac{\alpha}{L}; p^il_i = \frac{L}{\alpha};\\
&b_iP^i = \frac{\beta}{\alpha}; b_il^i = \frac{\beta}{L}
\end{aligned} \tag{2.2}$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by [13],

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_i p_j + p_i b_j + b_i b_j - \frac{\beta}{\alpha}p_i p_j = \frac{L}{\alpha}(a_{ij} - p_i p_j) + l_i l_j. \tag{2.3}$$

Theorem 2.1 ([4])**.** *For a Finsler space (M,F) consider the metric with the entries:*

$$Y^i_j = \sqrt{\frac{\alpha}{L}}(\delta^i_j - l^i l_j + \sqrt{\frac{\alpha}{L}}p^i p_j), \tag{2.4}$$

defined on TM. Then $Y_j = Y^i_j(\frac{\partial}{\partial y^i}), j \in 1,2,3,\ldots,n$ is a non holonomic frame.

Theorem 2.2 ([7])**.** *With respect to frame the holonomic components of the Finsler metric tensor $a_{\alpha\beta}$ is the Randers metric g_{ij}, i.e,*

$$g_{ij} = Y^\alpha_i Y^\beta_j a_{\alpha\beta}. \tag{2.5}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $y_i = a_{ij}y^j, \beta^i = a^{ij}b_j$, and so on. For a Finsler space with (α,β)-metric $F^2(x,y) = L\{\alpha(x,y),\beta(x,y)\}$ we have the Finsler invariants [9].

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial\alpha\partial\beta}; \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\right), \tag{2.6}$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with (α,β)-metric we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \tag{2.7}$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)–metric is given by [9].

$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\{b_i(x)y_j + b_j(x)y_i\} + \rho_{-2}y_i y_j. \tag{2.8}$$

From (2.8) we can see that $g_{ij}$is the result of two Finsler deformations:

$$I. \qquad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j).$$

$$II. \quad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-1} - \rho^2_{-1})b_i b_j. \tag{2.9}$$

The nonholonomic Finsler frame that corresponding to the I^{st} deformation (2.9) is according to the theorem (7.9.1) in [4], given by,

$$X^i_j = \sqrt{\rho}\delta^i_j - \frac{1}{\beta^2}\{\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}}\}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{2.10}$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X_i^k X_j^l a_{kl}. \tag{2.11}$$

Again the frame that corresponds to the II^{nd} deformation (2.9) given by,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \left(\frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}\right)}\}b^i b_j, \tag{2.12}$$

where $C^2 = h_{ij}b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula;

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \tag{2.13}$$

Theorem 2.3 ([4]). *Let $F^2(x,y) = L\{\alpha(x,y), \beta(x,y)\}$ be the metric function of a Finsler space with (α,β) metric for which the condition* (2.7) *is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (2.10) *and* (2.12) *respectively.*

3. Nonholonomic frames for Finsler geometry with (α, β)-metric

In this section we consider four cases of nonholonomic Finlser frames with special (α, β)-metrics, such a I^{st} Finsler frame product of (Randers metric + Kropina metric) and Riemannian metric ; II^{nd} Finsler frame product of (Randers metric + Kropina metric) and 1-form metric;III^{rd} Finsler frame product of approxomate Matsumoto metric and Riemannian metric ;IV^{th} Finsler frame product of approxomate Matsumoto metric and 1-form metric.

3.1 Nonholonomic frame for $L = (\alpha + \beta + \frac{\alpha^2}{\beta})\alpha = \alpha^2 + \alpha\beta + \frac{\alpha^3}{\beta}$

In the first case, for a Finsler space with the fundamental function $L = (\alpha + \beta + \frac{\alpha^2}{\beta})\alpha = \alpha^2 + \alpha\beta + \frac{\alpha^3}{\beta}$ the Finsler invariants (2.6) are given by

$$\begin{aligned} \rho_1 &= 1 + \frac{\beta}{2\alpha} + \frac{3\alpha}{2\beta}, \rho_0 = \frac{\alpha^2\beta - \alpha^3}{2\beta^2}, \\ \rho_{-1} &= \frac{\beta^2 - 3\alpha^2}{2\alpha\beta^2}, \quad \rho_{-2} = \frac{3\alpha^2 - \beta^2}{2\alpha^3\beta}, \\ B^2 &= \frac{(\beta^2 - 3\alpha)^2(\alpha^2 b^2 - \beta^2)}{4\alpha^4\beta^4}. \end{aligned} \tag{3.1}$$

Using (3.1) in (2.10) we have,

$$X^i_j = \sqrt{1+\frac{\beta}{2\alpha}+\frac{3\alpha}{2\beta}}\delta^i_j - \frac{(\beta^2-3\alpha^2)^2}{4\alpha^2\beta^6}[\sqrt{1+\frac{\beta}{2\alpha}+\frac{3\alpha}{2\beta}} \sqrt{1+\frac{\beta}{2\alpha}+\frac{3\alpha}{2\beta}}+\frac{2\alpha^3\beta^3}{(3\alpha^2-\beta)^2}](b^i-\frac{\beta}{\alpha^2}y^i)(b_j-\frac{\beta}{\alpha^2}y_j). \tag{3.2}$$

Again using (3.1) in (2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\{1 \pm \sqrt{1+\frac{2\beta^3C^2}{\alpha\beta^3-\alpha^3\beta-3\alpha^3+\alpha\beta^2}}\}b^ib_j, \tag{3.3}$$

where $C^2 = (1+\frac{\beta}{2\alpha}+\frac{3\alpha}{2\beta})b^2 - \frac{(3\alpha^2-\beta^2)}{2\alpha^3\beta^3}(\alpha^2b^2-\beta^2)^2$.

Theorem 3.1. *Let* $L = (\alpha+\beta+\frac{\alpha^2}{\beta})\alpha = \alpha^2+\alpha\beta+\frac{\alpha^3}{\beta}$ *be the metric function of a Finsler space with* (α,β) *metric for which the condition* (2.7) *is true. Then*

$$V^i_j = X^i_kY^k_j$$

is nonholonomic Finsler Frame with X^i_k *and* Y^k_j *are given by* (3.2) *and* (3.3) *respectively.*

3.2 Nonholonomic frame for $L = (\alpha+\beta+\frac{\alpha^2}{\beta})\beta = (\alpha+\beta+\frac{\beta^2}{\alpha})\alpha = \alpha^2+\alpha\beta+\beta^2$

In the second and third case, for a Finsler space with the fundamental function $L = (\alpha^2+\alpha\beta+\beta^2)$ are the same , the Finsler invariants (2.6) are given by

$$\begin{aligned} &\rho_1 = 1+\frac{\beta}{2\alpha}, \quad \rho_0 = 1,\\ &\rho_{-1} = \frac{1}{2\alpha}, \quad \rho_{-2} = \frac{-\beta}{2\alpha},\\ &B^2 = \frac{(\alpha^2b^2-\beta^2)}{4\alpha^4}. \end{aligned} \tag{3.4}$$

Using (3.4) in (2.10) we have,

$$\begin{aligned} X^i_j = &\sqrt{1+\frac{\beta}{2\alpha}}\delta^i_j - \frac{1}{4\alpha^2\beta}[\sqrt{1+\frac{\beta}{2\alpha}}\\ &+\sqrt{1+\frac{\beta}{2\alpha}}-2\alpha^3\beta](b^i-\frac{\beta}{\alpha^2}y^i)(b_j-\frac{\beta}{\alpha^2}y_j). \end{aligned} \tag{3.5}$$

Again using (3.4) in (2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\{1 \pm \sqrt{1+\frac{2\alpha\beta C^2}{1+2\alpha\beta}}\}b^ib_j, \tag{3.6}$$

where $C^2 = (1+\frac{\beta}{2\alpha})b^2 - \frac{1}{2\alpha^3\beta}(\alpha^2b^2-\beta^2)^2$.

Theorem 3.2. *Let* $L = (\alpha + \beta + \frac{\alpha^2}{\beta})\beta = (\alpha + \beta + \frac{\beta^2}{\alpha})\alpha = \alpha^2 + \alpha\beta + \beta^2$ *be the metric function of a Finsler space with (α,β) metric for which the condition* (2.7) *is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is nonholonomic Finsler Frame with X_k^i *and* Y_j^k *are given by* (3.5) *and* (3.6) *respectively.*

3.3 Nonholonomic frame for $L = (\alpha + \beta + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \beta^2 + \frac{\beta^3}{\alpha}$

In the fourth case, for a Finsler space with the fundamental function $L = (\alpha + \beta + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \beta^2 + \frac{\beta^3}{\alpha}$ the Finsler invariants (2.6) are given by

$$\begin{aligned} \rho_1 &= \frac{\alpha^2\beta - \beta^3}{2\alpha^3}, \quad \rho_0 = \frac{\alpha + 3\beta}{\alpha}, \\ \rho_{-1} &= \frac{\alpha^2 - 3\beta^2}{2\alpha^3}, \quad \rho_{-2} = \frac{3\beta^3 - \alpha\beta^2}{2\alpha^5}, \\ B^2 &= \frac{(\alpha^2 - 3\beta^2)^2(\alpha^2 b^2 - \beta^2)}{4\alpha^8}. \end{aligned} \tag{3.7}$$

Using (3.7) in (2.10) we have,

$$\begin{aligned} X_j^i = &\sqrt{\frac{\alpha^2\beta - \beta^3}{2\alpha^3}}\delta_j^i - \frac{(\alpha^2 - 3\beta^2)^2}{4\alpha^6\beta^2}[\sqrt{\frac{\alpha^2\beta - \beta^3}{2\alpha^3}} \\ &+ \sqrt{\frac{\alpha^2\beta - \beta^3}{2\alpha^3} + \frac{2\alpha^5\beta}{3\beta^2 - \alpha^2}}](b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j). \end{aligned} \tag{3.8}$$

Again using (3.7) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\{1 \pm \sqrt{1 + \frac{2\alpha\beta C^2}{\alpha^2 + 3\beta^2 + 2\alpha\beta}}\}b^i b_j, \tag{3.9}$$

where $C^2 = (\frac{\alpha^2\beta - \beta^3}{2\alpha^3})b^2 + \frac{(3\beta^2 - \alpha^2)}{\beta}(\alpha^2 b^2 - \beta^2)^2$.

Theorem 3.3. *Let* $L = (\alpha + \beta + \frac{\beta^2}{\alpha})\beta = \alpha\beta + \beta^2 + \frac{\beta^3}{\alpha}$ *be the metric function of a Finsler space with (α,β) metric for which the condition* (2.7) *is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is nonholonomic Finsler Frame with X_k^i *and* Y_j^k *are given by* (3.8) *and* (3.9) *respectively.*

4. Conclusion

Nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli and Bucataru [1, 2], have determined such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [10]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with (α,β)-metric, it appears a natural question: Does how many Finsler space with(α,β)-metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with(α, β)-metrics.

In this work, we consider the special Finsler (α,β) metrics,first approximate Matsomoto metric, Riemannian metric and 1-form metric we determine the nonholonomic Finsler frames. But, in Finsler geometry, there are many (α,β)-metrics, in future work we can determine the frames for them also.

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