DERIVED NUMBERS OF ONE VARIABLE MONOTONIC FUNCTIONS

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Abstract. The Fermat, Roll and Lagrange theorems are generalized into the class of nondifferentiable functions, the necessary and sufficient conditions for monotonicity one variable functions are given.

Keywords: derived number, periodic solution, almost periodic solution, nonsmooth analysis, Dini-Hölder derived number.

1. Introduction

In this paper, a method of periodic and almost periodic ordinary differential equations development is considered. It is based on the ideas of functional analysis. I.P. Natanson briefly outlined the theory of derived numbers [1]. Developing this theory, several theorems of mathematical analysis are proved. Implementation of this theory let reducing the restrictions on smoothness degree of the right-hand sides of the equations considered, which made it possible to extend the scope of the results obtained [2-11]. In many problems of classical and celestial mechanics, robotics and mechatronics, there are processes which the time dependence is not periodic in [12-21]. In this connection, the interest in derived theory implementation to the study of periodic and almost periodic solutions of differential equations and differential equations with almost periodic coefficients has arisen [22-26].

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2. Basic definitions

Let f be a function defined on an open interval (a, b), taking values in the set of real numbers R, i.e. $f : (a, b) \to R$, $a, b \in R$, $a < b$. Consider an arbitrary point x_0 in (a, b).

Let a number λ be a derived number of function f at x_0 if there exists a sequence $\{x_k\}$, such that $x_k \to x_0$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{x_k - x_0} = \lambda.$$

The fact that λ is the derived number of function f at x_0 is represented as $\lambda = \lambda[f](x_0)$.

The set of all derived numbers of function f at x_0 is denoted by $\Lambda[f](x_0)$

If in the definition of a derived number it is required the sequence $\{x_k\}$ to satisfy one more additional condition, which means that for all k the inequality $x_k - x_0 > 0$ is fulfilled, then such derived number is determined as the right derived number and denoted by $\lambda^+[f](x_0)$. If $x_k - x_0 < 0$ for all k, then such derived number is determined as the left derived number of function f at x_0 and denoted by $\lambda^-[f](x_0)$

Let the set of right derived number of function f at x_0 be denoted by $\Lambda^+[f](x_0)$, and the set of left derived number be denoted by $\Lambda^-[f](x_0)$.

It is clear that $\sup_{\lambda\in\Lambda^+[f](x_0)} \lambda$ determines $D^+f(x_0)$ that is the right upper derived number of a Dini function at a point x_0. Similarly, the remaining three derived number of Dini function at a point x_0 can be introduced.

Suppose

$$\lambda^\alpha = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\alpha}.$$

In this relation consider α such that for any $\varepsilon > 0$ the equalities $\lambda^{\alpha-\varepsilon} = 0$ and $\lambda^{\alpha+\varepsilon} = \infty$ are realised. If the function f is defined in some neighborhood of the point x_0, then such α obviously exists. The magnitude can depend only on the choice of convergence to x_0 of the subsequence $\{x_k\}$.

Let the number λ be called the derived number of a Hölder function at x_0 if there exist $\alpha \leq 0$ and a sequence $\{x_k\}$ converging to x_0, such that

$$\lambda = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\alpha},$$

and for any $\varepsilon > 0$

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha-\varepsilon}} = 0,$$

and

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha+\varepsilon}} = \infty.$$

Let the number α appearing in the definition of the Hölder derived number be called the exponent of this derived number.

The fact that λ is a Hölder derived number of function f at x_0 can be represented as following:

$$\lambda = \lambda_H[f](x_0).$$

The set of Hölder derived number of function f at x_0 is denoted by $\Lambda_H[f](x_0)$.

If in the definition of the Hölder derived number it is required that $x_k - x_0 > 0$ for all k, then such a derived number is determined as the right Hölder derived number and denoted by $\lambda_H^+[f](x_0)$. If $x_k - x_0 < 0$ for all k, then such a derived number is determined as the left Hölder derived number and denoted by $\lambda_H^-[f](x_0)$.

Let the set of all right Hölder derived numbers of function f at x_0 be denoted by $\Lambda_H^+[f](x_0)$, and the set of all left Hölder derived numbers at the same point be denoted by $\Lambda_H^-[f](x_0)$.

Let α^+ denote the minimal of the exponents of the derived numbers being into $\Lambda_H^+[f](x_0)$, and $\Lambda_H^{\alpha^+}[f](x_0)$ denote a set of derived numbers belonging to the set $\Lambda_H^+[f](x_0)$ and having the exponent α^+. Similarly, for a set $\Lambda_H^-[f](x_0)$, a number α^- and a set $\Lambda_H^{\alpha^-}[f](x_0)$ are introduced.

Let the number

$$\lambda = \sup_{\mu \in \Lambda_H^{\alpha^+}[f](x_0)} \mu$$

be called the right upper derivative of Dini-Hölder function f at x_0 and denoted by $DH^+[f](x_0)$.

Let the number

$$\lambda = \inf_{\mu \in \Lambda_H^{\alpha^+}[f](x_0)} \mu$$

be called the right lower derivative of Dini-Hölder function f at x_0.

Analogously, the notions of the left upper and left lower Dini-Hölder derivatives of function f at x_0 are introduced. These derivatives are denoted by $DH^-[f](x_0)$ and $DH_-[f](x_0)$, respectively. Let DH^*f denote any of the four Dini-Hölder derivatives of the function f.

3. Monotonic functions

Let the function f be called monotone if it follows from $x < y$ that $f(x) \leq f(y)$ or $f(x) \geq f(y)$. In the first case, the function f is called increasing, and in the second case decreasing. On the other hand, if from $x < y$ follows that $f(x) < f(y)$ or $f(x) > f(y)$, then f is called strictly monotonic. In oder to emphasize what kind of monotony is under consideration, then we stay that the function f strictly increases if $f(x) < f(y)$ follows from $x < y$, and strictly decreases if $f(x) \leq f(y)$ follows from $x < y$.

Theorem 1. *In order for the continuous function f to be strictly monotonic, it is necessary and sufficient that all its Dini derivatives be constant-sign and* *there is no interval at which at least one of the Dini derivatives of the function f is equal to zero.*

Proof. *Necessity.* Let f be continuous, and for definiteness assume that it is strictly increasing. Consider an arbitrary point x_0 and a sequence $\{x_k\}$ converging to it. Without loss of generality, a sequence $\{x_k\}$ can be chosen such that there exists a limit $\lim_{k\to\infty} \frac{f(x_k)-f(x_0)}{x_k-x_0}$.

By assumption, the function f strictly increases, and therefore for all k at once

$$\frac{f(x_k)-f(x_0)}{x_k-x_0} > 0.$$

Passing to the limit as $k \to \infty$ in this inequality, it follows that

$$\lim_{k\to\infty} \frac{f(x_k)-f(x_0)}{x_k-x_0} \geq 0.$$

This implies that all the Dini derivatives of the function f are constant-sign due to arbitrariness of the point x_0 and the sequence $\{x_k\}$. The case when f is strictly decreasing is treated similarly.

Let us show now that if f is strictly monotone and continuous, then the second condition of the theorem is also satisfied. Suppose, on the contrary, there exists an interval $[a, b]$ and at each point of it at least one of the Dini derivatives of the function f has a zero value. By continuity and monotonicity the function f transforms the interval $[a, b]$ to some interval $[\alpha, \beta]$ with measure $m[\alpha, \beta] = \beta - \alpha > 0$.

As follows from [4], if for some $p \geq 0$ the strictly monotonic function f at each point of the set $E \subset [a, b]$ has at least one Dini derivative such that

$$|D^*[f](x)| \leq p,$$

then

$$m^* f(E) \leq p \cdot m^* E,$$

where m^* denotes the outer measure of the corresponding set.

Applying this result to the considered case, the following is realised:
But this contradicts the inequality $|\beta - \alpha| > 0$ obtained above. Therefore the interval $[a, b]$ at each point of which at least one of the Dini derivatives of the function f vanishes can not exist.

Sufficiency. Let the conditions of the theorem be satisfied. Let us show that in this case the function f is strictly monotone. Suppose on the contrary that f is not monotonic. Then it is not monotonically increasing. Therefore, there exist two points x_1 and x_2, $x_1 < x_2$ such that $f(x_1) > f(x_2)$. Consider a straight line $l(x) = qx + r$ through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Note that q is necessarily less than zero since $f(x_1) > f(x_2)$.

Consider all situations which arise at such construction. Suppose first that there exists a decreasing sequence $\{x_k^{'}\}$ converging to x_1 and such that $f(x_k^{'}) \leq l(x_k^{'})$ for all k. Then, taking into account that $f(x_1) = l(x_1)$

$$\frac{f(x_k^{'}) - f(x_1)}{x_k^{'} - x_1} \leq \frac{l(x_k^{'}) - l(x_1)}{x_k^{'} - x_1} = q < 0.$$

Passing to the limit $k \to \infty$ in this inequality, it can be assumed without loss of generality to exist. Then it can be concluded that the function f at the point x_1 has a negative derivative number that does not exceed q.

Suppose that such sequence $\{x_k^{'}\}$ does not exist, but there exist an increasing sequence $\{x_k^{''}\}$ converging to x_2 and such that $f(x_k^{''}) \geq l(x_k^{''})$ for all k. Repeating the arguments given for the sequence $\{x_k^{'}\}$also for the sequence $\{x_k^{'}\}$, it follows that the function f at the point x_2 has a derivative number not exceeding q.

Finally, if neither sequence $\{x_k^{'}\}$ nor sequence $\{x_k^{''}\}$ exists, then this means that $f(x) > l(x)$ for all points $x > x_1$ from a sufficiently small neighborhood of x_1, and $f(x) < l(x)$ for all points $x < x_2$ from a sufficiently small neighborhood of x_1. Then, in view of continuity of the function f, there exist a point $x_0 \in (x_1, x_2)$ and a decreasing sequence $\{x_k^*\}$ convergent to x_0, such that $f(x_0) = l(x_0)$ and $f(x_k^*) \leq l(x_k^*)$ for all k. But this situation completely coincides with the situation with the sequence $\{x_k^{'}\}$ considered. Therefore, the same conclusion made for the point x_1 in the presence of a sequence $\{x_k^{'}\}$ is also valid for a point x_0.

Thus, in any of the possible situations, there exists at least one point in which the function f has a negative derivative number, and therefore there exists at least one point at which at least one of the Dini derivatives of the function f is negative.

Further, the function f that is monotonic is also not decreasing. Then there are two points y_1 and y_2, $y_1 < y_2$ such that $f(y_1) < f(y_2)$. Repeating for the points y_1 and y_2 the arguments given above, it follows that there exists at least one point at which at least one of the Dini derivatives of the function f is positive.

So, if the function f is not monotonic, then it necessarily has Dini derivatives of different signs, which contradicts the requirements of the theorem. Therefore, if the conditions of the theorem are satisfied, then the function f is monotone.

Let f be a monotonic function, but not strictly monotone. Then there are two points z_1 and z_2 such that $f(z_1) = f(z_2)$. Since f is monotonic, then $f(x) = f(z_1)$ for all $x \in [z_1, z_2]$. In this case, $D^*[f](x) = 0$ for all $x \in (z_1, z_2)$, which contradicts the assumption of the theorem about the absence of intervals of this type.

Thus, if the conditions of the theorem are satisfied, then the function f is necessarily strictly monotone.

From the presented proof and the fact that if any of the Dini derivatives of the function f continuous on $[a, b]$ is constant on (a, b), then the same condition is true for the other three Dini derivatives, it follows that Theorem 7 can be reformulated as follows:

In order for the function f continuous on $[a, b]$ to be strictly monotonic, it is necessary and sufficient that one of the Dini derivatives of this function be constant-sign on (a, b) and that there exists no interval on which the function f has a derivative equal to zero.

Theorem 2. *If at each point of interval $[a, b]$ the continuous function f has a positive right derivative, then f strictly increases on $[a, b]$.*

Proof. Suppose f is a function not increasing on $[a, b]$. Then, by continuity it reaches its local maximum at some point $x_0 \in [a, b)$. But as shown in the proof of Theorem 6 [27], the right derivative of the function f does not exceed zero at this point. This contradiction shows that f is an increasing function.

If f increases but not strictly, then, as shown in the proof of Theorem 7, there exists an interval at each point of which the right derivative of the function f is equal zero. From this fact it follows a contradiction again.

Thus, if the right derivative of the function f is positive, then f is necessarily strictly increasing.

Theorem 3. *Let f be defined on $[a, b]$. If $D_+ f \geq 0$ and f does not have jumps down, then it increases on $[a, b]$.*

Proof. Suppose f satisfies the conditions of the theorem. Let us construct a function ϕ by setting that

$$\phi(x) = f(x) + \varepsilon x, \qquad \varepsilon > 0.$$

It is clear that ϕ is defined on $[a, b]$ and also has no jumps down.

Assume $\phi(a) > \phi(b)$. Let us construct a straight line $y = \frac{1}{2}[\phi(a) + \phi(b)]$. Since ϕ has no jumps down, then there exist points on (a, b) that are roots of the equation $\phi(x) = y$. Let c denote an exact lower bound of a set of roots for this equation. $\phi(c) = y$ since if $\phi(c) > y$, then obviously the function ϕ has a right derived number equal to $-\infty$ at the point c. But this is impossible, since the definition of the function ϕ implies that all its right-derived numbers are nonnegative. If $\phi(c) < y$, then by the assumption that the function f has no downward jumps there must exist a point $c^{'} < c$ such that $\phi(c^{'}) = y$ which contradicts the choice of the point c.

Consider now the interval $[a, c]$. It's clear that $\phi(a) > \phi(c)$. Let us repeat for the interval $[a, c]$ the construction made for the interval $[a, b]$. As a result, the minimum point c_1 is obtained on (a, c) for which $\phi(c_1) = \frac{1}{2}[\phi(a) + \phi(c)]$.

Continuing this process, a decreasing sequence $c > c_1 > c_2 > \ldots$ is constructed. This sequence is bounded below by a number a and, consequently, has a limit. Let this limit be denoted by c_0. Likewise for the point c, it is shown that $\phi(c_0) = \phi(a)$.

From the construction given above it follows that for any n

$$\frac{\phi(c_n) - \phi(c_0)}{c_n - c_0} < 0.$$

This inequality implies that the function ϕ has a nonpositive lower right Dini derivative at the point c_0.

But as follows from the definition of the function ϕ, all its right derivatives in $[a, b]$ are not less then ε. This contradiction shows that the inequality $\phi(a) > \phi(b)$ is impossible. Hence, $\phi(a) \leq \phi(b)$ or $f(a) + \varepsilon a \leq f(b) + \varepsilon b$. Since ε is arbitrary, then passing to the limit in the last inequality as $\varepsilon \to 0$ it follows that $f(a) \leq f(b)$, which proves the theorem, since there could be any interval $[x, y] \subset [a, b]$ taken instead of $[a, b]$.

Note that if $D_+ f \geq 0$, then the condition that function f has no downward jump is satisfied, in particular, if $D_- f$ is bounded below.

As an example, consider the function f with a derivative defined on an interval $I = [0, 1]$ as follows:

$$f'(x) = 1, x \in E_1 0, x \in E_2, 1.2$$

where E_1 is a set of measure zero dense everywhere on I, and I.

According to Theorems 1 [27] and 1, by virtue of the density of E_1, the function f is continuous and strictly increasing in I. But then there is a chain of inequalities [4]:

$$m^* f(I) \leq m^* f(E_1) + m^* f(E_2) \leq 1 \cdot m^* E_1 + 0 \cdot m^* E_2 = 0,$$

where $m^* E$ denotes the outer measure of the set E. It obviously follows from the inequality obtained that an equality $mf(I) = 0$ holds, which contradicts the conclusion that the function f is strictly increasing on I, since if f is continuous and strictly increasing, then it transforms the interval $[0, 1]$ into some interval $[\alpha, \beta]$ with measure $m[\alpha, \beta] = \beta - \alpha > 0$.

Thus, these arguments lead to contradiction, since on the one hand, on the basis of Theorems 1 [27] and 1 the function f is continuous and strictly increasing on I, and on the other hand, according to the theory of monotone functions it follows from equality $mf(I) = 0$ that $f(0) = f\ (1)$, or in other words that f is constant on I. This contradiction shows that there is no function f. In particular, there is no function f which the Dirichlet function would play the role of the derivative for.

Let the function f be called almost continuous on the interval $[a, b]$ if there exists a continuous function g on $[a, b]$ such that $mE(f \neq g) = 0$, put this another way, if the changing the values of the function f on a set of measure zero implies that it can be made continuous.

Theorem 4. *If all the Dini derivatives of the function f are almost continuous and bounded on the interval $[a, b]$ functions, then f is continuously differentiable on $[a, b]$.*

Proof. If the conditions of the theorem are satisfied, then, the function $\Lambda[f](x)$ is obviously almost continuous. Let $\Lambda[f](x)$ be represented in the form

$$\Lambda[f](x) = g_1(x) + g_2(x) + g_3(x),$$

where g_1 is a function continuous on $[a, b]$, such that

$$\begin{aligned} &mE(\Lambda[f] \neq g) = 0, \\ &g_3(x) = \min(0, \Lambda[f](x)), \\ &g_2(x) = \Lambda[f](x) - g_1(x) - g_3(x). \end{aligned}$$

Let the function f can be represented as:

$$f = f_1 + f_2 + f_3,$$

where

$$\frac{df_1}{dx} = g_1, \qquad \frac{df_3}{dx} = g_3, \qquad \Lambda[f_2] = g_2.$$

It is easy to verify that the functions g_2 and g_3 are almost everywhere equal to zero, where g_3 is continuous, and g_2 does not take negative values. Thus, for each of the functions g_2 and g_3 the arguments given in the example above are applicable. Repeating these arguments, it follows that f_2 and f_3 are constants on $[a, b]$, which implies that

$$\frac{df}{dx} = \frac{df_1}{dx} = g_1,$$

so, f is continuously differentiable.

To complete the proof, it remains only to show that representation of the function f as a sum of three functions is possible. The validity of such a representation follows from the fact that the function $\bar{f} = f - f_1$ is continuous as a difference between two continuous functions, and the function $\Lambda[\bar{f}](x) = \Lambda[f](x) - g_1(x)$ are almost everywhere equal to zero.

4. Mean-value theorems

Thereafter it is assumed that all right-derived numbers of the function f are equal to each other, i.e. that the function f has a right derivative. Let us prove theorems that are analogous to the Rolle and Lagrange theorems for a considered class of functions.

Theorem 5. *If the function f is defined and continuous on the interval $[a, b]$, has a continuous right derivative (a, b) and has equal values at the ends of the interval, then there exists a point $x_0 \in (a, b)$ at which $f'^{+}(x_0) = 0$.*

Proof. Since the function f is continuous, then at some interior point x_0 of the interval $[a, b]$ it reaches its extremum. As stated in the theorem, f'^{+} is continuous in a neighborhood of point x_0, which implies that the right derivative vanishes at the point x_0 by virtue of Theorem 6 [27].

Theorem 6. *Suppose the function f is defined and continuous on the interval $[a,b]$ and has a right derivative in (a,b). Then there exist points $x_1 \in (a,b)$ and $x_2 \in (a,b)$ for which the following inequalities hold*

$$f'^{+}(x_1) \leq \frac{f(b)-f(a)}{b-a} \leq f'^{+}(x_2).$$

If f has a continuous right derivative in (a,b), then there exists a point $x_0 \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'^{+}(x_0).$$

Proof. Suppose that at each point of the open interval (a,b) the function f has a right derivative, and, contrary to the assertion of the theorem, the point $x_2 \in (a,b)$ is such that

$$\frac{f(b)-f(a)}{b-a} \leq f'^{+}(x_2),$$

does not exist. Then for any $x \in (a,b)$

$$\frac{f(b)-f(a)}{b-a} - f'^{+}(x) = \alpha(x) > 0.$$

Let us construct the following function

$$\phi(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(a) - f(x).$$

The function ϕ is continuous, since it is a linear combination of continuous functions. It is easy to verify that $f'^{+}(x) = \alpha(x)$ for all $x \in (a,b)$. By assumption, $\alpha(x) > 0$ for all $x \in (a,b)$. Therefore, the function ϕ is strictly increasing by virtue of Theorem 8. But if ϕ is continuous and strictly increasing, then inequality $\phi(a) < \phi(b)$ must necessarily be fulfilled, while direct substitution shows that $\phi(a) = \phi(b) = 0$. This contradiction proves the existence of the point $x_2 \in (a,b)$ considered in the theorem.

Similarly, by contradiction, the existence of the point $x_1 \in (a,b)$ is proved, where

$$f'^{+}(x_1) \leq \frac{f(b)-f(a)}{b-a}.$$

Combining these two inequalities, the desired result is obtained.

If in addition it is known that the function f'^{+} is continuous on (a,b), then the function ϕ constructed above obviously satisfies all the requirements of Theorem 11, and consequently, by virtue of this theorem there must exist a point x_0 in (a,b) at which ϕ'^{+} vanishes. But

$$\phi'^{+}(x_0) = \frac{f(b)-f(a)}{b-a} - f'^{+}(x_0),$$

which the required equality is obtained from.

Conclusion. The method of derived numbers to study periodic and almost periodic solutions of ordinary differential equations is developed. Necessary and sufficient conditions for the monotonicity of one variable functions are presented.

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