RING FORMS

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Abstract. Determinant on a commutative ring of characteristics p can be extended by a linear mapping to provide a trilinear alternating form. We show some basic properties of such forms.

If the underlying ring is a chain-ring, we compute dimensions of radicals of all vectors and thus prove nonequivalence of forms arising from chain-rings with different sizes of ideals. Moreover, in the case $p = 2$ we show that all three nondegenerate forms on dimension 6 are ring forms.

Keywords: trilinear alternating form, commutative ring, chain ring.

1. Introduction

Let $f : V^3 \to F$ be a trilinear form on a vector space V over a field F, $\dim V = n < \infty$. The form f is called *alternating* if $f(u, v, w) = 0$ whenever two of the input vectors are equal. Two forms f and g on V are *equivalent* if there exists an automorphism of V satisfying $f(u, v, w) = g(\phi(u), \phi(v), \phi(w))$ for all $u, v, w \in V$. Classification of classes of this equivalence seems to be a very difficult problem (unlike in the bilinear case) even for small dimensions of V and not much has been done in this respect. This classification was done for the case $n \leq 7$ in [1] for a large family of fields including all finite fields and Gurevitch [2], D. Djokovic [3] and L. Noui [4] solved the case $n = 8$ for $F = \mathbf{C}$, $F = \mathbf{R}$ and F algebraically closed field of arbitrary characteristic, respectively. The case of dimension 8 over $\mathbf{GF}(2)$ is solved in [7].

Trilinear alternating forms over the two-element field appear as important invariants of doubly even binary codes and thus the accent is put on the case of characteristic 2.

In this paper we study trilinear alternating forms that arise as extensions of determinant over a ring R with a linear mapping $l : R \to \mathbf{GF}(p)$. This construction is a generalization of so called trace-derived forms introduced in [8]. We show when such a form is nondegenerate and when it is decomposable. If the ring R is a chain-ring then there exists a mapping l yielding a nondegenerate form. Two chain-rings with different sizes (or number) of ideals give rise to two nonequivalent forms.

There are only three four-element rings of characteristic 2 and we show that they provide exactly the three nondegenerate forms on dimension 6 over **GF**(2).

2. Definitions

A trilinear alternating form f satisfies the equality:

$$f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) = \mathrm{sgn}(\sigma) f(v_1, v_2, v_3),$$

for every permutation $\sigma \in S_3$. Since this paper deals mainly with forms over the two-element field, this equation often collapses into symmetry.

We shall denote the bilinear form $f(v, -, -)$ by $f[v]$ and similarly $f[v_1, v_2]$ shall denote the linear form $f(v_1, v_2, -)$.

An automorphism ϕ of V is said to be an *automorphism of the form* f if

$$f(v_1, v_2, v_3) = f(\phi(v_1), \phi(v_2), \phi(v_3)) \text{ for all } v_1, v_2, v_3 \in V.$$

The group of automorphisms of f will be denoted by $\mathrm{Aut}(f)$.

Let f be a trilinear form on V. The set

$$\{x \in V; f[x] = 0\}$$

is called the *radical* of f and will be denoted by $\mathrm{Rad} f$. If $\mathrm{Rad} f$ is trivial (contains only the zero vector), then f is called *nondegenerate*.

Fix $v \in V$ and define the *radical* $\mathrm{Rad}_f(v)$ *of* v as:

$$\mathrm{Rad}_f(v) = \{u \in V; f[v, u] = 0\}.$$

If it is clear which form is meant, we shall omit the index f. The radical of any vector v is clearly a subspace of V. The *rank* of $v \in V$ is the codimension of $\mathrm{Rad}(v)$ in V

$$r(v) = n - \mathrm{dim}\mathrm{Rad}(v).$$

To capture the information about ranks of vectors of forms (over finite fields) we shall use an invariant introduced in [7], called the *radical polynomial*

$$P(f) = \sum_{v \in V} x^{r(v)} y^{n-r(v)}.$$

$P(f)$ is a homogenous polynomial of degree n and if written in the form

$$P(f) = \sum_{i=0}^{n} \alpha_i x^i y^{n-i} \tag{1}$$

then every α_i is a nonnegative integer and $\sum_{i=0}^{n-1} \alpha_i = q^n$. Since for every $u \in V$ we have $u \in \mathrm{Rad}(u)$, the rank $r(u)$ of any vector u is less than n and the sum in (1) can run only to $n-1$. Moreover, by Proposition 2.1 we get α_i is equal to zero whenever i odd.

Suppose that there is a fixed trilinear alternating form f on a vector space V. We say that nonzero vectors $u, v \in V$ are orthogonal, denoted by $u \perp v$, if $u \in \mathrm{Rad}(v)$. This relation is clearly reflexive and symmetric (the form is alternating) but is not necessarily transitive.

We shall use standard notation for forms: Let V be an n-dimensional vector space over a field F and fix a basis $B = \{b_1, \dots, b_n\}$ of V. Denote by $B^* = \{b_1^*, \dots, b_n^*\}$ its dual basis (defined as usual by $b_i^*(b_j) = \delta_{ij}$). Given B and B^* as above, a k-linear alternating form f can be expressed as

$$f_B = \sum_{1 \le i_1 < \dots < i_k \le n} f_{i_1 \dots i_k} b_{i_1}^* \wedge \dots \wedge b_{i_k}^*,$$

where the index B indicates the dependence of the presentation upon the chosen basis. Denote by Δ_f the set

$$\Delta_f = \{(i_1, \dots, i_k) \mid 1 \le i_1 < \dots < i_k \le n, f_{i_1 \dots i_k} \neq 0\}.$$

We shall shorten this notation to $f_B = \sum_{\Delta_f} f_{i_1 \dots i_k} \underline{i_1 \dots i_k}$.

If V is a vector space over the two element field $F = \mathbf{GF}(2)$ and $k = 3$, then to give a form f means to point out triples $\{i, j, k\}$ satisfying $f(b_i, b_j, b_k) = 1$, i.e., to give the set Δ.

In what follows we shall often use the well known characterization of bilinear alternating forms:

Proposition 2.1. *Let f be a bilinear alternating form on a vector space V of dimension n. Then there exists a basis $B = \{b_1, \dots, b_n\}$ and $k \le n$ such that*

$$f_B = \underline{12} + \underline{34} + \dots + \underline{(k-1)k}.$$

For two subspaces V_1 and V_2 of V we write $V_1 \perp V_2$ if $v_1 \perp v_2$ for any $v_1 \in V_1$ and $v_2 \in V_2$. We say that a nondegenerate form f on V is *decomposable* if $V = W_1 \oplus \dots \oplus W_m$, $m \ge 2$, and $W_i \perp W_j$ whenever $i \neq j$. Given an (orthogonal) decomposition of f, let π_i denote the projection of V onto W_i and f_i the restriction of f to W_i. Then we can express the form f as

$$f(u, v, w) = \sum_i f_i(\pi_i(u), \pi_i(v), \pi_i(w)),$$

and we shall write $f = \bigoplus f_i$. The difference between the bilinear case and k-linear case, $k \ge 3$, is that the finest decomposition of a nondegenerate multilinear form is unique, see [6]. On the other hand, there are many indecomposable trilinear forms if the dimension of V is at least six.

3. Ring forms

In this paper we are going to study trilinear forms arising primarily as determinants over a commutative ring R. Since some of the results need the ring to have the identity element, we consider only such rings. Moreover, the additive group $(R,+,-,0)$ of the ring is assumed to be elementary abelian of prime exponent p. In some cases we shall have $p=2$. The construction can be even more general:

Lemma 3.1. *Let p be a prime and R be a commutative ring with 1 satisfying $p\cdot r=0$ for all $r\in R$. Let M be an R-module and let $f: M^3\to R$ be a triadditive mapping. Let $l: R\to \mathbf{GF}(p)$ be a $\mathbf{GF}(p)$-linear mapping. Then $f_l=l\circ f: M^3\to GF(p)$ is a trilinear form over $GF(p)$.*

Proof. Straightforward. □

A special case, which we are going to study is when $M=R^3$ and f is the determinant. Throughout this paper we shall call such forms *ring forms* and shall denote them as $d(l)$. Since the determinant is alternating, any ring form is a trilinear alternating form.

Note that in this paper we study only trilinear forms, but the construction could be clearly generalized to k-linear forms.

Let $E_i(r): R\to R^3$ denote the mapping sending r to the triple (r_1,r_2,r_3), where $r_i=r$ and $r_j=0$, $j\neq i$.

A degenerate form f can be factored by its radical $\mathrm{Rad} f=\{u, f[u]\equiv 0\}$ to obtain a nondegenerate form on a lower dimension. Thus we first determine a condition under which a ring form is nondegenerate.

Lemma 3.2. *Let $f=d(l)$ be a ring form. Then f is nondegenerate iff*

$$rR^2=rR\not\subseteq \mathrm{Ker}\, l \text{ for every } 0\neq r\in R. \tag{2}$$

Proof. If $rR\subseteq \mathrm{Ker}\, l$ for some $r\in R$ then clearly $u=(r,0,0)$ is in the radical of f. On the other hand, if $u=(r_1,r_2,r_3)$ is in the radical of f, then $f(u,E_2(1),E_3(s))=l(r_1\cdot s)=0$ for every $s\in R$. □

Second step in the classification of trilinear forms is a decomposition to pairwise orthogonal subspaces. Propositions 3.3 and 3.6 show the connection between the decomposability of the ring form and the ring itself.

Proposition 3.3. *Let R_i be a ring of exponent p and let $d(l_i)$ be a nondegenerate ring form on $V_i=R_i^3$, $i\in\{1,2\}$. Then there exists a linear mapping $l: R_1\times R_2\to \mathbf{GF}(p)$ such that $d(l)$ is a nondegenerate form on $R_1\times R_2$ and $d(l)=d(l_1)\oplus d(l_2)$.*

Proof. Denote by W_i the kernel of l_i. By Lemma 3.2 we have $r_iR_i\not\subseteq W_i$ for every nonzero $r_i\in R_i$, $i\in\{1,2\}$. Choose $x_i\in R_i$ such that $R_i=W_i\vee\langle x_i\rangle$ as a vector space, $i\in\{1,2\}$. Put $W=W_1\vee W_2\vee\langle x_1+x_2\rangle$, which is clearly a hyperplane in $R_1 \times R_2$. Consider any nonzero $r = (r_1, r_2) \in R$. Without loss of generality we assume $r_1 \neq 0 \in R_1$. By assumption there exists an element $s_1 \in R_1$ such that $r_1 s_1 \notin W_1$. We prove that $(r_1, r_2) \cdot (s_1, 0) \notin W$. Suppose the contrary. Then

$$(r_1, r_2) \cdot (s_1, 0) = (r_1 s_1, 0) = a(w_1, 0) + b(0, w_2) + c(x_1, x_2), \tag{3}$$

$w_i \in W_i$ and $a, b, c \in \mathbf{GF}(p)$. From the second coordinate we get $0 = bw_2 + cx_2$ which implies $b = c = 0$ by the choice of x_2. Thus the equation (3) collapses to $r_1 s_1 = aw_1$, a contradiction. Finally, the mapping l can be any nonzero linear mapping with kernel W. □

Lemma 3.4. *Let $f = d(l)$ be a nondegenerate ring form on $V = R^3$. Then vectors $u = (r_1, r_2, r_3)$ and $v = (s_1, s_2, s_3)$ are orthogonal iff for every $i \neq j$ we have $r_i s_j - r_j s_i = 0$.*

Proof. Set $k = 6 - i - j$ and consider a vector $w = E_k(r)$. By the orthogonality of u and v we have $f(u, v, w) = l((r_i s_j - r_j s_i)r) = 0$ for any $r \in R$. Since f is assumed to be nondegenerate, using Lemma 3.2 yields the result. □

Lemma 3.5. *Let $R = R_1 \times R_2$ be a decomposable commutative ring. Then I is an ideal of R iff it is a direct sum $I_1 \times I_2$, I_i ideal of R_i, $i \in \{1, 2\}$. Moreover, I is principal iff both I_1 and I_2 are principal.*

Proposition 3.6. *Let $f = d(l)$ be a nondegenerate ring form on $V = R^3$. If f is decomposable then $R = R_1 \times R_2$ is a decomposable ring and $f = d(l_1) \oplus d(l_2)$, where l_i is a restriction of l to R_i, $i \in \{1, 2\}$.*

Proof. Let $V = V_1 \times V_2$ be an orthogonal decomposition of f. Then we can write the vector $E_1(1) = (1, 0, 0)$ as $(1, 0, 0) = (r_1, s_1, t_1) + (r_2, s_2, t_2)$, $u_i = (r_i, s_i, t_i) \in V_i$. We get $r_2 = 1 - r_1$, $s_2 = -s_1$ and $t_2 = -t_1$. Since the vectors u_1 and u_2 are orthogonal, we get by Lemma 3.4 $-r_1 s_1 - (1 - r_1)s_1 = 0$ and thus s_1 is equal to 0. By symmetry we get $t_1 = 0$. Denote r_1 by just r. Similarly we get the decomposition of $E_2(1)$ as $(0, 1, 0) = (0, s, 0) + (0, 1 - s, 0)$, $v_1 = (0, s, 0) \in V_1$ and $v_2 = (0, 1 - s, 0) \in V_2$. Using again Lemma 3.4 for pairs u_1, v_2 and u_2, v_1 we obtain equations

$$r(1 - s) = 0 \text{ and } s(1 - r) = 0, \tag{4}$$

respectively, and combining them we get $r = s$. Moreover, the equations (4) imply that r (and thus $1 - r$) is an idempotent and $R = rR \oplus (1 - r)R$.

It remains to prove that the restrictions of l to R_i satisfy the condition of Lemma 3.2. Since $\mathrm{Ker}\, l$ does not contain any principal ideal of R, we must have $\mathrm{Ker}\, l \cap R_1 \neq R_i$, otherwise $\mathrm{Ker}\, l$ would contain the ideal $R_1 \times 0$. Thus l_i is a nontrivial linear mapping and by Lemma 3.5 does not contain any principal ideal of R_i. □

Example. There are three nonisomorphic rings of order 4 satisfying the condition $r + r = 0$, namely $\mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{GF}(2)[x]/(x^2)$ and $\mathbf{GF}(4)$.

The ring $\mathbf{Z}_2 \times \mathbf{Z}_2$ is decomposable and thus by Proposition 3.6 yields a decomposable form whenever l satisfies the condition of Lemma 3.2. The kernel of l is by Proposition 3.3 equal to $\mathrm{Ker}\, l = \{0, 1\}$ and we get the form $f_3 = \underline{123} + \underline{456}$ (the numbers of forms in this example correspond to the numbers used in [7]). The radical polynomial is compatible with the direct sum of forms (see [7]). Thus the radical polynomial of this form is a product of two radical polynomials of simple determinant. Radical polynomials for the other two forms can be computed using Proposition 3.9, see Table 1.

Now, consider the ring $\mathbf{GF}(2)[x]/(x^2)$. Setting $b_1 = E_1(1)$, $b_2 = E_3(x)$, $b_3 = E_2(1)$, $b_4 = E_1(x)$, $b_5 = E_3(1)$, $b_6 = E_2(x)$ and $\mathrm{Ker}\, l = \{0, 1\}$ we get exactly the form $f_4 = \underline{123} + \underline{345} + \underline{156}$.

For the underlying ring equal to $\mathbf{GF}(4)$ with elements $0, 1, \alpha, \alpha + 1$ setting $b_1 = E_1(1)$, $b_2 = E_2(1)$, $b_3 = E_3(1)$, $b_4 = E_1(\alpha + 1)$, $b_5 = E_2(\alpha)$, $b_6 = E_3(\alpha)$ and $\mathrm{Ker}\, l = \{0, \alpha\}$ yields the form $f_{10} = \underline{123} + \underline{234} + \underline{345} + \underline{246} + \underline{156}$.

We see that all three nondegenerate forms on dimension 6 over $\mathbf{GF}(2)$ are ring forms.

R	f	$P(f)$
$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\underline{123} + \underline{456}$	$(y^3 + 7x^2y)(y^3 + 7x^2y)$
$\mathbf{GF}(2)[x]/(x^2)$	$\underline{123} + \underline{345} + \underline{156}$	$y^6 + 7x^2y^4 + 56x^4y^2$
$\mathbf{GF}(4)$	$\underline{123} + \underline{234} + \underline{345} + \underline{246} + \underline{156}$	$y^6 + 63x^4y^2$

Table 1: Nondegenerate (ring) forms on dimension 6

Example. Rings of order 8 provide forms on dimension 9 over $\mathbf{GF}(2)$. There are six commutative rings with identity of this order. Three decomposable $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{GF}(2)[x]/(x^2) \times \mathbf{Z}_2$, $\mathbf{GF}(4) \times \mathbf{Z}_2$ and three indecomposable $\mathbf{GF}(2)[x]/(x^3)$, $\mathbf{GF}(8)$ and $\mathbf{GF}(2)[x, y]/(x^2, y^2, xy)$. The ring $\mathbf{GF}(2)[x, y]/(x^2, y^2, xy)$ contains (two-element) principal ideals (x), (y) and $(x + y)$. Any hyperplane contains at least one of these ideals and thus this ring by Lemma 3.2 does not yield a nondegenerate form.

The decomposable forms are just direct sums of forms on dimension 6 with the determinant on b_7, b_8 and b_9, see Table 2, and thus we get the corresponding radical polynomials given in Table 3. Moreover, these three forms are the only (nondegenerate) decomposable forms on dimension 9, because there is no nondegenerate form on dimension 4 and thus the decomposition of the dimension must be $6 - 3$.

The rings $\mathbf{GF}(2)[x]/(x^3)$ and $\mathbf{GF}(8)$ are both chain rings and thus their radical polynomials can be computed using Proposition 3.9. The form arising from the field $\mathbf{GF}(8)$ has a transitive group of automorphisms (see [8]) and by

R	f
$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$	$\underline{123} + \underline{456} + \underline{789}$
$\mathbf{GF}(2)[x]/(x^2) \times \mathbf{Z}_2$	$\underline{123} + \underline{345} + \underline{156} + \underline{789}$
$\mathbf{GF}(4) \times \mathbf{Z}_2$	$\underline{123} + \underline{234} + \underline{345} + \underline{246} + \underline{156} + \underline{789}$
$\mathbf{GF}(2)[x]/(x^3)$	$\underline{149} + \underline{158} + \underline{167} + \underline{248} + \underline{257} + \underline{347}$
$\mathbf{GF}(8)$	$\underline{147} + \underline{148} + \underline{149} + \underline{157} + \underline{158} + \underline{167} + \underline{169} + \underline{247} +$ $+ \underline{248} + \underline{257} + \underline{259} + \underline{268} + \underline{347} + \underline{349} + \underline{358} + \underline{367}$

Table 2: Nondegenerate ring forms on dimension 9

(not yet published) classification of forms on dimension 9 over $\mathbf{GF}(2)$ we know that there is only one such form on this dimension.

R	$P(f)$
$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$	$(y^3 + 7x^2y)(y^3 + 7x^2y)(y^3 + 7x^2y)$
$\mathbf{GF}(2)[x]/(x^2) \times \mathbf{Z}_2$	$(y^6 + 7x^2y^4 + 56x^4y^2)(y^3 + 7x^2y)$
$\mathbf{GF}(4) \times \mathbf{Z}_2$	$(y^6 + 63x^4y^2)(y^3 + 7x^2y)$
$\mathbf{GF}(2)[x]/(x^3)$	$y^9 + 7x^2y^7 + 56x^4y^5 + 448x^6y^3$
$\mathbf{GF}(8)$	$y^9 + 511x^6y^3$

Table 3: Radical polynomials of ring forms on dimension 9

Now, we shall study forms arising from (commutative) chain rings - i.e., rings whose ideals form a chain. Every ideal is then principal. Moreover, any hyperplane not containing the minimal ideal satisfies the condition of Lemma 3.2, thus providing a nondegenerate form.

Let S be a subset of a ring R. The set $\{x \in R, xs = 0, \forall s \in S\}$ will be denoted by $\mathrm{Ann}(S)$ and is an ideal of R. We shall write $\mathrm{Ann}(s)$ instead of $\mathrm{Ann}(\{s\})$.

Lemma 3.7. *Let R be a commutative ring of exponent p and $r \in R$. Then* $\dim(rR) + \dim(\mathrm{Ann}(r)) = \dim R$.

Proof. The mapping $x \mapsto rx$ is a p-linear mapping from R to R, rR is its image and $\mathrm{Ann}(r)$ is its kernel. □

Lemma 3.8. *Let $d(l)$ be a nondegenerate chain ring form on $V = R^3$ and let $u = (r_1, r_2, r_3)$ be a vector in V. Then the size of* $\mathrm{Rad}(u)$ *is equal to* $|\,R\,| \cdot |\,\mathrm{Ann}(r_1, r_2, r_3)\,|^2$.

Proof. Since we compute only the size of the radical, we can assume without loss of generality that $r_3R \subseteq r_2R \subseteq r_1R$. Thus there are elements $x, y \in R$

such that $r_2 = r_1 x$ and $r_3 = r_1 y$. Consider a vector $v = (s_1, s_2, s_3)$. Then $d(l)(u, v, -)$ is equal to

$$l\left(\begin{vmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ - & - & - \end{vmatrix}\right) = l\left(\begin{vmatrix} r_1 & 0 & 0 \\ s_1 & s_2 + s_1 x & s_3 + s_1 y \\ - & - & - \end{vmatrix}\right).$$

By Lemma 3.4 vectors u and v are orthogonal if and only if $r_1(s_2+s_1x) = 0$ and $r_1(s_3 + s_1y) = 0$, in other words if both $s_2 + s_1x$, $s_3 + s_1y$ belong to $\mathrm{Ann}(r_1)$. Since x and y are fixed, s_1 can be arbitrary and there are $|\,\mathrm{Ann}(r_1)\,|$ suitable vectors s_2 to satisfy $s_2 + s_1x \in \mathrm{Ann}(r_1)$. Similarly for s_3. □

Proposition 3.9. *Let R be a finite chain ring of exponent p and $d(l)$ a non-degenerate form on n-dimensional vector space R^3. Let $0 = I_0 \subset I_1 \subset \ldots \subset I_m = R$ be the chain of all ideals of R and $0 = n_0 < n_1 < \ldots < n_m = n/3$ their respective dimensions as vector spaces over $\mathbf{GF}(p)$. Then the form $d(l)$ has only vectors of rank $2n_k$, $k \in \{0, \ldots, m\}$, and the number of such vectors is $p^{3n_k} - p^{3n_{k-1}}$, $k \in \{1, \ldots, m\}$.*

Proof. Consider a vector $u = (r_1, r_2, r_3)$ and let I_k be the ideal generated by $\{r_1, r_2, r_3\}$. The rank of u is $r(u) = n - \mathrm{Rad}(u)$, which is by Lemma 3.8 equal to $n - (n/3 + 2\dim(\mathrm{Ann}(r_1, r_2, r_3)))$. Moreover, by Lemma 3.7 we get $r(u) = n - (n/3 + 2(n/3 - \dim(I_k))) = 2n_k$. The number of nonzero vectors $u = (r_1, r_2, r_3)$ such that I_k is generated by $\{r_1, r_2, r_3\}$ is equal to $|\, I_k\,|^3 - |\, I_{k-1}\,|^3 = p^{3n_k} - p^{3n_{k-1}}$. □

The numbers $p^{3n_k} - p^{3n_{k-1}}$ together with 1 are the coefficients of the radical polynomial $\sum_{i=0}^{n-1} \alpha_i x^i y^{n-i}$ of the chain ring form. Thus every α_i, $i \in \{1, \ldots, n-1\}$, is divisible both by $p^{3n_{k-1}}$ and $p^3 - 1$, see Tables 1 and 3.

Proposition 3.10. *Let R_1 and R_2 be nonisomorphic chain rings of characteristics p. Then the corresponding ring forms are nonequivalent.*

Proof. It is well known, see for instance [5], that every chain ring R of characteristics p is isomorphic to $\mathbf{GF}(p^k)[x]/(x^t)$. Its size is p^{kt} and the size of the maximal ideal xR is $p^{k(t-1)}$. Nonisomorpic rings must have distinct numbers k (and t) and thus by Proposition 3.9 the numbers $p^{3kt} - p^{3k(t-1)}$ of vectors of rank $\frac{2n}{3}$ of the corresponding ring forms are distinct, too. □

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