MULTI-FUZZY GROUP INDUCED BY MULTISETS

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Abstract. Multisets can be used to represent real life problems where repetition(s) of elements is necessary. Such cases occur in database query, chemical structures and computer programming but to mention a few. In this paper, some properties of algebraic sum of multisets $\uplus$ and some previous results on selection are mentioned. This work also introduces a new way to construct fuzzy sets and fuzzy groups structure on multiset.

Keywords: multisets, functions on multiset, selection operation, multi-fuzzy set, submultiset.

1. Introduction

Many real life problems can be represented by multisets. Such cases occur in database query, chemical structures and computer programming but to mention a few.

The term multiset can be traced back to 1888, when Dedekind in [5] stated that the element of the set may belongs to more than one. Multisets are also considered in and replaced with various notions such as bag, fireset (**f**initely **r**epeated **e**lement **s**et), heap, bunches, etc.

In the recent time, Nazmul *et al* [6] has put algebraic structure on multisets of a set in order to be able to consider multigroup and other related algebraic properties as in the classical group. His work was extended by Shinoj *et al* [8].

In this paper, we present some results on the algebraic structure of multi-fuzzy sets.

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2. Preliminaries

In what follows, we shall use X to denote a non-empty set.

Definition 2.1 ([6]). *A* multiset *M drawn from a set X is denoted by the count function $C_M\colon X \longrightarrow N$ defined by $C_M(x) = n \in N$, the multiplicity or number of occurrence of x in M, where N is the set of non-negative integers.*

Definition 2.2 ([6]). *Let multisets A and B be drawn from X. A is said to be a* submultiset *of B and is denoted $A \subseteq B$ if $C_A(x) \leq C_B(x)\, \forall x, y \in X$.*

The root set *or* support *of a multiset M, which is denoted by M^*, is the set which contains the distinct elements in the multiset. Hence, M^* is the set of $x \in M$ such that $C_M(x) > 0$.*

Let us recall some notions concerning multisets. See e.g. [2] for more details.

A multiset M is called a *regular multiset* if $C_M(x) = C_M(y)\ \forall x, y \in M$. The count function of the intersection of two multisets A and B both drawn from X is denoted by $C_A(x) \cap C_B(x) = \min\{C_A(x), C_B(x)\}$ and that of their union is denoted $C_A(x) \cup C_B(x) = \max\{C_A(x), C_B(x)\}$.

Multisets A and B are said to be equal if and only if $C_A(x) = C_B(x)$. Denote by $[X]^\alpha$, all the multisets whose elements have the multiplicity not more than α and $MS(X)$ the *set of all multisets over* X. An empty multiset ϕ is such that $C_\phi(x) = 0,\ \forall x \in X$. *Cardinality* of a multiset M is denoted by $|M| = \sum C_M(x),\ \forall x \in M$. The *peak element* $x \in M$ is such that $C_M(x) \geq C_M(y),\ \forall y \in M$.

Definition 2.3 ([10]). *Let us consider $A \in MS(X)$.*

i. The insertion *of x into A results into a multiset denoted by $\mathcal{C} = x \uplus A$ which has the count function*

$$C_{\mathcal{C}}(y) = \begin{cases} C_A(y), & y \neq x \\ C_A(x) + 1, & y = x. \end{cases}$$

ii. The removal *of x from A results into a multiset denoted by $\mathcal{D} = A \ominus x$ which has the count function*

$$C_{\mathcal{D}}(y) = \begin{cases} \max\{C_A(y) - 1, 0\}, & y = x \\ C_A(y), & y \neq x. \end{cases}$$

Let us consider $A, B \in MS(X)$.

iii. The insertion *of A into B or of B into A results into a multiset which has the count function denoted by $\mathcal{C}(x) = C_A(x) + C_B(x)$.*

iv. The removal *of B from A results into a multiset which has the count function denoted by $C_{\mathcal{D}}(x) = \max\{C_A(x) - C_B(x), 0\}$.*

It is clear that the set of all multisets over X that is $MS(X)$ is commutative and associative with respect to the sum $\uplus$.

The removal operation is neither commutative nor associative. Apart from, it is also possible to make some kind of selection in multisets using the following operations.

Definition 2.4 ([10]). *Consider $A \in MS(X)$ and $B \subseteq X$.*

i. The multiset $\mathcal{E} = A \otimes B$ is such that $\mathcal{E}$ only contains elements of A which also occur in B. The count function of $\mathcal{E}$ is denoted by

$$C_{\mathcal{E}}(x) = \begin{cases} C_A(x), & x \in B \\ 0, & x \notin B. \end{cases}$$

ii. The multiset $\mathcal{F} = A \bigodot B$ is such that $\mathcal{F}$ only contains elements of A which do not occur in B. The count function of $\mathcal{F}$ is denoted by

$$C_{\mathcal{F}}(x) = \begin{cases} C_A(x), & x \notin B \\ 0, & x \in B. \end{cases}$$

Operations such as "$\otimes$" or "$\odot$" are called selection operations.

Definition 2.5 ([6]). *Let X be a group and $e \in X$ its identity. Then, $\forall x, y \in X$, a multiset M drawn from X is called a* multigroup *if*

i. $C_M(xy) \geq C_M(x) \wedge C_M(y)$,

ii. $C_M(x^{-1}) \geq C_M(x)$.

The immediate consequence is that $C_M(e) \geq C_M(x)$. The set $MG(X)$ is called the *set of all multigroups over X*. The next definition can be found e.g. in [2].

Definition 2.6. *Let $A \in MS(X)$, where X is a group.*

i. $A_n = \{x : C_A(x) \geq n\}$;

ii. We denote a multiset containing only one element x with multiplicity. It is called n as $[n]_x$–a simple multiset;

iii. The complement of the multiset $M \in [X]^\alpha$ *denoted by M' is such that* $C_{M'}(x) = \alpha - C_M(x)$;

iv. $nA = \{x^n,\ \forall x \in A$ *n is the multiplicity of each element that appears in A*$\}$.

Remark 2.1. For a multigroup A over a group X, A_n is a group, indeed the subgroup of X [6].

Proposition 2.1 ([6], p. 645). *Let $A, B \in MS(X)$ and $m, n \in \mathbb{N}$.*

i. *If $A \subseteq B$, then $A_n \subseteq B_n$;*

ii. *If $m \leq n$, then $A_m \supseteq A_n$;*

iii. $(A \cap B)_n = A_n \cap B_n$;

iv. $(A \cup B)_n = A_n \cup B_n$;

v. *$A = B$ if and only if $A_n = B_n$, $\forall n \in \mathbb{N}$.*

Definition 2.7 ([6]). *Let X and Y be two nonempty sets such that $f\colon X \longrightarrow Y$ is a mapping. Consider the multisets $M \in [X]^\alpha$ and $N \in [Y]^\alpha$. Then,*

i. *the image of M under f denoted $f(M)$ has the count function*

$$C_{f(M)}(y) = \begin{cases} \bigvee_{f(x)=y} C_M(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise}; \end{cases}$$

ii. *the inverse image of N under f denoted $f^{-1}(N)$ has the count function $C_{f^{-1}(N)}(x) = C_N[f(x)]$.*

The following propositions were proved in [6]. But we shall later show that the items (iv), (v) and (vii) are not true and that the Proposition 2.2 needs to be restated.

Proposition 2.2 ([6]). *Let X, Y and Z be three nonempty sets such that $f\colon X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are mappings. If $M_i \in [X]^\alpha$, $N_i \in [Y]^\alpha$, $i \in I$ then*

i. $M_1 \subseteq M_2 \Rightarrow f(M_1) \subseteq f(M_2)$;

ii. $f(\cup_{i\in I} M_i) = \cup_{i\in I} f(M_i)$;

iii. $N_1 \subseteq N_2 \Rightarrow f^{-1}(N_1) \subseteq f^{-1}(N_2)$;

iv. $f^{-1}(\cup_{i\in I} M_i) = \cup_{i\in I} f^{-1}(M_i)$;

v. $f^{-1}(\cap_{i\in I} M_i) = \cap_{i\in I} f^{-1}(M_i)$;

vi. $f(M_i) \subseteq N_j \Rightarrow M_j \subseteq f^{-1}(N_j)$;

vii. *$g[f(M_i)] = [gf](M_i)$ and $f^{-1}[g^{-1}(N_j)] = [gf]^{-1}(N_j)$.*

Definition 2.8 ([11]). *A fuzzy set A of a non-empty set X is a class of objects in X with the associated (or characteristic) membership function $\mu : X \longrightarrow [0, 1]$ which assigns to every $x \in X$ a real value between 0 and 1.*

The value of $\mu(x)$ restricted to A is actually the degree of membership of x in A. If $\mu(x) = 0$, it represents complete non-membership while $\mu(x) = 1$ represents complete membership. But since μ characterizes the fuzzy set A, we can simply refer to μ as fuzzy subset.

Definition 2.9 ([7]). *Let μ be a fuzzy subset of G. Then, μ is called a fuzzy subgroup of G if $\forall x, y \in G$*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x) = \mu(x^{-1})$

Proposition 2.3 ([7]). *A fuzzy subset μ of G is a fuzzy subgroup of G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ for every $x, y \in G$.*

In paper [5] defined a kind of multiset (Dedekind's multiset) denoted by M_f as follows: if there are n elements in a domain X (of a function f mapping X to Y) which are mapped to an element $y \in Y$, then y has frequency n so that it is an n-fold element of Y.

But, on multisets (mentioned as Definition 2.7(i)) fails for Dedekind's multisets. So let us redefine Nazmul *et al*'s definition of function as follows:

Definition 2.10 ([2]). *Let X and Y be two non-empty sets and $f\colon X \longrightarrow Y$ a mapping such that $M \in [X]^\alpha$. Then, $C_{f(M)}(y) = \sum_{f^{-1}(y)\neq\emptyset f(x)=y} C_M(x)$.*

For more details concerning this problem see [2].

3. Induced fuzzy group

In the Introduction we have recalled that $[X]^\alpha$ is the collection of all multisets drawn from X so that if $A \in [X]^\alpha$, then $C_A(x) \leq \alpha \ \forall x \in A$. Now, let $[[X]]^\alpha$ be a subset of $[X]^\alpha$ such that if $B \in [[X]]^\alpha$, $C_B(x) = \alpha \ \forall x \in B$, meaning that B is a regular multiset in which each element has the multiplicity α.

Proposition 3.1. *Let A be a regular multiset over a group X. Then A is a multigroup if and only if A^* is a group (i.e. a subgroup of X).*

Proof. Assume that A^* is a group. Then for any $x, y \in A^*, xy^{-1} \in A^*$. Then, $C_A(xy^{-1}) > 0$. Since A is regular, $C_A(xy^{-1}) = C_A(x) = C_A(y^{-1}) = C_A(y)$. Without loss of generality, $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y)$.

Conversely, assume that A is a multigroup. Since $A_1 = A^*$, by Remark 2.1, the proof is complete. □

Proposition 3.2. *Let A be a multigroup over a group X such that $A \in [X]^w$. Then A' is a regular multigroup over X if and only if A is a regular.*

Proof. Assume that A is a multigroup over X and regular. Then $\forall\ x, y \in A, C_A(x) = C_A(y)$. Since A is a multigroup, $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y) = C_A(x) = C_A(y)$. Also, since $C_A(x) \neq 0 \neq C_A(y)$, then $C_A(xy^{-1}) \neq 0$. Hence, we have $xy^{-1} \in A$. But, A is regular, $C_A(xy^{-1}) = C_A(x) = C_A(y)$. Without loss of generality, $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y)$ and $w - C_A(xy^{-1}) \geq w - C_A(x) \wedge w - C_A(y)$. Thus, $C_{A'}(xy^{-1}) \geq C_{A'}(x) \wedge C_{A'}(y)$.

Conversely, assume that A' is a regular multigroup over X. Since $C_{A'}(xy^{-1}) = C_{A'}(x) = C_{A'}(y)$, then we can have both $C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y)$ and $C_A(x) = C_A(y) \forall x, y \in A$. □

There are some exceptions to what happens in classical algebra of set in multiset. If $A \in [X]^w$ is regular, it is possible to have the following:

(i) $A = A'$;

(ii) $A \cap A' \neq \emptyset$;

(iii) $A \subseteq A'$ or $A' \subseteq A$;

(iv) Also, $A \biguplus A' = B \in [[X]]^w$.

Example 3.1. Let $X = Z_6, A = \{0,0,2,2,4,4\}$ and $B = \{0,0,0,2,2,2,4,4,4\}$. Then the complement $A' = \{0,0,0,0,2,2,2,2,4,4,4,4\}$ and $B' = \{0,0,0,2,2,2,4,4,4\} = B$.

Definition 3.1. *Let $\bar{B} \in [X]^\alpha$ and $B \in [[X]]^\alpha$ such that $\bar{B} \subseteq B$. Then the degree to which the multiplicity of $b \in \bar{B}$ is near to the multiplicity of $b \in B$ or how near $\bar{B}$ is to B is defined by $\gamma = \frac{C_{\bar{B}}(b)}{\alpha}$.*

Remark 3.1. Then, any such $\bar{B}$ and $C_{\bar{B}}$ induce a *multi-fuzzy group* structure with membership function $\mu_{\frac{B}{\bar{B}}}(x) = \gamma$.

For simplicity, we shall use $\mu_{\mathcal{B}}$ in place of $\mu_{\frac{B}{\bar{B}}}$ and $\mu_{\bar{\mathcal{B}}}$ in place of $\mu_{\frac{B}{\bar{\bar{B}}}}$. This multiset has a structure similar to that defined by Syropoulos [9] in that every $x \in X$ which has a multiplicity in $\bar{B}$ is such that it has only one membership degree and one multiplicity. The support of $\mu_{\mathcal{B}}$ yields a fuzzy subset of X.

It should be noted that if X is a group with identity e and $[X]^\alpha$ is the collection of all multigroups drawn from X such that for any $\bar{A} \in [X]^\alpha$ and $\forall x \in \bar{A}, C_{\bar{A}}(x) \leq \alpha$. Let $A \in [[X]]^\alpha$ be the collection of all regular multigroups drawn from X. Then, any such multigroup $\bar{A}$ from $[X]^\alpha$ induces a *multi-fuzzy group* with membership degree $\mu_{\mathcal{A}}$. It can be said that $\mu_{\mathcal{A}}$ is *fuzzy relative to* A. The support of $\mu_{\mathcal{A}}$ is a fuzzy subgroup of X. All the operations on fuzzy set such as intersection, union, inclusion and complement can be seen to hold for $\mu_{\mathcal{A}}$.

Example 3.2. Let

$$X = S_3 = \{e, (12) = a, (13) = b, (23) = c, (123) = d, (132) = f\}$$

be the group with identity. Let $A \in [X]^4$ be $A = \{e,e,e,e,d,d,d,d,f,f,f,f\}$, $\bar{A} = \{e,e,e,d,d,f,f\}$ and $\bar{\bar{A}} = \{e,d,d,f\}$. Observe the following:

(i) $\mu_{\mathcal{A}} = \{(e,0.75),(e,0.75),(e,0.75),(d,0.5),(d,0.5),(f,0.5),(f,0.5)\}$ is a multi-fuzzy group.

(ii) $\mu_{\bar{\mathcal{A}}} = \{(e,0.25),(d,0.5),(d,0.5),(f,0.25)\}$ is a multi-fuzzy set.

(iii) $\bar{A}$ is a multigroup and $\mu_{\mathcal{A}}$ is a multi-fuzzy group. The support of $\mu_{\mathcal{A}}$ is the fuzzy subgroup $\{(e,0.75),(d,0.5),(f,0.5)\}$ of X.

(iv) Considering $\mu_{\mathcal{A}}$ and $\mu_{\bar{A}}$ as multi-fuzzy sets,

$$\mu_{\mathcal{A}} \cap \mu_{\bar{A}} = \{(e, 0.25), (d, 0.5), (d, 0.5), (f, 0.25)\}$$

and

$$\mu_{\mathcal{A}} \cup \mu_{\bar{A}} = \{(e, 0.75), (e, 0.75), (e, 0.75), (d, 0.5), (d, 0.5), (f, 0.5), (f, 0.5)\}.$$

Proposition 3.3. *Let X be a group, $\bar{A} \in [X]^\alpha$ be a multigroup over X and $\mu_{\mathcal{A}}$ the multi-fuzzy group induced by $\bar{A}$ and $C_{\bar{A}}$. Then, $\forall x \in X$ the following hold:*

(*i*) $\mu_{\mathcal{A}}(e) \geq \mu_{\mathcal{A}}(x)$,

(*ii*) $\mu_{\mathcal{A}}(x^{-1}) = \mu_{\mathcal{A}}(x)$.

Proof. (i) Since $\bar{A}$ is a multigroup, $C_{\bar{A}}(e) \geq C_{\bar{A}}(x)$. This implies that $\frac{C_{\bar{A}}(e)}{\alpha} \geq \frac{C_{\bar{A}}(x)}{\alpha}$. Thus, $\mu_{\mathcal{A}}(e) \geq \mu_{\mathcal{A}}(x)$.

(ii) The proof is similar. □

Proposition 3.4. *Let X be a group, $A \in [[X]]^\alpha$ and $\bar{A} \in [A]^\alpha$. Then, $\mu_{\mathcal{A}}$ has a fuzzy group structure.*

Proof. Let $x, y^{-1} \in \bar{A}$. Since $\bar{A}$ is a multigroup, $C_{\bar{A}}(y) = C_{\bar{A}}(y^{-1}) = \alpha$. But, $\mu_{\mathcal{A}}(x) = \frac{C_{\bar{A}}(x)}{\alpha}$ and $\mu_{\mathcal{A}}(y) = \mu_{\mathcal{A}}(y^{-1}) = \frac{C_{\bar{A}}(y)}{\alpha}$. Since $\bar{A}$ is a multigroup, $C_{\bar{A}}(xy^{-1}) \geq \min\{C_{\bar{A}}(x), C_{\bar{A}}(y^{-1})\}$. Since α is a natural number, we can have that $\frac{C_{\bar{A}}(xy^{-1})}{\alpha} \geq \min\{\frac{C_{\bar{A}}(x)}{\alpha}, \frac{C_{\bar{A}}(y^{-1})}{\alpha}\}$. Hence,

$$\mu_{\mathcal{A}}(xy^{-1}) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y^{-1})\}. \quad \square$$

The intersection of this multi-fuzzy group is also a multi-fuzzy group but the union is not necessarily a multi-fuzzy group. These properties are established by the following examples.

Example 3.3. Let $X = \{e, a, b, ab\}$ the Klein's 4-group. Consider the multigroups $A \in [X]^3$ such that $A = \{e, e, e, a, a, a, b, b, b, ab, ab, ab\}$. Let $\bar{A} = \{e, e, e, a, a\}$ and $\bar{\bar{A}} = \{e, e, b, b\}$. $\mu_{\mathcal{A}} = \{(e, 1), (e, 1), (e, 1), (a, \frac{2}{3}), (a, \frac{2}{3})\}$ and $\mu_{\bar{\mathcal{A}}} = \{(e, \frac{2}{3}), (e, \frac{2}{3}), (b, \frac{2}{3}), (b, \frac{2}{3})\}$. The intersection of $\mu_{\mathcal{A}}$ and $\mu_{\bar{\mathcal{A}}}$ is $\{(e, \frac{2}{3}), (e, \frac{2}{3})\}$, which is a trivial multi-fuzzy group. Their union is

$$\{(e, 1), (e, 1), (e, 1), (a, \frac{2}{3}), (a, \frac{2}{3}), (b, \frac{2}{3}), (b, \frac{2}{3})\},$$

which is obviously not a fuzzy group, since

$$0 = \mu_{\mathcal{A} \cup \bar{\mathcal{A}}}(ab) \ngeq \min\{\mu_{\mathcal{A} \cup \bar{\mathcal{A}}}(a), \mu_{\mathcal{A} \cup \bar{\mathcal{A}}}(b)\} = \frac{2}{3}.$$

Conclusion

It is known that the theory of multisets is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set just once. Given a regular multiset $A \in [X]$, where X is a group, we can fuzzify, relative to A, any multiset B which is a submultiset of A. Another interesting approach to multisets can be found e.g. in [1, 3, 4].

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