ON THE NUMBER OF CYCLIC SUBGROUPS IN FINITE GROUPS

Keyan Song
Wei Zhou*
School of Mathematics and Statistics
Southwest University
Chongqing, 400715
China
skyysh@outlook.com
zh_great@swu.edu.cn

Abstract. It is proved that a finite group G has $|G|-3$ cyclic subgroups if and only if $G \cong D_{10}$ or Q_8.
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1. Introduction

Let G be a finite group and $C(G)$ be the poset of cyclic subgroups of G. Sometimes $C(G)$ can decide the structure of G. For example, G is an elementary abelian 2-group if and only if $|C(G)| = |G|$. Tărnăuceanu [3, 4] classified the groups G such that $|G| - |C(G)| = 1$ or 2.

In this paper, we shall continue this study by describing the finite groups G such that

$$|C(G)| = |G| - 3.$$

We prove that there are just two such groups: D_{10} and Q_8.

For any finite group G, denote by $\pi_e(G)$ the set of all element orders of G, and denote by $\pi(G)$ the set of all prime divisors of $|G|$. For convenience, let $\pi_c(G) = \pi_e(G) - (\pi(G) \cup \{1\})$. For any $i \in \pi_e(G)$, denote by $C_i(G)$ the set of all cyclic subgroups of order i in G, and denote $c_i(G) = |C_i(G)|$.

2. The main result

Throughout this section, let $c_i = c_i(G)$ for each $i \in \pi_e(G)$.

Lemma 2.1. *Let* $|G| = p_1^{a_1} \cdots p_r^{a_r}$ *with* $p_1 < \cdots < p_r$. *If* $r \geq 3$, *then* $|G| - c(G) > p_r$.

*. Corresponding author
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Proof. For any finite group G, we know that

$$|G| = \sum_{k\in\pi_e(G)} c_k(\phi(k)),$$
$$|C(G)| = \sum_{k\in\pi_e(G)} c_k,$$

where ϕ is the Euler function. Hence

$$|G| - |C(G)| = \sum_{k\in\pi_e(G)} c_k(\phi(k)-1). \tag{1}$$

Let G be a group such that $|G| - |C(G)| \leq p_r$. By (1), we see that

$$\sum_{k\in\pi_e(G)} c_k(\phi(k)-1) \leq p_r.$$

By Cauchy theorem, $c_{p_i} \geq 1$ for all $i \leq r$. Hence we get that

$$c_{p_1}(p_1-2) + c_{p_2}(p_2-2) + \cdots + c_{p_r}(p_r-2) + \sum_{s\in\pi_c(G)} c_s(\phi(s)-1) \leq p_r.$$

Since $r \geq 3$, we get that $p_r \geq 5$. Thus $c_{p_r} = 1$ and

$$c_{p_1}(p_1-2) + c_{p_2}(p_2-2) + \cdots + c_{p_{r-1}}(p_{r-1}-2) + \sum_{s\in\pi_c(G)} c_s(\phi(s)-1) \leq 2.$$

So we get $r = 3$, $p_2 = 3$ and $p_1 = 2$. It follows that

$$c_3 + \sum_{s\in\pi_c(G)} c_s(\phi(s)-1) \leq 2.$$

If $c_3 = 2$, then $\pi_c(G) = \emptyset$. Let X_1 and X_2 be the two cyclic subgroups of order 3. Consider the action of G on $\{X_1, X_2\}$, we see that X_1 is normalized by a Sylow p_r-subgroup, which implies $3p_r \in \phi_c(G)$, a contradiction. Hence $c_3 = 1$. Similarly, we get that $3p_r \in \pi_c(G)$. But $c_{3p_r}(\phi(3p_r) - 1) \geq 2(p_r - 1) - 1 \geq 2(5-1) - 1 = 7$, a contradiction. □

Lemma 2.2. *Suppose that $|G| = p^a q^b$, where p, q are primes such that $p < q$. Then $|G| - c(G) > q$ if $G \ncong D_{2q}$, C_6, D_{12}, C_6 or S_3, and*

$$|D_{2q}| - |C(D_{2q})| = q-2,$$
$$|C_6| - |C(C_6)| = 2,$$
$$|D_{12}| - |C(D_{12})| = 2,$$
$$|C_6| - |C(C_6)| = 2,$$
$$|S_3| - |C(S_3)| = 1.$$

Proof. Let G be a group such that $|G| - c(G) \leq q$. By (1), we have

$$c_p(p-2) + c_q(q-2) + \sum_{s \in \pi_c(G)} c_s(\phi(s) - 1) \leq q. \tag{2}$$

i) $q \geq 5$. Then $c_q = 1$ and

$$c_p(p-2) + \sum_{s \in \pi_c(G)} c_s(\phi(s) - 1) \leq 2.$$

If $p \neq 2$, then $\pi_c(G) = \emptyset$, and $c_p(p-2) \leq 2$. It follows that $p = 3$ and $c_3 \leq 2$. But we can find an element of order $3q$, a contradiction. It follows that $p = 2$, and we see that $\pi_e(G) = \{1, 2, q\}$ or $\{1, 2, 2^2, q\}$. Thus G has only one Sylow q-subgroup Q which is isomorphic to C_q.

If $\pi_e(G) = \{1, 2, 4, q\}$, then $c_4 \leq 2$. Thus Q normalizes a cyclic subgroup of order 4. This implies that $4q \in \pi_c(G)$, a contradiction. Hence $\pi_e(G) = \{1, 2, q\}$. If $a \geq 2$, by considering the conjugate action of a Sylow 2-group on Q, we can find an element of order $2q$, a contradiction. Hence $|G| = 2q$, and $G = \langle u, v | u^q = 1, v^2 = 1, u^v = u^{-1} \rangle \cong D_{2q}$. In this case, $|G| - |C(G)| = q - 2$.

ii) $q = 3$ and $p = 2$. Now (2) becomes that

$$c_3 + \sum_{s \in \pi_c(G)} c_s(\phi(s) - 1) \leq 3.$$

It follows that $c_3 \leq 3$ and $\pi_e(G) \subseteq \{1, 2, 3, 4, 6\}$.

If $\pi_e(G) = \{1, 2, 3, 4, 6\}$, then $c_3 = c_4 = c_6 = 1$. And we get $12 \in \pi_e(G)$, a contradiction. If $\pi_e(G) = \{1, 2, 3, 4\}$, then $c_4 \leq 2$. We can get that $12 \in \pi_e(G)$ if $c_4 = 1$. Hence $c_4 = 2$. Therefore, a Sylow 3-group will normalizes a cyclic subgroup of order 4, and we get that $12 \in \pi_e(G)$, a contradiction.

If $\pi_e(G) = \{1, 2, 3, 6\}$, then $c_3 + c_6 \leq 3$. Then $c_3 = 1$ or $c_6 = 1$. Thus we can get a normal cyclic subgroup $X = \langle x \rangle$ of order 3. Thus $|G : C_G(x)| \leq 2$. If $a \geq 3$, then $C_G(x)$ will contain a subgroup $L \cong C_2 \times C_2 \times C_3$. Since $c_6(L) = 3$, we get a contradiction. Hence $a \leq 2$. From $c_3 \leq 2$, we get that $b = 1$. Hence $|G| \leq 12$, and $G \cong C_6$ or D_{12}.

Now we need to consider the case that $\pi_e(G) = \{1, 2, 3\}$. Thus $c_3 \leq 3$, and there are at most 6 nontrivial 3-element. It follows that a Sylow 3-subgroup is isomorphic to C_3. By Sylow theorem, $c_3 = 1$. Thus the Sylow 3-subgroup Q is normal in G. Since $6 \notin \pi_e(G)$, $a \leq 1$, and $|G| = 6$. Since $|C_6| - |C(C_6)| = 2$ and $|S_3| - |C(S_3)| = 1$, we get that $G \cong C_6$ or S_3 in this case. □

Lemma 2.3. *Let* $|G| = 2^a$. *If* $|G| - |C(G)| = 2^a - 3$, *then* $G \cong Q_8$.

Proof. Since $\phi(8) = 4$, from (1), $exp(G) \leq 4$. If $exp(G) = 2$, then $c(G) = |G|$, a contradiction. Hence $exp(G) = 4$, and $c_4 = 3$. We find a normal cyclic subgroup $X = \langle x \rangle$ of order 4. Let $C = C_G(X)$. Then $|C| = 2^a$ or 2^{a-1}.

We claim that C/X is an elementary 2-group. Otherwise, there exists an element $g \in C$ such that gX is an element of order 4 in C/X. Hence $|g| = 4$. Let $D = \langle g, x\rangle$. Then D is abelian. Since $|gX| = 4$, we get that $\langle g\rangle \cap X = 1$, and $D \cong C_4 \times C_4$. But $c_4(C_4 \times C_4) > 3$, a contradicton.

Hence the Frattini subgroup $\Phi(C) \leq X$, and C is a 2-group with cyclic Frattini subgroup. Suppose that C is non-abelian. By [1, Theorem 4.4], if $|\Phi(C)| > 2$, then there exists an element of order $2|\Phi(C) \geq 8$, a contradiction. Hence $\Phi(C) = C' \cong C_2$. By [1, Lemma 4.2], $C = EZ(C)$ and $|E \cap Z(C)| = 2$, where E is an extra-special 2-groups. By [2, Theorem 3.13.8], $E = A_1 * \cdots * A_m$, the central product of A_i, where $A_i \cong D_8$ or Q_8. Note that $c_4(Q_8) = 3$ and $c_4(D_8) = 1$ and $c_4(D_8 * D_8) > 3$. We get that $E = D_8$ for $X \not\leq E$. Let $y \in E$ with $|y| = 4$. Since $c_4 = 3$, we see that $Z(C) \cong C_4$ or $C_4 \times C_2$. If $Z(C) \cong C_4 \times C_2$, then $c_4(\langle y, Z(C)\rangle) > 3$, a contradiction. Hence $Z(C) = X \cong C_4$. But now $c_4(C) > 3$, a contradiction. So we see that C is abelian.

Since $c_4 = 3$, we get that $C \cong C_4$ or $C_4 \times C_2$. Suppose that $C \cong C_4 \times C_2$. Since $c_4(C) = 2$ and $c_4 = 3$, there exists $u \in G - C$ such that $|u| = 2$. Let $C = \langle x\rangle \times \langle w\rangle$, where $|w| = 2$. Thus $x^u = x^{\pm 1}$, and $w^u = w$ or wx^2, and we get 4 groups, but none of the them satisfy $c_4 = 3$. Hence $C \cong C_4$, and $G \cong Q_8$. $\square$

Now by Lemma 2.1, 2.2 and 2.3, we get our main result.

Theorem 2.4. *If* $|C(G)| = |G| - 3$, *then* $G \cong D_{10}$ *or* Q_8.

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