θ-CLOSURE AND $T_{2\frac{1}{2}}$ SPACES VIA IDEALS

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Abstract. We introduce θ-closure of a set with respect to an ideal using the local closure function and obtain some properties.We also introduce θ-convergence of a filter and $T_{2\frac{1}{2}}$ spaces with respect to an ideal and by using these concepts and other separation axioms obtain the sufficient conditions for a set to be θ-closed with respect to an ideal and also obtain some characterizations of local closure function. Finally, the sufficient conditions for the equivalence of θ-closure with respect to an ideal and closure in $*$-topology are given.

Keywords: $\mathcal{I}_\theta$-closed, T_2 mod $\mathcal{I}$, $T_{2\frac{1}{2}}$ mod $\mathcal{I}$, $\mathcal{I}$-compact, $\mathcal{I}$-regular, almost-$\mathcal{I}$-regular, $\mathcal{I}$-QHC, $\mathcal{I}_\theta$ convergence, ideal.

1. Introduction

In [15], Veličko introduced strong form of closed sets called θ-closed sets and in [7], Janković utilized these sets to obtain new characterizations of separation axioms. On the other hand in [1], Al-Omari and Noiri defined the local closure function stronger than the local function with respect to ideal topological space and obtained various properties of it. The concept of ideals has arisen due to Kuratowski [9] to study various topological properties. An ideal $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X such that it is closed downwards (i.e. every subset of member of $\mathcal{I}$ is in $\mathcal{I}$) and closed under finite union. This concept was further studied by Vaidyanathaswamy who obtained a new topology $\tau^*(\mathcal{I}, \tau)$ called the $*$-topology which is generally finer than the original topology having the corresponding Kuratowski closure operator $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [13], where $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every open nhd. U of x in $X\}$ called a local function[9] of A with respect to $\mathcal{I}$ and τ and $\beta = \{V - A : V \in \tau$ and $A \in \mathcal{I}\}$ is a basis for the $*$-topology τ^*. We will write A^* for $A^*(\mathcal{I}, \tau)$ and $\tau^*(\mathcal{I})$ or τ^* for $\tau^*(\mathcal{I}, \tau)$.

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For a topological space (X,τ) and any subset A of X, a point $x \in X$ is said to be in the θ-closure of A if for every open nhd. U of x in X, $\overline{U} \cap A \neq \emptyset$ denoted by $cl_\theta(A)$[15] and in θ-interior of A if there exists a nhd. U of x such that $U \subset \overline{U} \subset A$ denoted by $int_\theta(A)$. The subset A is said to be θ-closed (θ-open) if $cl_\theta(A) = A$ ($int_\theta(A) = A$). Also the collection of all θ-open sets forms a topology denoted by τ_θ which is generally weeker than the original topology. And for an ideal topological space $(X,\tau,\mathcal{I})$ for any subset A of X, the local closure function $\Gamma(A)(\mathcal{I},\tau)$ with respect to $\mathcal{I}$ and τ is given as $\Gamma(A)(\mathcal{I},\tau) = \{x \in X : \overline{U} \cap A \notin \mathcal{I}$ for every τ-nhd. U of x in X $\}$ [1] and the operator $\Psi_\Gamma(A) = X - \Gamma(X - A) = \{x \in X :$ there exist τ-nhd. U of x such that $\overline{U} - A \in \mathcal{I}$ $\}$. Also note that the collection $\sigma = \{A \subset X : A \subset \Psi_\Gamma(A)\}$ forms a topology for X [1]. Further a topological space (X,τ) is said to be S_2 [2] if for any two distinct points x, y of X, whenever one of them has an open set not containing the other, then there exist disjoint open subsets containing them and a space is said to be Urysohn space or $T_{2\frac{1}{2}}$ space [5] if for every distinct points x, y of X there exist open subsets U, V containing x, y respectively such that $cl(U) \cap cl(V) = \emptyset$. Also an ideal space $(X,\tau,\mathcal{I})$ is said to be T_2 mod $\mathcal{I}$ [11] if for any two distinct points x, y of X, there exist open sets U and V such that $x \in U, y \in V$ and $U \cap V \in \mathcal{I}$. An ideal space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-compact [10] if for every open cover $\{G_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $X - \bigcup\{G_\alpha : \alpha \in \Delta_0\} \in \mathcal{I}$ and is said to be $\mathcal{I}$-regular [4] if for any closed subset F of X and any point $x \in X$ whenever $x \notin F$, there exist disjoint open subsets U, V such that $x \in U$ and $F - V \in \mathcal{I}$. Also we have the following:

Definition 1.1 ([6])**.** Let $(X,\tau,\mathcal{I})$ be an ideal space then $\mathcal{I}$ is said to be codense if $\tau \cap \mathcal{I} = \emptyset$.

Example 1.1. Let $X = \{a,b,c\}$ with $\tau = \{\emptyset, \{b,c\}\}$. Then the following are codense ideals:

(a) $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a,c\}\}$

(b) $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

(c) $\mathcal{I} = \{\emptyset, \{a\}\}$

(d) $\mathcal{I} = \{\emptyset, \{b\}\}$

(e) $\mathcal{I} = \{\emptyset, \{c\}\}$

Definition 1.2 ([15])**.** Let (X,τ) be any topological space and $\mathcal{F}$ be any filter on X. Then for any point $a \in X$, $\mathcal{F}$ *is said to be θ-convergent to a* denoted by $\mathcal{F} \to_\theta a$ if for every open set U containing a, $\overline{U} \in \mathcal{F}$.

Lemma 1.1 ([1])**.** *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then for any subset A of X the following holds:*

(a) $A^* \subset \Gamma(A)(\mathcal{I},\tau) \subset cl_\theta(A)$.

(*b*) $\Gamma(A)(\mathcal{I},\tau) = cl(\Gamma(A)(\mathcal{I},\tau))$.

Lemma 1.2 ([6]). *Let $(X,\tau,\mathcal{I})$ be an ideal space. then the following are equivalent :*

(*a*) *$\mathcal{I}$ is codense.*

(*b*) *$X = X^*$.*

(*c*) *For every $U \in \tau$, $U \subseteq U^*$.*

Theorem 1.1 ([11]). *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then X is $\mathcal{I}$-regular if and only if for each $x \in X$ and open set U containing x , there is an open set V containing x such that $\overline{V} - U \in \mathcal{I}$.*

Notation. Throughout this paper (X,τ) will denote topological space and for an ideal $\mathcal{I}$ on X, $(X,\tau,\mathcal{I})$ is called an ideal space. When there is no chance of confusion, by a open subset (open nhd.) of X, we will mean open set in the topological space (X,τ). For a subset A of X, $cl(A)$ or $\overline{A}$ and $int(A)$ will denote the closure of A, interior of A in (X,τ), respectively, $cl^*(A)$ and $int^*(A)$ will denote the closure of A, interior of A in (X,τ^*), respectively, and $X - A = A^C$ will denote the complement of A in X.

In section 2 of this paper, firstly for any subset of an ideal topological space $(X,\tau,\mathcal{I})$, we define θ-closure of a set using the local closure function and obtain its relationship with other θ-closure (closure) of a set. We prove that unlike in the case of local function, the local closure function of any subset A of X with respect to τ and τ^* need not be same (Example 2.2 below), but in case of codense ideal the local closure function of any subset A of X with respect to τ and τ^* coincide (Theorem 2.1 below). Further in Section 3, we introduce θ-convergence of a filter and $T_{2\frac{1}{2}}$ spaces with respect to an ideal and obtain various properties. We also obtain the characterization of local closure function in terms of $\mathcal{I}_\theta$ convergence of a filter (Theorem 3.2 below) and the characterization of $T_{2\frac{1}{2}}$ space with respect to an ideal in terms of $\mathcal{I}_\theta$ convergence of a filter and local closure function (Theorems 3.5 and 3.6 below). Finally, the sufficient conditions for the equivalence of $\mathcal{I}_\theta$-closure and closure in $*$-topology (Theorems 2.4 and 2.5 below), for the $\mathcal{I}_\theta$-closedness of a set (Theorems 2.3 and 3.8 below) and for the $\mathcal{I}_\theta$-closedness of $\mathcal{I}_\theta$-closure of a set (Theorem 2.6 below) are obtained. Examples are given throughout the paper to give counterexamples and illustrations.

2. Results

We begin by defining the θ-closure of a set with respect to an ideal (briefly $\mathcal{I}_\theta$-closure) for any subset A of X in an ideal topological space $(X,\tau,\mathcal{I})$.

Definition 2.1. Let $(X,\tau,\mathcal{I})$ be an ideal space. For any subset A of X, θ-closure of A with respect to an ideal $\mathcal{I}$ is given by $cl_{\mathcal{I}_\theta}(A) = A \cup \Gamma(A)(\mathcal{I},\tau)$. The subset A is said to be $\mathcal{I}_\theta$-closed if $cl_{\mathcal{I}_\theta}(A) = A$.

Remark 2.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. For any subset* A *of* X*, define* $Int_{\mathcal{I}_\theta}(A) = X - cl_{\mathcal{I}_\theta}(X - A)$ *so that* $Int_{\mathcal{I}_\theta}(A) = \{x \in A : \overline{U} - A \in \mathcal{I}$ *for some* τ*-nhd.* U *of* x *in* X $\}$*. The subset* A *is said to be* $\mathcal{I}_\theta$*-open if* $Int_{\mathcal{I}_\theta}(A) = A$*. It can be easily checked that the collection of* $\mathcal{I}_\theta$*-open sets forms a topology. In our further results we denote it by* $\tau_{\mathcal{I}_\theta}$*. Also note that in view of Lemma 1.1(a), we have* $\tau_\theta \subset \tau_{\mathcal{I}_\theta} \subset \tau^*$*.*

Even though using Lemma 1.1(b), it follows that $cl_{\mathcal{I}_\theta}(A)$ is closed subset of X, but the following Example 2.1, shows that it need not be $\mathcal{I}_\theta$ closed.

Example 2.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $cl_{\mathcal{I}_\theta}(\{b\}) = \{b, c\}$ and $cl_{\mathcal{I}_\theta}(\{b, c\}) = \{a, b, c\}$. Hence $\{b\}$ is not $\mathcal{I}_\theta$ closed.

In [6], Janković and Hamlett proved that for any subset A of X in an ideal space $(X, \tau, \mathcal{I})$, $A^*(\mathcal{I}, \tau) = A^*(\mathcal{I}, \tau^*(\mathcal{I}))$. So the natural question arises is the result true for the local closure function. The following Theorem 2.1 shows that for codense ideals the result also holds for the local closure function. Before this we prove the following Lemma:

Lemma 2.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. Then* $cl^*(G) - I \subset cl^*(G - I)$ *for any open subset* G *of* X *and* $I \in \mathcal{I}$*.*

Proof. Let G be any open subset of X and $I \in \mathcal{I}$. Let $x \in X$ be such that $x \in cl^*(G) - I$ and H be any τ^*-nhd. of x, then $H - I$ is also τ^*-nhd. of x (since every $I \in \mathcal{I}$ is τ^*-closed and $x \notin I$). Therefore, $x \in cl^*(G)$ implies that $(H - I) \cap G \neq \emptyset$ and so $H \cap (G - I) \neq \emptyset$. Hence $x \in cl^*(G - I)$. □

Theorem 2.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space where* $\mathcal{I}$ *is codense. Then* $\Gamma(A)(\mathcal{I}, \tau) = \Gamma(A)(\mathcal{I}, \tau^*)$ *for any subset* A *of* X*.*

Proof. Let A be any subset of X. Then it can be seen easily that $\Gamma(A)(\mathcal{I}, \tau^*) \subset \Gamma(A)(\mathcal{I}, \tau)$ since $\tau \subset \tau^*$ and $\mathcal{I}$ is closed downwards. Conversely, let $x \in X$ such that $x \notin \Gamma(A)(\mathcal{I}, \tau^*)$, so there exist τ^* nhd. G of x such that $cl^*(G) \cap A \in \mathcal{I}$. Since $\beta = \{V - A : V \in \tau$ and $A \in \mathcal{I}\}$ is a basis for the $*$-topology τ^*, so there exist basic open set $U - I$ such that $x \in U - I \subset G$ and so $cl^*(U - I) \cap A \in \mathcal{I}$. Now using Lemma 2.1, it follows that $cl^*(U) \cap A \in \mathcal{I}$. Further $\mathcal{I}$ is codense, so $cl^*(U) = cl(U)$ for every open subset U of X. Hence $x \notin \Gamma(A)(\mathcal{I}, \tau)$. □

The following Example 2.2 shows that if the ideal is not codense then Theorem 2.1 need not be true.

Example 2.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So $\tau^* = \wp(X)$. Then $\Gamma\{c\}(\mathcal{I}, \tau^*) = \{c\}$ and $\Gamma\{c\}(\mathcal{I}, \tau) = \{a, b, c\}$.

Theorem 2.2. *The topology* $\tau^*(\mathcal{I}, \tau_\theta)$*, the local function of which is given by* $A^*(\mathcal{I}, \tau_\theta) = \{x \in X : U \cap A \notin \mathcal{I}$ *for every nhd.* U *of* x *in* $(X, \tau_\theta)\}$ *is generally coarser than* $\tau_{\mathcal{I}_\theta}$*.*

Proof. For this we will prove $cl_{\mathcal{I}_\theta}(A) \subset cl^*(A)(\mathcal{I}, \tau_\theta)$. Let $x \in X$ be any element such that $x \notin cl^*(A)(\mathcal{I}, \tau_\theta)$. So there exist τ_θ open subset U of x such that $U \cap A \in \mathcal{I}$. Therefore, U is θ-open implies that there exist open subset V of x such that $cl(V) \subset U$ and so $cl(V) \cap A \in \mathcal{I}$. Hence $x \notin cl_{\mathcal{I}_\theta}(A)$. □

Corollary 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then for any subset A of X, $cl^*(A)(\mathcal{I}, \tau) \subset cl_{\mathcal{I}_\theta}(A) \subset cl^*(A)(\mathcal{I}, \tau_\theta) \subset cl_\theta(A)$. So every τ_θ-open set is $(\tau_\theta)^*$-open, $(\tau_\theta)^*$-open set is $\mathcal{I}_\theta$-open and hence τ^*-open.*

Here a natural question arise given any ideal space $(X, \tau, \mathcal{I})$, is there can be any relationship between $*$-topology of θ-open sets and θ-open sets with respect to τ^*. The following Example 2.3 shows that there is no relationship between the topological spaces $\tau^*(\mathcal{I}, \tau_\theta)$ and $(\tau^*)_\theta$, where $(\tau^*)_\theta$ means θ-open sets with respect to τ^*.

Example 2.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. So $\tau^* = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then $\tau_\theta = \{\emptyset, X\}$ and so $\tau^*(\mathcal{I}, \tau_\theta) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. But $(\tau^*)_\theta = \{\emptyset, \{b\}, \{a, c\}, X\}$.

In an ideal space $(X, \tau, \mathcal{I})$ even though $cl^*(A)(\mathcal{I}, \tau) \subset cl_{\mathcal{I}_\theta}(A)$ i.e. every $\mathcal{I}_\theta$-open set is τ^*-open but the following example shows that there is no relationship between $\mathcal{I}_\theta$-open and τ-open subset of X.

Example 2.4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then it can be easily checked that $\tau_{\mathcal{I}_\theta} = \{\emptyset, \{b, c\}, X\}$.

In [6], Janković and Hamlett proved that $A^*(\mathcal{I}_f, \tau) = A^d$ if and only if (X, τ) is T_1 (where $\mathcal{I}_f$ denotes the ideal of finite subsets of X and A^d denotes the derived set of A in (X, τ)). Therefore, we have the immediate results in the following:

Note 1. Let $(X, \tau, \mathcal{I}_f)$ be an ideal space and (X, τ) be T_1. Then $cl(A) = cl^*(A) \subset cl_{\mathcal{I}_\theta}(A) \subset cl^*(\mathcal{I}, \tau_\theta)(A) \subset cl_\theta(A)$ for any subset A of X.

Further in [6], it is shown that for an ideal space $(X, \tau, \mathcal{I}_{cd})$, where $\mathcal{I}_{cd} = \{A \subset X : A^d = \emptyset\}$, $A^d \subset A^*$. So we have

Note 2. Let $(X, \tau, \mathcal{I}_{cd})$ be an ideal space. Then $cl(A) = cl^*(A) \subset cl_{\mathcal{I}_\theta}(A) \subset cl^*(\mathcal{I}, \tau_\theta)(A) \subset cl_\theta(A)$ for any subset A of X.

Therefore, the above Notes 1 and 2 give the relationship between closed sets and $\mathcal{I}_\theta$ closed sets for particular ideal of finite sets and ideal of closed and discrete sets.

Further we will give characterizations of $\mathcal{I}_\theta$-closed sets using separation axioms.

It is well known that every compact set in T_2 space is closed and in [7], Janković proved the stronger result that a space is T_2 if and only if every compact set is θ-closed. On the other hand in [11], Sivaraj and Renukadevi proved that an $\mathcal{I}$-compact set in T_2 mod $\mathcal{I}$ space is τ^*-closed. Therefore, analogously the following Theorem 2.3 shows the stronger result that every $\mathcal{I}$-compact set in T_2 space is $\mathcal{I}_\theta$ closed.

Theorem 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space and (X, τ) be T_2. Then for any $\mathcal{I}$-compact subset A of X, A is $\mathcal{I}_\theta$ closed.*

Proof. We have to prove that $cl_{\mathcal{I}_\theta}(A) \subset A$. Let $x \in X$ be any element such that $x \notin A$. Since X is T_2, so for all $y \in A$, there exist disjoint open subsets U_y, V_y containing x and y respectively. This implies that $A \subset \bigcup_{y\in A} V_y$. Now A is $\mathcal{I}$-compact, so there exist finite subset of A such that $A - \bigcup_{i=1}^{n} V_{y_i} \in \mathcal{I}$. Let $V = \bigcup_{i=1}^{n} V_{y_i}$ and $U = \bigcap_{i=1}^{n} U_{y_i}$, then $U \cap V = \emptyset$ and so $\overline{U} \cap V = \emptyset$. Therefore, $(\overline{U} \cap V) \bigcup (A - V) \in \mathcal{I}$ and so $\overline{U} \cap A \in \mathcal{I}$. Hence $x \notin cl_{\mathcal{I}_\theta}(A)$. $\square$

The following Example shows that the result need not be true if we replace T_2 space by T_2 mod $\mathcal{I}$ space.

Example 2.5. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then X is T_2 mod $\mathcal{I}$ space but not T_2. Also X is finite, so every subset of X is $\mathcal{I}$-compact. But $cl_{\mathcal{I}_\theta}(\{c\}) = \{a, b, c\}$. Hence $\{c\}$ is not $\mathcal{I}_\theta$ closed.

The following Example 2.6 shows that converse of Theorem 2.3 need not be true.

Example 2.6. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then it can be easily checked that every subset of X is $\mathcal{I}_\theta$ closed. But X is not T_2.

The following Theorem 2.4 gives the sufficient condition for the equivalence of $*$-closure and $\mathcal{I}_\theta$-closure.

Theorem 2.4. *Let $(X, \tau, \mathcal{I})$ be S_2 ideal space, then $cl^*(A) = cl_{\mathcal{I}_\theta}(A)$ for every $\mathcal{I}$-compact subset A of X.*

Proof. Let A be any $\mathcal{I}$-compact subset of X. Since $cl^*(A) \subset cl_{\mathcal{I}_\theta}(A)$, we only need to prove $cl_{\mathcal{I}_\theta}(A) \subset cl^*(A)$. Also if $A \in \mathcal{I}$, then trivially $cl^*(A) = cl_{\mathcal{I}_\theta}(A)$. Therefore, consider the case when $A \notin \mathcal{I}$. Let $y \notin cl^*(A)$, so there exist open set U containing y such that $U \cap A \in \mathcal{I}$ and so $U \cap A = I$ for some $I \in \mathcal{I}$. This implies that $U \cap (A - I) = \emptyset$. Now X is S_2, so for all $z \in A - I$, y has an open set U not containing z, so there exist open sets V_z and W_z such that $z \in V_z, y \in W_z$ and $V_z \cap W_z = \emptyset$. Therefore, $A \subset U \cup \bigcup_{z\in A-I} V_z$, but A is $\mathcal{I}$-compact implies that there exist finite subset of $A - I$ such that $A - \bigcup_{i=1}^{n} V_{z_i} \in \mathcal{I}$ (without loss of generality, we can remove the open subset U, since $U \cap A \in \mathcal{I}$ implies that $(A - \bigcup_{i=1}^{n} V_{z_i}) \cap U \in \mathcal{I}$). Let $G = \bigcup_{i=1}^{n} V_{z_i}$ and $H = \bigcap_{i=1}^{n} W_{z_i}$. Now for all

$i = 1, 2, ...n$, $V_{z_i} \cap W_{z_i} = \emptyset$ implies that $G \cap H = \emptyset$ and so $\overline{H} \cap G = \emptyset$. Since $A - G \in \mathcal{I}$, so $\overline{H} \cap A \in \mathcal{I}$. Therefore, H is the required open set containing y such that $\overline{H} \cap A \in \mathcal{I}$ and so $y \notin cl_{\mathcal{I}_\theta}(A)$. Hence $cl^*(A) = cl_{\mathcal{I}_\theta}(A)$ for every $\mathcal{I}$-compact A of X. □

Even though we have seen in Example 2.4 that there is no relationship between closed and $\mathcal{I}_\theta$-closed sets. The following Corollary 2.2 gives the sufficient condition for a closed set to be $\mathcal{I}_\theta$-closed.

Corollary 2.2. *Let $(X, \tau, \mathcal{I})$ be S_2 ideal space, then $cl_{\mathcal{I}_\theta}(A) \subset cl(A)$ for every $\mathcal{I}$-compact subset A of X.*

Proof. Proof follows from Theorem 2.4. □

The following Theorem 2.5 shows that for $\mathcal{I}$-regular spaces the concept of $*$-closure and $\mathcal{I}_\theta$-closure coincide.

Theorem 2.5. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then X is $\mathcal{I}$-regular if and only if $\Gamma(A)(\mathcal{I}, \tau) \subset A^*$ for any subset A of X. Hence in particular, X is $\mathcal{I}$-regular if and only if $cl_{\mathcal{I}_\theta}(A) = cl^*(A)$ for any subset A of X.*

Proof. Firstly, let X be $\mathcal{I}$-regular space and $x \in X$ be any element such that $x \notin A^*$. So there exist open subset U of x in X such that $U \cap A \in \mathcal{I}$. Since X is $\mathcal{I}$-regular, so by Theorem 1.1 there exist open subset G of X such that $x \in G \in \overline{G}$ and $\overline{G} - U \in \mathcal{I}$. Therefore, $(\overline{G} - U) \bigcup (U \cap A) \in \mathcal{I}$ and so $\overline{G} \cap A \in \mathcal{I}$. Hence $x \notin \Gamma(A)(\mathcal{I}, \tau)$. Conversely, let F be any closed set and $a \in X$ such that $a \notin F$. Since F is closed and hence τ^*-closed, so $\Gamma(F)(\mathcal{I}, \tau) \subset F^* \subset F$ implies that $a \notin \Gamma(F)(\mathcal{I}, \tau)$. Therefore, there exist open subset G of a in X such that $\overline{G} \cap F \in \mathcal{I}$. Hence G and $(\overline{G})^C$ are the required disjoint open subsets of X such that $a \in G$ and $F - (\overline{G})^C \in \mathcal{I}$ and so X is $\mathcal{I}$-regular. □

Corollary 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space and X be $\mathcal{I}$-regular space, then $cl_{\mathcal{I}_\theta}(A)$ is $\mathcal{I}_\theta$ closed for every subset A of X.*

Proof. Proof follows from Theorem 2.5, since X is $\mathcal{I}$-regular implies $cl^*(A) = cl_{\mathcal{I}_\theta}(A)$ for any subset A of X. □

Corollary 2.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then X is $\mathcal{I}$-regular if and only if $cl_{\mathcal{I}_\theta}(A) \subset cl(A)$ for any subset A of X.*

Proof. Proof follows from Theorem 2.5 and the fact that $cl^*(A) \subset cl(A)$ for any subset A of X. □

Lemma 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $cl_{\mathcal{I}_\theta}(A) = \bigcap_{G \in \tau} \{cl_{\mathcal{I}_\theta}(G) : A \cap G^C \in \mathcal{I}\}$ for any subset A of X.*

Proof. Let $\bigcap_{G\in\tau}\{cl_{\mathcal{I}_\theta}(G) : A \cap G^C \in \mathcal{I}\} = T$. Firstly, let $x \notin T$, then there exist open G such that $A \cap G^C \in \mathcal{I}$ and $x \notin cl_{\mathcal{I}_\theta}(G)$. Therefore, there exist open subset V containing x such that $\overline{V} \cap G \in \mathcal{I}$ and so $(A \cap G^C) \cup (\overline{V} \cap G) \in \mathcal{I}$. Hence $\overline{V} \cap A \in \mathcal{I}$ implies that $x \notin cl_{\mathcal{I}_\theta}(A)$. Conversely, let $x \notin cl_{\mathcal{I}_\theta}(A)$, then there exist open V containing x such that $\overline{V} \cap A \in \mathcal{I}$ and so $(\overline{V}^C)^C \cap A \in \mathcal{I}$. Therefore, $\overline{V} \cap \overline{V}^C = \emptyset$ implies that $x \notin cl_{\mathcal{I}_\theta}(\overline{V}^C)$ and so $x \notin T$. Hence $cl_{\mathcal{I}_\theta}(A) = \bigcap_{G\in\tau}\{cl_{\mathcal{I}_\theta}(G) : A \cap G^C \in \mathcal{I}\}$. □

Now we will prove the stronger result than Corollary 2.3 that $\mathcal{I}_\theta$-closure of any subset A of X is $\mathcal{I}_\theta$ closed even for almost-$\mathcal{I}$-regular spaces where

Definition 2.2. An ideal space $(X, \tau, \mathcal{I})$ is said to be almost-$\mathcal{I}$-regular if for any point x and a regular closed set F not containing x there exist disjoint open subsets G, H such that $x \in G$, $F - H \in \mathcal{I}$.

It can be seen easily that every $\mathcal{I}$-regular space is almost-$\mathcal{I}$-regular but the converse need not be true, since every regular closed set is closed and closed set need not be regular closed.

Theorem 2.6. *Let $(X, \tau, \mathcal{I})$ be an ideal space and X be almost-$\mathcal{I}$-regular space, then $cl_{\mathcal{I}_\theta}(A)$ is $\mathcal{I}_\theta$ closed for any subset A of X.*

Proof. Let A be any subset of X. We only need to prove that $cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(A)) \subset cl_{\mathcal{I}_\theta}(A)$. Firstly, we will prove that $cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(U)) \subset cl_{\mathcal{I}_\theta}(U)$ for any open subset U of X. Let $x \notin cl_{\mathcal{I}_\theta}(U)$. It can be easily checked that $cl_{\mathcal{I}_\theta}(U)$ is regular closed set, so there exist disjoint open sets G, H such that $x \in G$ and $cl_{\mathcal{I}_\theta}(U) - H \in \mathcal{I}$, since X is almost-$\mathcal{I}$-regular space. Now $G \cap H = \emptyset$ and so $\overline{G} \cap H = \emptyset$. Therefore, $(cl_{\mathcal{I}_\theta}(U) \cap H^C) \bigcup (\overline{G} \cap H) \in \mathcal{I}$ and so $\overline{G} \cap (cl_{\mathcal{I}_\theta}(U)) \in \mathcal{I}$. This implies that $x \notin cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(U))$ and so $cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(U)) \subset cl_{\mathcal{I}_\theta}(U)$ for any open subset U of X. Now let $x \notin cl_{\mathcal{I}_\theta}(A)$, then by Lemma 2.2 there exist open subset G of X such that $A \cap G^C \in \mathcal{I}$ and $x \notin cl_{\mathcal{I}_\theta}(G)$ and so $x \notin cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(G))$. Also from Lemma 2.3, it follows that $cl_{\mathcal{I}_\theta}(A) \subset cl_{\mathcal{I}_\theta}(G)$ and so $cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(A)) \subset cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(G))$. Hence $x \notin cl_{\mathcal{I}_\theta}(cl_{\mathcal{I}_\theta}(A))$. □

3. θ-convergence and $T_{2\frac{1}{2}}$ spaces with respect to an ideal

Now we will discuss $\mathcal{I}_\theta$-convergence of a filter.

Definition 3.1. Let $(X, \tau, \mathcal{I})$ be any ideal space and the filter $\mathcal{F}$ on X with $\mathcal{F} \cap \mathcal{I} = \emptyset$. Let $a \in X$ be any element then $\mathcal{F}$ is said to be $\mathcal{I}_\theta$-convergent to a, denoted by $\mathcal{F} \rightarrow_{\mathcal{I}_\theta} a$ if for every open subset U containing a, there exists $F \in \mathcal{F}$ such that $F - \overline{U} \in \mathcal{I}$. We denote the collection of all such points by $\mathcal{I}_\theta$-lim $\mathcal{F}$.

It can be seen easily that $\mathcal{F} \rightarrow_\theta a$ implies $\mathcal{F} \rightarrow_{\mathcal{I}_\theta} a$, but the converse need not be true as can be seen from the example below:

Example 3.1. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, \{b\}, X\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$ and $\mathcal{F} = \{X\}$. Then $\mathcal{F} \rightarrow_{\mathcal{I}_\theta} a$, but $\mathcal{F}$ does not θ-converge to a, since $cl\{a\} = \{a, c\} \notin \mathcal{F}$.

Remark 3.1. Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I} = \emptyset$, then $\mathcal{F} \rightarrow_\theta a$ if and only if $\mathcal{F} \rightarrow_{\mathcal{I}_\theta} a$.

Even though we have seen above $\mathcal{F} \rightarrow_{\mathcal{I}_\theta} a$ does not imply $\mathcal{F} \rightarrow_\theta a$, but the following result shows that in case of ultrafilter both concepts coincide.

Theorem 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\mathcal{F}$ be an ultrafilter with $\mathcal{F} \cap \mathcal{I} = \emptyset$, then $\mathcal{F} \rightarrow_{\mathcal{I}_\theta} a$ if and only if $\mathcal{F} \rightarrow_\theta a$.*

Proof. Let G be open subset of X containing a. Then there exists $F \in \mathcal{F}$ such that $F - \overline{G} \in \mathcal{I}$ since $\mathcal{F} \rightarrow_{\mathcal{I}_\theta} a$. Therefore, $F - \overline{G} = I$ for some $I \in \mathcal{I}$ and so $F \subset \overline{G} \cup I$. Thus $\overline{G} \cup I \in \mathcal{F}$. Further, $\mathcal{F}$ is ultrafilter implies that $\overline{G} \in \mathcal{F}$ as $I \notin \mathcal{F}$. Hence $\mathcal{F} \rightarrow_\theta a$. □

For an ideal space $(X, \tau, \mathcal{I})$ and any subset A of X, the following theorem gives various characterizations for a point to be in the local closure function of A in terms of θ-convergence ($\mathcal{I}_\theta$-convergence) of a filter.

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space and A be any subset of X. Then the following conditions are equivalent:*

(*a*) *$x \in \Gamma(A)(\mathcal{I}, \tau)$.*

(*b*) *there exists a filter $\mathcal{F}$ containing A with $\mathcal{F} \cap \mathcal{I} = \emptyset$ such that $\mathcal{F} \rightarrow_\theta x$.*

(*c*) *there exists a filter $\mathcal{F}$ containing A with $\mathcal{F} \cap \mathcal{I} = \emptyset$ such that $\mathcal{F} \rightarrow_{\mathcal{I}_\theta} x$.*

Proof. (a)⇒(b): Let $x \in \Gamma(A)(\mathcal{I}, \tau)$, so for every open subset G containing x, $\overline{G} \cap A \notin \mathcal{I}$. Consider the filter $\mathcal{F}$ generated by the filterbase $\mathcal{F}(\mathcal{B}) = \{\overline{G} \cap A : G$ is open subset of X containing x $\}$. Therefore, $\overline{G} \cap A \subset \overline{G}$ for every open subset G containing x and $\overline{G} \cap A \subset A$ implies that $\overline{G} \in \mathcal{F}$ and $A \in \mathcal{F}$. Hence $\mathcal{F}$ is the required filter containing A such that $\mathcal{F} \rightarrow_\theta x$.

(b)⇒(c): is obvious.

(c)⇒(a): Let G be open subset of X containing x. Then by (c), there exists $F \in \mathcal{F}$ such that $F - \overline{G} \in \mathcal{I}$ and so $(A \cap F) - \overline{G} \in \mathcal{I}$. On contrary, let $\overline{G} \cap A \in \mathcal{I}$ then $\overline{G} \cap A \cap F \in \mathcal{I}$ and so $((A \cap F) - \overline{G}) \cup ((A \cap F) \cap \overline{G}) \in \mathcal{I}$. This implies that $A \cap F \in \mathcal{I}$, which contradicts that $\mathcal{F} \cap \mathcal{I} = \emptyset$. Hence $\overline{G} \cap A \notin \mathcal{I}$ and so $x \in \Gamma(A)(\mathcal{I}, \tau)$. □

Further, we introduce $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ spaces and obtain its various properties and characterizations.

Definition 3.2. An ideal space $(X, \tau, \mathcal{I})$ is said to be $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ if for any two distinct points x, y of X, there exist open sets U and V such that $x \in U, y \in V$ and $\overline{U} \cap \overline{V} \in \mathcal{I}$.

Since $\emptyset \in \mathcal{I}$, it can be easily seen that every $T_{2\frac{1}{2}}$ space is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$, but the following Example 3.6 shows that the converse need not be true.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ but not $T_{2\frac{1}{2}}$.

Theorem 3.3. *If an ideal space $(X, \tau, \mathcal{I})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ and $\mathcal{I} \subset \mathcal{J}$ then $(X, \tau, \mathcal{J})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{J}$.*

Proof. Proof is obvious and hence is omitted. □

The following Example 3.3 shows that if (X, τ^*) is $T_{2\frac{1}{2}}$, then X need not be $T_{2\frac{1}{2}}$ mod $\mathcal{I}$.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So $\tau^* = \wp(X)$ and hence (X, τ^*) is obviously $T_{2\frac{1}{2}}$, but X is not $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. Since $\overline{\{a\}} \cap \overline{\{b\}} = \{a, c\} \cap \{b, c\} = \{c\} \notin \mathcal{I}$.

Note: It can be seen easily that if (X, τ) is $T_{2\frac{1}{2}}$, then (X, τ^*) is also $T_{2\frac{1}{2}}$. But the above Example 3.3 shows that the converse need not be true.

Even though we have seen that if (X, τ^*) is $T_{2\frac{1}{2}}$, then X need not be $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. but the following Theorem 3.4 shows that for codense ideals (X, τ^*) is $T_{2\frac{1}{2}}$ implies X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$.

Theorem 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense and (X, τ^*) be $T_{2\frac{1}{2}}$ then X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$.*

Proof. Let $x, y \in X$ be any two distinct elements then (X, τ^*) is $T_{2\frac{1}{2}}$ implies there exists basic open subsets $G - I, H - J$ where G, H are open in X and $I, J \in \mathcal{I}$ such that $x \in G - I, y \in H - J$ and $cl^*(G - I) \cap cl^*(H - J) = \emptyset$ and so by Lemma 2.1, $[cl^*(G) - I] \cap [cl^*(H) - J] = \emptyset$. This implies that $(cl^*(G) \cap cl^*(H)) - (I \cup J) = \emptyset$. Therefore, $cl^*(G) \cap cl^*(H) \subset (I \cup J) \in \mathcal{I}$. Also $\mathcal{I}$ is codense implies that $cl^*(G) = cl(G)$ for every open subset G of X. Hence $cl(G) \cap cl(H) \in \mathcal{I}$ and so X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. □

The following Example 3.4 shows that if X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$, then (X, τ^*) need not be $T_{2\frac{1}{2}}$, even if $\mathcal{I}$ is codense. Hence in particular, (X, τ) is not $T_{2\frac{1}{2}}$.

Example 3.4 ([12])**.** Consider the space $X = \Re^2$ with an additional point 0^* with double origin topology τ is given as follows: Neighbourhoods of points other than 0 and 0^* are the usual open sets of $\Re^2 - \{0\}$ and for the basis of neighbourhoods of 0 and 0^*, take $V_n(0) = \{(x, y) \in \Re^2 : x^2 + y^2 < \frac{1}{n^2}, y > 0\} \cup \{0\}$ and $V_n(0^*) = \{(x, y) \in \Re^2 : x^2 + y^2 < \frac{1}{n^2}, y < 0\} \cup \{0^*\}$, where $n \in \mathcal{N}$ and $\mathcal{I} = \wp(\Re) \equiv$ collection of all subsets of real numbers. Then X is not $T_{2\frac{1}{2}}$, since $\{0\}$ and $\{0^*\}$ do not have disjoint closed nhds. as any two nhds. of $\{0\}$ and $\{0^*\}$ contain a segment of the x-axis in the intersection of their closures. But $\mathcal{I} = \wp(\Re)$ implies that X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. Further it can be easily seen that nhds. of $\{0\}$ and $\{0^*\}$ are same in the given topology τ and its $*$ topology τ^* (since τ^* has basis $\beta = \{V - A : V \in \tau \text{ and } A \in \mathcal{I}\}$ and any two nhds. of $\{0\}$ and $\{0^*\}$ do not contain a segment of the x-axis). Also $\mathcal{I}$ is codense implies $cl^*(U) = cl(U)$ for every open subset U of X. Hence (X, τ^*) is not $T_{2\frac{1}{2}}$.

Theorem 3.5. *An ideal space $(X, \tau, \mathcal{I})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ if and only if for all convergent filter $\mathcal{F}$ with $\mathcal{F} \cap \mathcal{I} = \emptyset$, $\mathcal{I}_\theta$-lim $\mathcal{F}$ is unique.*

Proof. Firstly, let X be $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ and $\mathcal{F}$ be any convergent filter with $\mathcal{F} \cap \mathcal{I} = \emptyset$ Let $x \neq y$ be any two elements of X such that $x, y \in \mathcal{I}_\theta$-lim $\mathcal{F}$. Therefore, X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ implies that there exist open subsets U, V of X containing x, y respectively such that $\overline{U} \cap \overline{V} \in \mathcal{I}$. Further $x, y \in \mathcal{I}_\theta$-lim $\mathcal{F}$ implies that there exists $F_1, F_2 \in \mathcal{F}$ such that $F_1 - \overline{U}, F_2 - \overline{V} \in \mathcal{I}$ and so $(F_1 \cap F_2) - (\overline{U} \cap \overline{V}) \in \mathcal{I}$. Thus $((F_1 \cap F_2) - (\overline{U} \cap \overline{V})) \cup (\overline{U} \cap \overline{V}) \in \mathcal{I}$ and so $F_1 \cap F_2 \in \mathcal{I}$, which contradicts that $\mathcal{F}$ does not contain the members of $\mathcal{I}$. Hence $\mathcal{I}_\theta$-lim $\mathcal{F}$ is unique.

Conversely, let $x \neq y$ be any two elements of X and there does not exist any open subsets U, V containing x, y respectively such that $\overline{U} \cap \overline{V} \in \mathcal{I}$. Therefore, it can be easily checked that $\mathcal{F}$ is the filter generated by the filterbase $\mathcal{F}(\mathcal{B}) = \{\overline{U} \cap \overline{V} : U, V$ are open subsets of X containing x, y respectively $\}$. Further for every open subsets U, V containing x, y respectively, $\overline{U} \cap \overline{V} \subset \overline{U}$ and $\overline{U} \cap \overline{V} \subset \overline{V}$ implies that $\overline{U} \in \mathcal{F}$ and $\overline{V} \in \mathcal{F}$. This implies that $x, y \in \mathcal{I}_\theta$-lim $\mathcal{F}$, contradicting the fact that $\mathcal{I}_\theta$-lim $\mathcal{F}$ is unique. Hence X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. □

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent:*

(a) *$(X, \tau, \mathcal{I})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$.*

(b) *If $x \in X$, then for each $y \neq x$, $y \notin \Gamma(\overline{G})(\mathcal{I}, \tau)$ for some open subset G containing x.*

(c) *$\bigcap\{\Gamma(\overline{G})(\mathcal{I}, \tau) : G$ is open subset containing $x\} = \emptyset$ or $\{x\}$ for all $x \in X$.*

Proof. (a)⇒(b): Let $x \in X$ and $y \neq x$ be any element. Then (a) implies that there exists open subsets G and H such that $x \in G, y \in H$ and $\overline{G} \cap \overline{H} \in \mathcal{I}$. Therefore, $y \notin \Gamma(\overline{G})(\mathcal{I}, \tau)$.

(b)⇒(c): Let $x \in X$ and $y \neq x$ be any element. Then by (b), $y \notin \Gamma(\overline{G})(\mathcal{I}, \tau)$ for some open subset G containing x and so $y \notin \bigcap\{\Gamma(\overline{G})(\mathcal{I}, \tau) : G$ is open subset containing $x\}$. Hence (c) holds.

(c)⇒(a): Let x, y be two distinct elements of X.

Then by (c), $y \notin \bigcap\{\Gamma(\overline{G})(\mathcal{I}, \tau) : G$ is open subset containing $x\}$ and so there exist open subset G containing x such that $y \notin \Gamma(\overline{G})(\mathcal{I}, \tau)$. This implies that there exist open subset H containing y such that $\overline{H} \cap \overline{G} \in \mathcal{I}$. Hence, $(X, \tau, \mathcal{I})$ is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$. □

In [3], Gupta and Noiri introduced QHC spaces with respect to an ideal written $\mathcal{I}$-QHC(where An ideal space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-QHC if for every open cover $\{G_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $X - \bigcup\{cl(G_\alpha) : \alpha \in \Delta_0\} \in \mathcal{I}$). We now characterize $\mathcal{I}$-QHC spaces in terms of $\mathcal{I}_\theta$ closure of a set and also prove that every $\mathcal{I}$-QHC set in $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ space is $\mathcal{I}_\theta$ closed.

Theorem 3.7. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following are equivalent:*

(*a*) *$(X,\tau,\mathcal{I})$ is $\mathcal{I}$-QHC.*

(*b*) *for every filter $\mathcal{F}$ with $\mathcal{F} \cap \mathcal{I} = \emptyset$, $\bigcap_{F\in\mathcal{F}} \Gamma(F)(\mathcal{I},\tau) \neq \emptyset$.*

Proof. (a)⇒(b): Let $\mathcal{F}$ be any filter with $\mathcal{F}\cap\mathcal{I} = \emptyset$ such that $\bigcap_{F\in\mathcal{F}} \Gamma(F)(\mathcal{I},\tau) = \emptyset$. Therefore, for all $x \in X$ there exist open set G_x containing x and $F_x \in \mathcal{F}$ such that $\overline{G_x} \cap F_x \in \mathcal{I}$. Now $X = \bigcup_{x\in X} G_x$, so (a) implies that there exist finite subset of X such that $X - \bigcup_{i=1}^n \overline{G_{x_i}} \in \mathcal{I}$. Let $G = \bigcup_{i=1}^n \overline{G_{x_i}}$ and $F = \bigcap_{i=1}^n F_{x_i}$, then $G \cap F \in \mathcal{I}$. Therefore, $(G \cap F) \cup (X - G) \in \mathcal{I}$ and so $F \in \mathcal{I}$ contradicting the fact that $\mathcal{F} \cap \mathcal{I} = \emptyset$ (since finite intersection of members of $\mathcal{F}$ is also in $\mathcal{F}$). Hence $\bigcap_{F\in\mathcal{F}} \Gamma(F)(\mathcal{I},\tau) \neq \emptyset$.

(b)⇒(a): Let $\{G_\alpha : \alpha \in \Delta\}$ be an open cover of X such that there does not exist any finite subset Δ_0 of Δ such that $X - \bigcup_{\alpha\in\Delta_0} \overline{G_\alpha} \in \mathcal{I}$. Therefore, for every finite subset Δ_0 of Δ, $\bigcap_{\alpha\in\Delta_0} (\overline{G_\alpha})^C \notin \mathcal{I}$. Let $\mathcal{F}(\mathcal{B}) = \{\bigcap_{\alpha\in\Delta_0} (\overline{G_\alpha})^C : \Delta_0$ is finite $\}$. Then it can be easily checked that $\mathcal{F}(\mathcal{B})$ is a filterbase not containing the members of $\mathcal{I}$. Now consider the filter $\mathcal{F}$ generated by $\mathcal{F}(\mathcal{B})$ and let $x \in X = \bigcup_\alpha G_\alpha$ and so $x \in G_\alpha$ for some $\alpha \in \Delta$. Therefore, $\overline{G_\alpha} \cap (\overline{G_\alpha})^C = \emptyset$ implies that $x \notin \bigcap_{F\in\mathcal{F}} \Gamma(F)(\mathcal{I},\tau)$ (since $(\overline{G_\alpha})^C \in \mathcal{F}$) contradicting (b). Hence $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-QHC. □

Theorem 3.8. *Let $(X,\tau,\mathcal{I})$ be an ideal space and X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ then every $\mathcal{I}$-QHC set is $\mathcal{I}_\theta$ closed.*

Proof. Let K be an $\mathcal{I}$-QHC subset of X. We have to prove that $cl_{\mathcal{I}_\theta}(K) \subset K$. Let $x \in X$ such that $x \notin K$, then for all $y \in K$, $y \neq x$ and so X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ implies that there exist open subsets G_y and H_y containing x, y respectively such that $\overline{G_y} \cap \overline{H_y} \in \mathcal{I}$. Now $K \subset \bigcup_{y\in K} H_y$ but K is $\mathcal{I}$-QHC implies that there exist finite subset of K such that $K - \bigcup_{i=1}^n \overline{H_{y_i}} \in \mathcal{I}$. Let $G = \bigcap_{i=1}^n G_{y_i}$ and $H = \bigcup_{i=1}^n \overline{H_{y_i}}$ then $\overline{G} \subset \bigcap_{i=1}^n \overline{G_{y_i}}$ implies that $\overline{G} \cap H \in \mathcal{I}$ and so $(\overline{G} \cap H) \cup (K - H) \in \mathcal{I}$. Therefore, $\overline{G} \cap K \in \mathcal{I}$ and so $x \notin cl_{\mathcal{I}_\theta}(K)$ since G is open subset of x. Hence K is $\mathcal{I}_\theta$ closed. □

Corollary 3.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space and X is $T_{2\frac{1}{2}}$ mod $\mathcal{I}$ then every $\mathcal{I}$-compact set is $\mathcal{I}_\theta$ closed.*

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