AN ONC-CHARACTERIZATION OF A_{14} AND A_{15}

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Abstract. Let G be a finite group, $o_1(G)$ denote the largest element order of G, $n_1(G)$ the number of the elements of order $o_1(G)$. Assume that G totally has r elements of order $o_1(G)$, whose centralizers have distinct orders, say, they are $c_i(G)$, $i = 1,\ 2,\ \cdots,\ r$. The following quantity is called the 1st $ONC-$degree of G

$$ONC_1(G) = \{o_1(G);\ n_1(G);\ c_1(G),\ c_2(G),\ \cdots,\ c_r(G)\},$$

denoted as $ONC_1(G)$. It has been proved that K_3-simple groups, $L_2(q)$($q = 8$, 11, 13, 17, 19, 23, 29), Mathieu simple groups, Janko Groups and alternating groups $A_n(5 \leq n \leq 13)$ can be characterized by their 1st $ONC-$degrees, but unfortunately $L_2(q)(q = 16,\ 25)$ cannot be characterized by the 1st $ONC-$degree. Since the $ONC-$degree of an alternating group usually contains only 3 numbers, so it is interesting to study if an alternating group can be characterized by the 1st $ONC-$degree. We shall prove that A_{14} can be characterized by the 1st $ONC-$degree, but we can not prove A_{15} does by using our approaches. We shall prove if the prime graph of G is not connected and $ONC_1(G) = ONC_1(A_{15})$, then $G \cong A_{15}$.
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1. Introduction

Professor W. J. Shi put forward the famous conjecture in 1989:

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Shi's Conjecture. Let G be a finite group, M a finite simple group, then $G \cong M$ if and only if $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the set of element orders in G (see [1]).

The conjecture is recorded as Problem 12.39 in Unsolved Problems in Group Theory (see [7]). Research on Shi's conjecture opened the era of quantitative characterization of finite simple groups since 1980's. In 2009, Shi's conjecture was completely proved. Afterwards, an interesting topic is trying to weaken conditions of Shi's conjecture since the set of element orders seems containing too many numbers. The last two authors defined the 1st $ONC-$ degree in [2]. Let G be a finite group, $o_1(G)$ denote the largest element order of G, $n_1(G)$ the number of the elements of order $o_1(G)$. Assume that G totally has r elements of order $o_1(G)$, whose centralizers have distinct orders, say, $c_i(G)$, $i = 1,\ 2,\ \cdots,\ r$. The following quantity is called the 1st $ONC-$degree of G

$$ONC_1(G) = \{o_1(G);\ n_1(G);\ c_1(G),\ c_2(G),\ \cdots,\ c_r(G)\},$$

denoted as $ONC_1(G)$. Notice $ONC_1(G)$ is not a set, but a series of numbers.

Because in many groups, orders of centralizers of elements having the largest order are the same, so the 1st $ONC-$ degree often contains only three numbers, for example, in alternating groups or symmetric groups. Hence the 1st $ONC-$ degree contains less numbers than Shi's conjecture in some cases. The reason why orders of centralizers of elements of largest orders are considered in the 1st $ONC-$ degree is that they almost determine the prime graph of a finite group. Hence, it is meaningful to study if a finite group, especially a finite simple group, can be characterized by the 1st $ONC-$degree.

Li-Guan He characterized some non-abelian simple groups by the 1st $ONC-$degree in his doctoral dissertation , such as K_3 simple groups , A_5, A_6, $L_2(8)$, $L_3(3)$ and $L_2(17)$ (see [2] and [4]). Apart from those, by comparing the second order or other special numbers, he also characterized some other sporadic simple groups in [2]. Later, Li-Guan He and Gui-Yun Chen continued to study the 1st $ONC-$degree. For example, it has been proved that Mathieu groups and Janko Groups can be characterized by the 1st $ONC-$degree in [2] and [3]. In [5], it is proved that $L_2(q)$ for $q = 11,\ 13,\ 19,\ 23,\ 29$ can be characterized by the 1st $ONC-$degree and $L_2(16)$ and $L_2(25)$ cannot, but a classification of finite groups G such that $ONC_1(G) = ONC_1(L_2(16))$ or $ONC_1(L_2(25))$ is given. For alternating groups, Li-Guan He proved that alternating groups A_n, $n \leq 13$, can be characterized by the 1st $ONC-$degree in [6]. In this paper, we continue to discuss the 1st $ONC-$degree characterization of alternating groups, and prove that A_{14} can be characterized by the 1st $ONC-$degree, A_{15} can be characterized by the 1st $ONC-$degree and its prime graph.

2. Prelimiaries

In this section we present some lemmas which are required in Section 3.

Lemma 2.1 ([10], Theorem A). *If G is a finite group whose prime graph has more than one component, then G has one of the following structures:*

(1) *G is a Frobenius group or a $2-Frobenius$ group;*

(2) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1 groups, K/H is a simple group, H is a nilpotent group, where $2 \in \pi_1$. And $|G/K|\big||Out(K/H)|$.*

Lemma 2.2. *Let G and H be two groups. Suppose that G acts on H co-primely, then for every prime $p\big||G|$, H has a $G-$invariant $p-$Sylow subgroup.*

Lemma 2.3. *Let G be a $p-$group of order p^n and K an elementary commutative group of order p^n, then $|Aut(K)|$ is divided by $|Aut(G)|$.*

Lemma 2.4. *Let G be a p group of order p^n, and G act on a q-group H of order q^α, where p and q are distinct primes. If $|G| \nmid \prod_{i=1}^{\alpha}(q^i-1)$, then $pq \in \pi_e(G \ltimes H)$.*

Proof. A group G acts group H, so $G/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$, then

$$|G/C_G(H)|\big||Aut(H)|\big|q^{\frac{\alpha(\alpha-1)}{2}} \cdot \prod_{i=1}^{\alpha}(q^i-1).$$

Since $(|p|,\ q^{\frac{\alpha(\alpha-1)}{2}}) = 1$ and $|G| \nmid \prod_{i=1}^{\alpha}(q^i-1)$, we have $|G| \nmid |Aut(H)|$. Therefore $C_G(H) \neq 1$, which concludes that $p\big||C_G(H)|$, $pq \in \pi_e(G \ltimes H)$. □

Lemma 2.5 ([10], Corollary). *If G is solvable with a non-connected prime graph. Then G is either a Frobenius or a $2-Frobenius$ group. Moreover if G is a $2-Frobenius$ group, then G has exactly two components, and one of which consists of primes dividing the lower Frobenius complement.*

Lemma 2.6. (1) *Let G be a Frobenius group (not a $2-Frobenius$ group) with Frobenius kernel H and Frobenius complement K. then it has a non-connected prime graph, and the vertex sets of prime graph components of G are exactly $\{\pi(H),\ \pi(K)\}$.*

(2) *Let G be a $2-Frobenius$ group, then $G = ABC$, where A is normal in G, AB is a Frobenius group with Frobenius kernel A and Frobenius complement B, BC is a Frobenius group with Frobenius kernel B and Frobenius complement C, where B and C are cyclic groups. In addition, $|C|\big||Aut(B)|$. The vertex sets of prime graph components of G are exactly $\{\pi(A) \cup \pi(C),\ \pi(B)\}$.*

Proof. At first, we prove (2). By the definition of $2-Frobenius$ group, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that G/H and K are *Frobenius* groups with *Frobenius* kernels K/H and H respectively. Hence, let $K = HL$, where L is the *Frobenius* complement of K. Based on the generalized *Frattini* Argument, we have that $G = N_G(L)K = N_G(L)H$. Because $N_G(L) \cap H = N_H(L) = 1$, $N_G(L) \cong G/H$ is a *Frobenius* group. (2) follows from Lemma 2.5.

Now we prove (1). Since H is a nilpotent group, vertexes in $\pi(H)$ belong to one component and K acts fixed-point-freely on H, so $\pi(H)$ must be an independent prime graph component. If K is unsolvable, owing to [8], we know that K has a normal subgroup K_0 which is isomorphic to $SL(2,\ 5)$, $|K : K_0| \le 2$, $K_0 \cong Z \times SL(2,\ 5)$, $(|Z|,\ 30) = 1$. Because the prime graph of $SL(2,\ 5)$ is connected, the prime graph of K must be connected. If K is solvable and has more than one prime graph component, by Lemma 2.5, K is a $Frobenius$ or a $2-Frobenius$ group, consequently G is a $2-Frobenius$ group, G has the structure in (2). Now (1) follows. □

Theorem 2.7. *Let G be a finite group, $M = A_{14}$, then $G \cong M$ if and only if $ONC_1(G)= ONC_1(M)$.*

Proof. The necessity is obvious. It is enough to prove the sufficiency.

Because the largest element order of A_{14} is 45, the number of elements of order 45 in A_{14} is $\frac{P_{14}^9}{9} \cdot \frac{P_5^5}{5} = 2^{11} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13$, where P_m^n is the number of permutations of m letters taking from n letters, and all elements of order 45 in A_{14} are self-centralized, thus, $ONC_1(G)= ONC_1(A_{14})=\{45;\ 2^{11} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13;\ 45\}$. We may assume that elements of order 45 in G are partitioned into t conjugacy classes. Due to lengths of conjugacy classes of any two elements of order 45 in G are equal to $\frac{|G|}{45}$, then $t \cdot \frac{|G|}{45} = 2^{11} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13$. Hence $|G|\big|2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$ and $|G| > 2^{11}\cdot 3^3\cdot 5\cdot 7^2\cdot 11\cdot 13$. Now $45\big||G|$ yields $3, 5\in \pi(G)$. If $2\notin \pi(G)$, then $|G|$ divides $3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$, so $|G| < 2^{11} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13$, a contradiction. Hence $2\in \pi(G)$. Noticing $2^{11} \cdot 3^5 \cdot 5^2 \cdot 11 \cdot 13 < 2^{11} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13$, we get $7 \in \pi(G)$; By $2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 < 2^{11} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13$, we get that $\{11,\ 13\} \cap \pi(G) \ne \varnothing$.

Now we divide our proof into several steps.

(1) We prove that $\{11,\ 13\} \subset \pi(G)$.

It is enough to show both $13\in \pi(G)$, $11\notin \pi(G)$ and $13\notin \pi(G)$, $11\in \pi(G)$ are impossible.

(1.1) If $13\in \pi(G)$ and $11\notin \pi(G)$, then $|G| > 2^{11} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13$ yields $7^2\big||G|$. We assert that: G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a non-abelian simple group and $\{7,\ 13\} \subset \pi(K/H)$.

In fact, considering chief series of G: $1 = G_k \trianglelefteq G_{k-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = G$, we may assume $\{7,\ 13\}\cap\pi(G_i) \ne \varnothing$ and $\{7,\ 13\}\cap\pi(G_{i+1}) = \varnothing$. Let $H = G_{i+1}$ and $K = G_i$, then $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ is a normal series of G, K/H is a minimal normal subgroup of G/H, and further K/H is the direct product of isomorphic simple groups. Now we prove that $\{7,\ 13\} \subset \pi(K/H)$, hence K/H is a direct product of nonabelian simple groups.

Otherwise, either $7 \in \pi(K/H)$, $13 \notin \pi(K/H)$ or $7 \notin \pi(K/H)$, $13 \in \pi(K/H)$. If the former one holds, considering the action of an element of order 13 of G/H on K/H by conjugation, and noticing $13 \nmid (7-1)(7^2-1) = 2^5 \cdot 3^2$, we conclude that $91 \in \pi_e(G)$ by Lemma 2.2, 2.3 and 2.4. Hence $91 > o_1(G) = 45$, a contradiction. Therefore $13 \in \pi(K/H)$ while $7 \in \pi(K/H)$. Similarly, we can get a contradiction too while $7 \notin \pi(K/H)$, $13 \in \pi(K/H)$. Therefore $\{7,\ 13\} \subset \pi(K/H)$, and K/H is a direct product of non-abelian simple groups. Further by $13\big|\big||G|$, we get that K/H is a nonabelian simple group, the assertion follows.

Now $|G|\big|2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$. Checking simple groups of orders divided by 13 and dividing $2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 13$ in [9], one gets that K/H may be one of the following groups:

$$L_2(13)(2^2\cdot 3\cdot 7\cdot 13),\ L_2(27)(2^2\cdot 3^3\cdot 7\cdot 13),\ S_z(8)(2^6\cdot 5\cdot 7\cdot 13),$$
$$L_2(64)(2^6\cdot 3^2\cdot 5\cdot 7\cdot 13).$$

For above groups, it always holds that $7\big|\big||Aut(K/H)|$. Since $7^2\big||G|$ and $7\nmid|H|$, we have $7\in\pi(C_{G/H}(K/H))$ by $G/H/C_{G/H}(K/H)\leq Aut(K/H)$, so $7\times 13=91\in\pi_e(G)$, which contradicts $o_1(G)=45$. Thus $11\in\pi(G)$ while $13\in\pi(G)$.

(1.2) If $11\in\pi(G)$, $13\notin\pi(G)$, by the same reasoning as above, we conclude that $\{7,\ 11\}\subset\pi(K/H)$, and K/H is a non-abelian simple group. Checking simple groups of orders divided by 11 and dividing $2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11$ in [9], we get that K/H is one of the following groups:

$$M_{22}(2^7\cdot 3^2\cdot 5\cdot 7\cdot 11),$$
$$A_{11}(2^7\cdot 3^4\cdot 5^2\cdot 7\cdot 11),$$
$$A_{12}(2^9\cdot 3^5\cdot 5^2\cdot 7\cdot 11).$$

By the same reasoning as in (1.1), we come to $7\in\pi(C_{G/H}(K/H))$, hence $77\in\pi_e(G)$, again a contradiction to $o_1(G)=45$. Therefore $13\in\pi(G)$ while $11\in\pi(G)$, which concludes (1).

(2) We prove that $G\cong A_{14}$.

It follows by that $\pi(G)=\{2,\ 3,\ 5,\ 7,\ 11,\ 13\}$, $|G|\big|2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$, G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$, where K/H is a non-abelian simple group and $\{11,\ 13\}\subset\pi(K/H)$. By [9], we get that K/H may be one of $A_{13}(2^9\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13)$ and $A_{14}(2^{10}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13)$.

If $K/H=A_{13}$, then $7\big|\big||Aut(K/H)|$. If $7^2\big||G|$, it follows that $7\big||H|$ or $7\in\pi(C_{G/H}(K/H))$ by $G/H\big/C_{G/H}(K/H)\leq Aut(K/H)$. While $7\big||H|$, H has a $7-Sylow$ subgroup of order 7, and $13\notin\pi(H)$. Consider the action of an element of order 13 of G on H by conjugation, we get $91\in\pi_e(G)$ by Lemma 2.2, which contradicts $o_1(G)=45$. Hence $7\in\pi(C_{G/H}(K/H))$, which means $91\in\pi_e(G)$, a contradiction. Thus $7\ \big|\big||G|$, so $|H|\big|4$ by comparing orders of G and A_{13}. Then $|Aut(H)|\big|6$, which implies that $|G/C_G(H)|\big|6$, hence $\{2,\ 3,\ 5,\ 7,\ 11,\ 13\}\subseteq\pi(C_G(H))$. However $H\leq C_G(H)\trianglelefteq G$ yields $(C_G(H)/H)\cap K/H=1$ or K/H. Therefore $C_G(H)\cap K=H$ or K. Obviously, the former one is impossible. For the latter, it follows $H\leq Z(K)$. But A_{13} has an element of order 35, thus K must have an element of order 70, a contradiction.

Now we have that $K/H=A_{14}$ and then $|H|\big|2$. If $|G|=2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13$. If $|H|=1$, then $K=A_{14}$, then $G=S_{14}$ or $G=A_{14}\times C_2$, both of which has an element of order 90, a contradiction. Therefore $|H|=2$, let an element of order 45 of G act on H by conjugation, we conclude $90\in\pi_e(G)$, which contradicts $o_1(G) = 45$. Hence $|G| = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$. Therefore $H = 1$ and $G = K = A_{14}$. This concludes the theorem. □

Theorem 2.8. *Let G be a finite group with non-connected prime graph, $M = A_{15}$, then $G \cong M$ if and only if $ONC_1(G) = ONC_1(M)$.*

Proof. The necessity is obvious. It is enough to show the sufficiency.

Because the largest element order of A_{15} is $3 \times 5 \times 7$, the number of elements of order 105 in A_{15} is $\frac{P_{15}^3}{3} \cdot \frac{P_{12}^5}{5} \cdot \frac{P_7^7}{7} = 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, and every element of the largest order is self-centralized, thus

$$ONC_1(G) = ONC_1(A_{15}) = \{105;\ 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13;\ 105\}.$$

Assume that elements of order 105 in G are divided into t conjugacy classes. Due to lengths of conjugacy classes of elements of order 105 in G are the same and equal to $\frac{|G|}{105}$, so $t \cdot \frac{|G|}{105} = 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, which implies that $|G| \big| 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ and $|G| > 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Obviously 3, 5, $7 \in \pi(G)$. If $2 \notin \pi(G)$, then $|G| \leq 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 < 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, a contradiction. Therefore $2 \in \pi(G)$. If $(11 \times 13,\ |G|) = 1$, then $2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 < 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, hence $\{11,\ 13\} \cap \pi(G) \neq \varnothing$. By $|G| > 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ and trivial comparing, we have that $2^8 \big| |G|$, $3^5 \big| |G|$ and $5^2 \big| |G|$.

Now we divide the proof into several steps.

(1) to prove G is neither a Frobenius group nor a 2-Frobenius group.

(1.1) If G is a *Frobenius* group with *Frobenius* kernel H and *Frobenius* complement K, then $G = HK$ and either $105 \big| |H|$ or $105 \big| |K|$ by Lemma 2.7. If $105 \big| |H|$, then $\pi(H) = \{3,\ 5,\ 7\}$. Otherwise, $o_1(G) \geq o_1(H) > 105$. As $\{11,\ 13\} \cap \pi(G) \neq \varnothing$, so $2^8 \cdot 11$ or $2^8 \times 13 \big| |K|$. Now consider the action of $2 - Sylow$ subgroup of K on the $7 - Sylow$ subgroup of H, one comes to that G has an element of order 14, consequently the prime graph of G is connected, a contradiction. If $105 \big| |K|$, then K has an element of order 105. If $11 \big| |H|$ or $13 \big| |H|$, we consider the action of an element of order 7 in K on the $11 - Sylow$ subgroup of H, which concludes that there exists an element of order 77 or an element of order 91 in G, both contradict Lemma 2.6. So H is just a $2-$group and $2^8 \big| |H| \big| 2^{11}$. Notice $5^2 \big| |K|$ and consider the action of a $5 - Sylow$ subgroup of G on H by conjugation, we see by Lemma 2.4 that there exists an element of order 10 in G, a contradiction to Lemma 2.6.

(1.2) If G is a $2 - Frobenius$ group, then $G = ABC$, where $A,\ B,\ C$ are as in Lemma 2.6. Since $\pi(B)$ is a vertex set of a prime graph component of G, so either $105 \big| |B|$ or $105 \big| |A||C|$. Hence if the former holds, then $5^2 \times 3^5 \times 7 \big| |G|$. But B is a cyclic group, so B has an element of order $\geq 5^2 \cdot 3^5 \cdot 7$, a contradiction. If the latter case holds, then $2^8 \big| |B|$ and at least one of $11 \big| |B|$ and $13 \big| |B|$ holds. Since B is a cyclic group, then $|B| \leq 105$, so $2^8 \nmid |B|$, $11 \times 13 \nmid |B|$. Hence $|B| = 11$ or 13, and then $|C| \big| 10$ or 12 by Lemma 2.6. Thus $7 \big| |A|$. Note that the $7-$Sylow subgroup D of A is a normal subgroup of G, where $|D| \big| 7^2$. We observe the action of B on D by conjugation and come to that there exists elements of order 77 or 91 in G by Lemma 2.4, which contradicts that AB is a *Frobenius* group. (1) follows.

Now by Lemma 2.1, we have the following:

(2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H ia a non-abelian simple group, and H is a nilpotent group, $\pi(K/H) \subset \{2, 3, 5, 7, 11, 13\}$.

(3) It is impossible that $11 \in \pi(G)$, $13 \notin \pi(G)$ and $11 \big| |K/H|$.

Otherwise, $11 \big| |K/H| \big| 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$. Checking simple groups of order divided by 11 and dividing $|K/H| \big| 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$ in [9], we get that K/H may be one of the following groups: $L_2(11)(2^2 \cdot 3 \cdot 5 \cdot 11)$, $M_{11}(2^4 \cdot 3^2 \cdot 5 \cdot 11)$, $M_{12}(2^6 \cdot 3^3 \cdot 5 \cdot 11)$, $M_{22}(2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$, $U_5(2)(2^{10} \cdot 3^5 \cdot 5 \cdot 11)$, $A_{11}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11)$, $HS(2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11)$, $A_{12}(2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11)$, $M^cL(2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11)$.

(3.1) to prove that $5,\ 7 \notin \pi(H)$.

Note that $|G| \big| 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ and $11 \big| |K/H| \big| 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, one has $|H| \big| 2^9 \cdot 3^5 \cdot 5^2 \cdot 7^2$.

If $7 \in \pi(H)$, let an element of order 11 of G act on H by conjugation, since $11 \nmid \prod_{i=1}^{2}(7^i - 1) = 2^5 \cdot 3^2$, we get $7 \times 11 \in \pi_e(G)$ by Lemma 2.4. If $2 \in \pi(H)$, then the $2-$Sylow subgroup of H is of order dividing 2^9. Since $11 \nmid \prod_{i=1}^{9}(2^i - 1) = 3^5 \cdot 5^2 \cdot 7^2 \cdot 31 \cdot 73 \cdot 127$, it follows $11 \times 2 \in \pi_e(G)$ by Lemma 2.4 and considering the action of an element of order 11 on 2-Sylow subgroup of H, this implies that G has a connected prime graph, a contradiction. Hence $2 \notin \pi(H)$, thus we can consider the co-prime action of subgroup of order 2^8 of G on H by conjugation. Since $2^8 \nmid \prod_{i=1}^{2}(7^i - 1) = 2^5 \cdot 3^2$, we come to $2 \times 7 \in \pi_e(G)$ by Lemma 2.2, 2.3 and 2.4, again G has a connected prime graph, a contradiction. Therefore $7 \notin \pi(H)$.

If $5 \in \pi(H)$, by $11 \nmid \prod_{i=1}^{2}(5^i - 1) = 2^5 \cdot 3$, we get $5 \times 11 \in \pi_e(G)$. In this case, if $2 \in \pi(H)$, we can prove by the same approach that $11 \times 2 \in \pi_e(G)$, then G has a connected prime graph, a contradiction. So $2 \notin \pi(H)$. Consider the action of a subgroup of order 2^8 of G on H by conjugation and notice $2^8 \nmid \prod_{i=1}^{2}(5^i - 1) = 2^5 \cdot 3$, we get $2 \times 5 \in \pi_e(G)$ by Lemma 2.4, this implies that G has a connected prime graph, a contradiction. This concludes (3.1).

(3.2) to prove that $K/H \neq L_2(11)$, M_{11}, M_{12} or $U_5(2)$.

If K/H is one of $L_2(11)$, M_{11}, M_{12} and $U_5(2)$, then $7 \notin \pi(K)$. Let an element of order 7 of G act on K by conjugation, because $7 \nmid (11 - 1) = 2 \times 5$, G has an element of order 77 by Lemma 2.4.

If $2 \in \pi(H)$, consider the action of an element of order 11 of G on the $2-$*Sylow* subgroup of H by conjugation, by $11 \nmid \prod_{i=1}^{9}(2^i - 1) = 3^5 \cdot 5^2 \cdot 7^2 \cdot 31 \cdot 73 \cdot 127$ and Lemma 2.4, we get $2 \times 11 \in \pi_e(G)$. Furthermore, the prime graph of G is connected, a contradiction. Hence $\{2,\ 5,\ 7\} \notin \pi(H)$ by (3.1). Therefore H is a $3-$group.

If $K/H = L_2(11)(2^2 \cdot 3 \cdot 5 \cdot 11)$, then $2^2 \big\| |K|$. Let an element of order 7 act on K by conjugation, we get $7 \times 2 \in \pi_e(G)$, which means that the prime graph of G is connected, a contradiction. Hence $K/H \neq L_2(11)$.

If K/H is one of $M_{11}(2^4\cdot 3^2\cdot 5\cdot 11)$, $M_{12}(2^6\cdot 3^3\cdot 5\cdot 11)$ and $U_5(2)(2^{10}\cdot 3^5\cdot 5\cdot 11)$, then $|H|\big||3^4$. Assume $H\neq 1$, considering the action of an element of order 77 on H by conjugation, we get by $(77,\ \prod_{i=1}^{4}(3^i-1))=(77,\ 2^9\cdot 5\cdot 13)=1$ and Lemma 2.4 that G has an element of order 77×3, which contradicts $o_1(G)=105$. So $H=1$ and K is one of $M_{11},\ M_{12}$ and $U_5(2)$. Because $G/C_G(K)\leq Aut(K)$. Notice that $|Out(K)|=1$ or 2, we come to $|Aut(K)||2^{11}\cdot 3^4\cdot 5\cdot 11$. However $5^2\big||G|$, hence $5\in\pi(C_G(K))$, this implies that $5\times 2,\ 5\times 11\in\pi_e(G)$, which implies that G has a connected prime graph, a contradiction.

(3.3) to prove $K/H\neq A_{11},\ HS,\ A_{12}$ or M^cL.

If K/H is one of $A_{11}(2^7\cdot 3^4\cdot 5^2\cdot 7\cdot 11)$, $HS(2^9\cdot 3^2\cdot 5^3\cdot 7\cdot 11)$, $A_{12}(2^9\cdot 3^5\cdot 5^2\cdot 7\cdot 11)$ and $M^cL(2^7\cdot 3^6\cdot 5^3\cdot 7\cdot 11)$. It follows by (3.1) that $5,\ 7\notin\pi(H)$. Hence $|H|\big|2^4\cdot 3^4$. If $2\big||H|$, noticing $5^2\nmid\prod_{i=1}^{4}(2^i-1)=3^2\cdot 5\cdot 7,\ 11\nmid\prod_{i=1}^{4}(2^i-1)=3^2\cdot 5\cdot 7$, considering the actions of subgroups of order 5^2 and 11 in G on H by conjugation respectively, we conclude that $5\times 2,\quad 11\times 2\in\pi_e(G)$. Hence the prime graph of G is connected, a contradiction. This also means that if $H\neq 1$ then $|H|\big|3^4$. Because $11\nmid\prod_{i=1}^{4}(3^i-1)=2^9\cdot 5\cdot 13$, considering action of element of order 11 on H by conjugation, we can get $3\times 11\in\pi_e(G)$ by Lemma 2.4. Since $G/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$, and $|Aut(H)|\big|3^6\cdot\Pi_{i=1}^{4}(3^i-1)=2^9\cdot 3^6\cdot 5\cdot 13$, we have that $7\big||C_G(H)|$. If $2\in\pi(C_G(H))$, then $2\times 3\in\pi_e(G)$, the prime graph of G is connected, a contradiction. Therefore $2\notin\pi(C_G(H))$, any subgroup of order 2^8 in G acts on $C_G(H)$ co-primely by conjugation. Since $7\big||C_G(H)|$, there exists a $G-$invariant $7-Sylow$ subgroup of $C_G(H)$. But $2^8\nmid\prod_{i=1}^{2}(7^i-1)=2^5\cdot 3^2$, so Lemma 2.4 implies $2\times 7\in\pi_e(G)$, which concludes the prime graph of G is connected, a contradiction. Therefore $H=1$, and K is one of $A_{11},\ HS,\ A_{12}$ and M^cL. Moreover $G/C_G(K)\leq Aut(K)$. If $C_G(K)\neq 1$, G has a connected prime graph, a contradiction. Hence $C_G(K)=1,\ G\leq Aut(K)$. But $|Out(K)|=2$ for these groups, hence $|Aut(K)|\big|2^{10}\cdot 3^6\cdot 5^3\cdot 7\cdot 11$, which contradicts $|G|>2^{11}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$.

(4) It is impossible that $11\in\pi(G)$, $13\notin\pi(G)$ or $11\nmid|K/H|$.

Otherwise, by [9] and step (2), K/H is one of the following groups: $A_5(2^2\cdot 3\cdot 5)$, $L_3(2)(2^3\cdot 3\cdot 7)$,$A_6(2^3\cdot 3^2\cdot 5)$, $L_2(8)(2^3\cdot 3^2\cdot 7)$, $A_7(2^3\cdot 3^2\cdot 5\cdot 7)$, $U_3(3)(2^5\cdot 3^3\cdot 7)$, $A_8(2^6\cdot 3^2\cdot 5\cdot 7)$, $U_4(2)(2^6\cdot 3^4\cdot 5)$, $L_2(49)(2^4\cdot 3\cdot 5^2\cdot 7^2)$, $U_3(5)(2^4\cdot 3^2\cdot 5^3\cdot 7)$, $A_9(2^6\cdot 3^4\cdot 5\cdot 7)$, $J_2(2^7\cdot 3^3\cdot 5^2\cdot 7)$, $L_3(4)(2^6\cdot 3^2\cdot 5\cdot 7)$, $S_6(2)(2^9\cdot 3^4\cdot 5\cdot 7)$, $A_{10}(2^7\cdot 3^4\cdot 5^2\cdot 7)$, $U_4(3)(2^7\cdot 3^6\cdot 5\cdot 7)$.

For the above groups, 11 does not divide the order of the outer automorphism group of any group above. Hence $11\in\pi(H)$, hence $11\big||H|\big|2^9\cdot 3^5\cdot 5^3\cdot 7^2\cdot 11$. By Lemma 2.1, H is nilpotent. Considering the action of a subgroup of order 2^2 and a subgroup of order 3^2 of G on the $11-Sylow$ subgroup of H by conjugation, we come to $11\times 2,\ 11\times 3\in\pi_e(G)$, which implies that the prime graph of G is connected, a contradiction. Step (4) follows.

(5) It is impossible that $13\in\pi(G)$ or $11\notin\pi(G)$.

Here we mention the proved fact that $5^2\big||G|$, $3^4\big||G|$ and $2^8\big||G|$.

If $13 \notin \pi(K/H)$, then $\pi(K/H)$ does not contain 11 and 13, which is a case as Step (4), a contradiction.

If $13 \in \pi(K/H)$, then K/H is one of the following groups by [9]: $L_2(13)(2^2 \cdot 3 \cdot 7 \cdot 13)$, $L_3(3)(2^4 \cdot 3^3 \cdot 13)$, $L_2(25)(2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(27)(2^2 \cdot 3^3 \cdot 7 \cdot 13)$, $S_z(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$, $U_3(4)(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(64)(2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$, $G_2(3)(2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $L_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, ${}^2F_4(2)(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$, $L_3(9)(2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13)$.

For all simple groups above, $|H|\big|2^9 \cdot 3^5 \cdot 5^3 \cdot 7^2$ always follows.

(5.1) to prove $5,\ 7 \notin \pi(H)$**.**

If $7 \in \pi(H)$, let an element of order 13 in G act on the $7-Sylow$ subgroup of H by conjugation, then $7 \times 13 \in \pi_e(G)$ by $13 \nmid \prod_{i=1}^{2}(7^i - 1) = 2^5 \cdot 3^2$ and Lemma 2.4. In this case, if $2 \in \pi(H)$, we consider action of an element of order 13 in G on the $2-Sylow$ subgroup of H by conjugation, by $13 \nmid \prod_{i=1}^{9}(2^i - 1) = 3^5 \cdot 5^2 \cdot 7^2 \cdot 31 \cdot 73 \cdot 127$ and Lemma 2.4, we conclude that $13 \times 2 \in \pi_e(G)$, thus the prime graph of G is connected, a contradiction. So $2 \notin \pi(H)$. Considering the action of a subgroup of order 2^8 in G on the $7-Sylow$ subgroup of H by conjugation, and noticing $2^8 \nmid \prod_{i=1}^{2}(7^i - 1) = 2^5 \cdot 3^2$, we get by Lemma 2.4 that $2 \times 7 \in \pi_e(G)$. Hence G has a connected prime graph, a contradiction. Therefore $7 \notin \pi(H)$.

If $5 \in \pi(H)$, by $13 \nmid \prod_{i=1}^{2}(5^i - 1) = 2^5 \cdot 3$ and Lemma 2.4, we have $5 \times 13 \in \pi_e(G)$. Now if $2 \in \pi(H)$, then $2 \times 13 \in \pi_e(G)$, the prime graph of G is connected, a contradiction. So $2 \notin \pi(H)$. Now let a subgroup of order 2^8 in G act on the $5-Sylow$ subgroup of H by conjugation, since $2^8 \nmid \prod_{i=1}^{2}(5^i - 1) = 2^5 \cdot 3$, it follows $2 \times 5 \in \pi_e(G)$ by Lemma 2.4, so the prime graph of G is connected, a contradiction.

(5.2) to prove $K/H \neq L_2(13),\ L_3(3),\ L_2(27)$ **or** $G_2(3)$**.**

If $K/H{=}L_2(13)(|Out(K/H)| = 2)$, $L_3(3)(|Out(K/H)| = 2)$,

$L_2(27)(|Out(K/H)| = 6)$ or $G_2(3)$ $(|Out(K/H)| = 2)$, then $5 \notin \pi(Aut(K/H))$. By $G/H/C_{G/H}(K/H) \leqslant Aut(K/H)$, we get $5 \in \pi(C_{G/H}(K/H)$ by (5.1). Consequently, G has a connected prime graph, a contradiction.

(5.3) to prove $K/H \neq L_2(25)$**,** $U_3(4)$**,** $L_4(3)$ **or** ${}^2F_4(2)$**.**

Otherwise, if K/H is one of above groups, then $|Out(K/H)|\big|4$, $7 \notin \pi(K/H)$. Thus $7 \notin \pi(Aut(K/H))$, then $7 \in \pi(C_{G/H}(K/H)$ by (5.1), which implies that the prime graph of G is connected, a contradiction.

(5.4) to prove that K/H cannot be any one of rest groups listed in beginning of the proof.

If K/H is one of rest groups, then $|H|\big|2^5 \cdot 3^6$. If $2 \in \pi(H)$, by $13 \nmid \prod_{i=1}^{5}(2^i - 1) = 3^2 \cdot 5 \cdot 7 \cdot 31$, $5^2 \nmid \prod_{i=1}^{5}(2^i - 1) = 3^2 \cdot 5 \cdot 7 \cdot 31$ and Lemma 2.4, we come to $13 \times 2 \in \pi_e(G)$, $5 \times 2 \in \pi_e(G)$, which implies the prime graph of G connected, a contradiction.

If $H \neq 1$, then H is a $3-$ group. For possible choice of K/H, it always follows that $|Out(K/H)| = 2$ or 2^2, or 3, hence $|Aut(K/H)|\big|2^9 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$. But $5^2\big||G|$ yields $5 \in \pi(C_{G/H}(K/H))$, then $5 \times 2 \in \pi_e(G)$, $5 \times 13 \in \pi_e(G)$, so G has only one prime graph component, a contradiction.

(6) If $\{11,\ 13\} \subset \pi(G)$**, then** $G \cong A_{15}$.

It follows from (3)-(5) that $\pi(G) = \{2,\ 3,\ 5,\ 7,\ 11,\ 13\}$.

Now we assert $\{11,\ 13\} \subset \pi(K/H)$. Otherwise, it follows by (4) and (5) that $11 \in \pi(K/H)$ and $13 \notin \pi(K/H)$, hence either $13 \big| |H|$ or $13 \big| |G/K|$. If the former holds, we consider the conjugate action of an element of order 11 of G on the $13-Sylow$ subgroup of H and get $11 \times 13 = 143 \in \pi_e(G)$, a contradiction. Hence $13 \in \pi(G/K)$. Since $(13,\ |K/H|) = 1$, we consider the action of an element of order 13 in G/H on K/H by conjugation and come to some $11-Sylow$ subgroup of K/H is fixed. So $11 \times 13 = 143 \in \pi_e(G)$, a contradiction.

Hence $\pi(K/H) = \{2,\ 3,\ 5,\ 7,\ 11,\ 13\}$. Comparing the orders of K/H and G, we have that $K/H = A_{13}(2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$, $A_{14}(2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13)$ or $A_{15}(2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)$.

For these groups, $|H| \big| 2^2 \cdot 3 \cdot 5 \cdot 7$. We assert that $7 \notin \pi(H)$. Otherwise if $7 \in \pi(H)$, since $13 \nmid (7-1) = 6$, $11 \nmid (7-1) = 6$, viewing the actions of element of order 11 and 13 of G on the $7-Sylow$ subgroup of H by conjugation, we come to $13 \times 7,\ 11 \times 7 \in \pi_e(G)$ by Lemma 2.4. If $2 \in \pi(H)$, then $11 \times 2 \in \pi_e(G)$, and G has only one prime graph component, a contradiction. Hence $2 \notin \pi(H)$. Now consider the action of subgroup of order 2^2 of G on the $7-Sylow$ subgroup of H and noticing that $2^2 \nmid (7-1) = 6$, we get by Lemma 2.4 that $7 \times 2 \in \pi_e(G)$, which implies that the prime graph of G is connected, a contradiction.

Suppose $5 \in \pi(H)$, considering actions of elements of order 11 and 13 of G on the $5-Sylow$ subgroup of H by conjugation respectively, we get $13 \times 5,\ 11 \times 5 \in \pi_e(G)$. Noticing that 5 is adjacent to $2,\ 3,\ 5,\ 7$ in A_{13}, A_{14} and A_{15}, we come to that the prime graph is connected, a contradiction. Hence $5 \notin \pi(H)$. Similarly, we can prove that $2 \notin \pi(H)$ and $3 \notin \pi(H)$. Hence $H = 1$.

Now $K = A_{13},\ A_{14}$ or A_{15}. By the prime graph of G is connected, it follows that $C_G(K) = 1$. Thus $K \leq G \leq Aut(K) = S_{13},\ S_{14}$ or S_{15}. at last by $o_1(G) = 105$, we come to $G = K = A_{15}$. □

Remark 2.9. Because the 1st $ONC-$degree of an alternating group usually contains only three numbers, so such kind of characterization involved not many numbers for alternating groups. But it is for this reason, counterexamples may appear in the alternating groups. Since we cannot show A_{15} can be characterized by the 1st $ONC-$degree, it is worth to study whether A_{15} is a counterexample. Surely if we add the condition that the prime graph of G is not connected, then we can prove $G \cong M$ for many alternating groups M while $ONC_1(G) = ONC_1(M)$. In fact, we checked for $M = A_n(n = 17,\ 18,\ 19,\ 20)$, if G is a finite group with non-connected prime graph, then $G \cong M$ if and only if $ONC_1(G) = ONC_1(M)$. Because of the length of the paper, we will not describe the results here.

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